Generalized Hadamard Matrices from Generalized Orthogonal Matrix

Sudha Singh¹, M. K. Singh², D. K. Singh³

Abstract-A new generalization of matrix orthogonality is introduced. It is shown that from generalized orthogonal matrices some known as well as a few new complex H-matrices with circulant blocks can be obtained. The orders of new complex H-matrix are 26, 36, 50 and 82.

IndexTerms- Circulant matrix, Hadamard matrix, Generalization of Hadamard matrix, Quaternion, Associative algebra of matrices, generalized orthogonal matrix.

I. INTRODUCTION

First we recall the following definitions:

Circulant Matrix: It is an n×n matrix of the form
\[
\begin{pmatrix}
a_1 & a_2 & a_3 & \cdots & a_n \\
a_n & a_1 & a_2 & \cdots & a_{n-1} \\
a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_2 & a_3 & a_4 & \cdots & a_1
\end{pmatrix}
\]
which is denoted as \(\text{Circ}(a_1, a_2, a_3, \ldots, a_n)\).

Hadamard matrix (or an H-matrix): It is an n×n matrix H with entries +1, -1 such that \(HH^T = nI_n\), where \(I_n\) is the n×n identity matrix.

Complex H-matrix: It is an n×n matrix \(H = [H_{ij}]\), where \(H_{ij}\) are complex numbers with \(|H_{ij}| = 1\) for \(i, j = 1, 2, \ldots, n\), satisfying \(HH^* = nI\), where \(I\) is the identity matrix and \(H^*\) denotes the Hermitian transpose of \(H\). A complex H-matrix is called dephased if elements of its first row and column are 1.

Butson H-matrix: It is an n×n complex Hadamard matrix with elements belonging to the set of \(m^{th}\) roots of 1 and is denoted as \(BH(m, n)\).

Unimodular complex H-matrix: It is an n×n complex H-matrix whose elements are of the form \(\exp(i\theta)\). An \(m\)-parameter-affine complex Hadamard family (or orbit) \(H(R)\) stemming from a dephased n×n complex Hadamard matrix H is the set of matrices A satisfying \(AA^* = nI\), associated with an m-dimensional subspace R of a space of all real n×n matrices with zeros in the first row and column.

Weighing matrix \(W(n, w)\): A \(W(n, w)\) of order \(n\) and weight \(w\) is an n×n (0, 1, -1)-matrix such that \(WW^T = wI\), where \(w\) is a positive integer.

Conference matrix: It is a weighing matrix \(W(n, n-1)\) with \(0\) occurring only on the diagonal.

Quaternion: A number of the form \(q = a_1 + bi + cj + dk\), where \(i^2 = j^2 = k^2 = -1\), \(k = ij = -ji\), \(a, b, c, d\) are real numbers, is called a quaternion or a hypercomplex number. \(q\) reduces to a complex number when \(c = d = 0\) and to a real number when \(b = c = 0\). If \(i, j, k\) are taken as \(2 \times 2\) matrices
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & i \\
i & 0
\end{pmatrix}
\]
respectively we get two dimensional complex matrix representation of the quaternion \(q = \begin{pmatrix}
a - ci \\
b + di
\end{pmatrix}\), \(\begin{pmatrix}
-a + ci \\
-b - di
\end{pmatrix}\).

Replacing 1 by \(\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}\) and \(i\) by \(\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}\) in the matrices 1, i, j, k we get four dimensional real matrix representation of the quaternion \(q = \begin{pmatrix}
a & -c & b & d \\
c & a & -d & b \\
b & d & a & c \\
-d & -b & -c & a
\end{pmatrix}\).

The Hermitian conjugate of a quaternion \(q = a_1 + ib + jc + kd\) is \(\bar{q} = a_1 - ib - jc - kd\) and the modulus of \(q\) is
\[|q| = \sqrt{a^2 + b^2 + c^2 + d^2}.
\]

Associative algebra of matrices: Let \(R\) be a ring of complex numbers and \(A\) be a vector space of \(v \times v\) matrices over \(R\) with basis matrices \(I = A_0, A_1, \ldots, A_m\). A is called an associative algebra with unity \(I\) if they satisfy
\[A_iA_j = \sum_{k=0}^{m} p_{ij}^k A_k ,\]
where \(p_{ij}^k\) are in general complex numbers.
In what follows we assume that the elements of $A_i$ are only 0, 1 and $-1$ and $p^2$ are integers called multiplication coefficients of the algebra.

Example 1 Algebra of quaternions spanned by the matrices 

$$I, i, j, k$$ of order $n$, that span will be denoted as $w^i = \begin{bmatrix} 0,1,0,0 \end{bmatrix}$.

Example 2 Algebra of circulant matrices spanned by $A_i = w^i = \begin{bmatrix} 0,1,0, \ldots, 0 \end{bmatrix}$, $i=1,2,\ldots,m$ satisfying $A_0+A_1+\ldots+A_m = I$, (all 1 matrix) and (1), where $A_0 = I$, and $p^2$ are nonnegative integers.

(A_0, A_1, \ldots, A_m) defines an $m$-class association scheme (or $m$-AS) with parameters $p^2$.

A 2-AS is also called strongly regular graph and its parameters satisfy $p^2 + p^2 = n$, $i=1,2$, and $p^2$ satisfy $p^2 - p^2 = \delta$, $n_1+n_2 = \nu - 1$, and $p^2 + p^2 = \delta$, $i=1,2$, where $\delta = 0$ for $i=j$ and $\delta = 1$ for $i=j$. (see Raghavarao[4]).

**Generalized orthogonal matrix (GOM):** Let $A$ be an $m \times n$ matrix whose entries are the element of an associative algebra of matrices over a ring of complex numbers. The conjugate of an element $a = \sum_{i \in G} a_i A_i A_i^T$ will be denoted by $\overline{a} = \sum_{i \in G} a_i^* A_i A_i^T$, where $a_i^*$ is the complex conjugate of $a_i$, and $T$ stands for transpose.

A will be called a generalized orthogonal matrix if the dot product of any two rows $R_i, R_j = (a_{i1}, a_{i2}, \ldots, a_{im})(b_{j1}, b_{j2}, \ldots, b_{jn})$

$$= \sum_{k=1}^{n} a_{ik} b_{jk}$$

$$= \begin{cases} \lambda I, & \text{if } i \neq j \\ \alpha_i + \lambda_i \sum_{i=1}^{m} A_i & \text{if } i = j. \end{cases}$$

where $\lambda$, $\lambda_0$, $\lambda_i$ are parameters independent of $i$ and $j$. Here $\lambda$, $\lambda_0$, $\lambda_i$ will be called parameters of orthogonal matrix $A$.

The purpose of this paper is to show that notion of generalized orthogonal matrix provides a general framework for constructing several classical real H-matrices of Paley[3], Williamson[7] and Ito[1] as well as some new Butson H-matrices and GDG H-matrices through special methods or computer search. We also identify some Butson H-matrices which admit non-Dita-type affine complex Hadamard family (or orbit) (vide szollosi [5]) Such matrices are recently being used in quantum information theory and quantum tomography.

Notations: The circulant matrix $circ(0,1,0,\ldots,0)$ will be denoted as $w_n$. The direct product of $w_m, w_n$ will be denoted as $w_{mxw_n}$.

**II. CONSTRUCTION OF COMPLEX H-MATRICES FROM GENERALIZED ORTHOGONAL MATRICES**

**A. Construction of some H-matrices with circulant blocks**

**Construction of certain Paley type-1 H-matrices** (for definition see page 12, chapter 2 of 10)

Theorem I: Let $p = 4t-1$ be a prime. If $(d_1, d_2, d_3, \ldots, d_k) \mod p$ be a difference set[11,10], then GO-matrix $A = [w p d_1 + w p d_2 + \ldots + w p d_k]$, where $w_p = Circ(0,1,0,0,\ldots,0)$ gives the core of a H-matrix of order $4t$, if we replace 0 by -1 in $A$.

**Construction of H-matrices of Williamson’s form**

Williamson H-matrix of order $4(2m+1)$ is itself a $1 \times 1$ generalized orthogonal matrix $H = 1 \times A + i \times B + j \times C + k \times D$, where $1, i, j, k$ are 4x4 matrix representation of basic quaternions,

$$1 = I_{4, i} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}$$

and $A, B, C, D$ are (+1,-1) suitable linear combinations of (0,1)-circulant matrices $W_1, W_2, W_3, \ldots, W_m$ of order $n$, that span a Bose-mesner algebras (see Raghavarao[4], for details of Bose-mesner algebra).

**Construction of whiteman’s type H-matrices of order** ($pq+1$) where $p$ and $q$ are twin primes (vide whiteman[8])

Theorem II: Let $p$ and $q$ be two primes, $q = p+2$.

An 1x1 generalized orthogonal matrix $A = [(1+w_p+w_p^2+\ldots+w_p^{p-1})I_q + (w_p + w_p q)^{d_1} + (w_p + w_p q)^{d_2} + \ldots + (w_p + w_p q)^{d_k}]$ where $w_p = Circ(0,1,0,0,\ldots,0)$

$$w_q = Circ(0,1,0,0,\ldots,0)$$

and $d$ satisfies $d_k \equiv 1 \mod p,$

$$d_k \equiv 1 \mod q,$$

$$k = \frac{(p-1)(q-1)}{2}$$

gives the core of H-matrix of order ($pq+1$), if we replace 0 by -1 everywhere in $A$.

**Construction of an H-matrix of order 36**

We consider on $3 \times 7$ rectangular generalized orthogonal matrix $A=$
\[
\begin{pmatrix}
\omega^4 + \omega^2 + \omega^3 & 0 & 0 & 1_5 & 1_5 & 1_5 \\
1_5 & 1_5 & \omega^4 + \omega^2 + \omega^3 & 0 & 0 & 0 \\
0 & 0 & 1_5 & 1_5 & 1_5 & \omega^4 + \omega^2 + \omega^3
\end{pmatrix},
\]

where
\[
\omega = \text{cis}(\frac{2\pi}{5}), 0 \times 5 \times 5 \text{ null matrix and } 1_5 \times \text{ unit matrix}.
\]

Then replacing 3 by 0 and 0 by -1 in \(A^T A\), we get the core of a H-matrix (see Horadam[10] for definition and details of core) of order 36.

B. H-matrices from generalized orthogonal matrices arising from BIBDs (see Hall [12] for BIBDs)

**Theorem III**: Existence of a BIBD with parameters 
\[
v = 2n^2 - n, \quad b = 4n^2 - 1, \quad r = 2n + 1, \quad k = n, \quad \lambda = 1
\]

implies the existence of an H-matrix of order 4\(n^2\).

**Method of construction**: Let \(N\) be the incidence matrix of BIBD with parameters mentioned in theorem III. \(N'N\) is a \(b \times b\) square matrix. Let \(A\) be the (1,-1) matrix obtained from \(N'N\) by replacing diagonal entries by -1, 1 by 1 and 0 by -1. Then \(A\) is a 1×1 generalized orthogonal matrix and
\[
A = \begin{pmatrix} -1 & e \end{pmatrix}, \quad \begin{pmatrix} e^T \end{pmatrix} \text{ is a H-matrix of order } 4n^2 \text{ where } e \text{ is a } 1 \times (4n^2 - 1) \text{ matrix of 1's, } e^T \text{ is the transpose of } e.
\]

Example 4: We consider the BIBD
Parameters: \(v = 6, b = 15, r = 5, k = 2, \lambda = 1\).

Let \(N\) be the incidence matrix of BIBD with given parameters.
Its dual \(N'\) is
\[
\begin{pmatrix}
000011 \\
110000 \\
101000 \\
100100 \\
100010 \\
100001 \\
011000 \\
010100 \\
010010 \\
010001 \\
001100 \\
001010 \\
001001 \\
000110 \\
000101 
\end{pmatrix}
\]

The product of \(N\) and \(N'\) is
\[
\begin{pmatrix}
211111110100000 \\
121111000110000 \\
112110100010110 \\
111210010101011 \\
110002111100000 \\
110010101010101 \\
1001011211101110 \\
1001101120101211 \\
1010011001111201 \\
1010101011111111 \\
1001010100011120 \\
1010101101201111 \\
1001010101011121 \\
1000101010101010 \\
0001010101010101 \\
0001100110111111
\end{pmatrix}
\]

This matrix attains an affine orbit by lemma 3.4 (see SZOLLOSI [5]).

Example 5: We consider the BIBD
Parameters: \(v = 15, b = 35, r = 7, k = 3, \lambda = 1\).

Let \(N\) be the incidence matrix of BIBD with given parameters. Its dual \(N'\) is
\[
\begin{pmatrix}
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{pmatrix}
\]

This matrix attains an affine orbit by lemma 3.4 (see SZOLLOSI [5]).

Fig 1: MATRIX-1
The product of $N$ and $N'$ is

$$= 3A_0 + 1A_1 + 0A_2$$

where $I = A_0, A_1, A_2$ span Bose-Mesner algebra. From $NN'$ we can obtain a $1 \times 1$ generalized orthogonal matrix $A$ by replacing 3 by 0 and 0 by -1. Adjoining a row of all 1's and a column of all 1's we get the following $36 \times 36$ H-matrix

**Fig 2: MATRIX-2**

This matrix attains an affine orbit by lemma 3.4 (see SZOLLOS [5]).

**C. Construction of some new Butson H-matrices**

**Example 6:** Butson H-matrices can be constructed from the following circulant representation of some generalized orthogonal matrices:

(i) $A_5 = [w + w^4, w^2 + w^3]$, where $w = w_5 = \text{circ}(01000)$

(ii) $A_{13} = [w + w^3 + w^9, w^2 + w^5 + w^6]$, where $w = w_{13} = \text{circ}(0100...0)$ of order 13

(iii) $A_5 = [\text{circ}(1 + w^4, 0,0,1) \text{circ}(1 + w^3, 0, w^2, 1,0)]$, where $w = w_5 = \text{circ}(01000)$

(iv) $A_{41} = [w + w^{37} + w^{16} + w^{18} + w^{10}, w^8 + w^9 + w^5 + w^{21} + w^{39}]$, where $w = \text{circ}(010...0)$ of order 41.

**Method of Construction:** Let $A$ be any of the matrices above in (i), (ii), (iii) or (iv).

a) Obtain the symmetric square matrix $A'A$.

b) In $A'A$ replacing diagonal element by 1, 0 by -i and 1 by i, we get Butson H-matrices $BH(4,2n)$ for $2n = 10, 26, 50$ and 82.

c) In $A'A$ replacing diagonal elements by 0, we get a conference matrix.

**Remark 1** The matrices of above orders constructed from circulant matrices of order 5, 13 and 41 appears to be different from those arising from well-known constructions from Galois fields of order 25, 49 and 81.

**Remark 2:** Since Hadamard matrices obtained in the above theorem are derivable from conference matrices, each matrix $A$ is non Dita-type and admits an affine family of complex Hadamard matrices of at least one parameter which contains $A$ (vide sz"oll"osi’s theorems 4.1 and 4.2 in [5]).

**Remark 3:** In the recent catalogue [6] only Dita-type matrices were considered in dimensions $N = 10$ and 14. Sz"oll"osi [5] presents non Dita-type matrix of order 10. In view of Theorem 4.1 and of 4.2 of sz"oll"osi we can now present new parametric families of non Dita-type complex Hadamard matrices of order 26, 50, 82.

(1) $BH(4,26)$

A Butson H-matrix of order 26 obtained by the above method is:

**Fig 4: MATRIX-4**

(2) $BH(4,50)$

A Butson H-matrix of order 50 obtained by the above method is:

**Fig 5: MATRIX-5**

**D. Some Butson H-matrices $BH(m,n)$ for $m = 3, 6$.**

Following Butson H-matrices are obtained from suitable generalized orthogonal matrices

(i) $BH(3,6)$:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & A & 1 & 1 & 1 \\
1 & 1 & A & 1 & 1 \\
1 & 1 & 1 & A & 1 \\
1 & 1 & 1 & 1 & A
\end{pmatrix}$$

where the core $A$ is given by $A = [I_5 + w(w_5 + w_3^2) + w^2(w_5^2 + w_3^2)]$,

where $w_5 = \text{circ}(01000)$ and $w$ is an imaginary cube root of unity.

(ii) $BH(3,9)$

$$\begin{pmatrix}
w & w & w & w & w & w & w & w & w \\
w & w & w & w & w & w & w & w & w \\
w & w & w & w & w & w & w & w & w \\
w & w & w & w & w & w & w & w & w \\
w & w & w & w & w & w & w & w & w \\
w & w & w & w & w & w & w & w & w \\
w & w & w & w & w & w & w & w & w \\
w & w & w & w & w & w & w & w & w \\
w & w & w & w & w & w & w & w & w
\end{pmatrix}$$
(iii) BH(6, 7)

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -w^2 & -1 & -1 & w & -w & -w \\
1 & -1 & -w^2 & -1 & w & w & w \\
1 & -1 & w & w & -1 & -1 & -1 \\
1 & w & -w & w & -1 & -1 & -1 \\
1 & -w & w & w & -1 & -1 & -w^2
\end{pmatrix}
\]

III. Conclusion

Butson H-matrices are constructed from generalized orthogonal matrices by replacement or minor changes. During constructions we get new complex H-matrices of orders 26, 36, 50 and 82, which is not equivalent to existing complex Hadamard matrices of same order. We hope that in future generalized orthogonal matrices will provide insights to construct more matrices of combinatorial and practical interests.

IV. References

Fig 3: Matrix-3

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
Fig 4: Matrix-4

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}
\]
Fig 5: Matrix-5

\[ i \mid l - 1 - 1111 - 11 - 11 - 1 - 1 - 1 - 1 - 1111 - 111111 - 111111 - 11 - 1 - 111111 - 1 - 11 - 11 - 11 \\
1 \mid l - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-111l - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-1111l - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-1111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-111111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-11111111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-1111111111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-111111111111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-1111111111111111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-111111111111111111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
-11111111111111111111111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 \\
1111 - 111111 - 1 - 111111 - 1 - 1 - 11 - 11 - 11 - 11111 - 111111 - 111111 - 11 - 1 - 111111 - 11 - 11 - 11 - 11 
\]