Global Journal of Researches in Engineering: E
Civil And Structural Engineering
Volume 14 Issue 1 Version 1.0 Year 2014
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4596 & Print ISSN: 0975-5861

C^0- Continuity Isoparametric Formulation using Trigonometric Displacement Functions for One Dimensional Elements

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GJRE-E Classification : FOR Code: 090599, 090506

C0CONTINUITYISOPARAMETRICFORMULATIONUSINGTRIGONOMETRICDISPLACEMENTFUNCTIONSFORONEDIMENSIONALELEMENTS

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C^0- Continuity Isoparametric Formulation using Trigonometric Displacement Functions for One Dimensional Elements

Esmaeil Asadzadeh [α] & Mehtab Alam [σ]

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I. Introduction

Finite element method (FEM) is the approximate piecewise analysis in the domain of interest, researchers have put in efforts to select an appropriate interpolating function which can very closely approximate the field variable and converge to the exact solution. Polynomials have been studied for many years, starting in the 19th century, and they have shown to have mostly good approximation properties. Nevertheless, they are not ''good for all seasons'' [1]. In [2], it was shown that for differential equations with rough coefficients, the finite element method using polynomial shape functions can lead to arbitrarily ''bad'' results. Effective shape functions should have good approximation properties in entire domain of the interest. To increase the accuracy of the solution various procedures for error estimation have been devised and mesh refinement is used. Various procedures exist for the refinement of finite element (FE) solutions. More researches have been reported on the references [4-14].

By considering the linear-strain triangular (LST) element it can be seen that the development of element matrices and equations expressed in terms of a global coordinate system becomes an enormously difficult task [15]. The isoparametric method may appear somewhat tedious (and confusing initially), but it leads to a simple computer program formulation, and it is generally applicable for one-, two- and three-dimensional stress analysis and for nonstructural problems. The isoparametric formulation allows elements to be created that are nonrectangular and have curved sides [16].

In this paper, we first illustrate the trigonometric isoparametric formulation to develop the shape functions of C^0 continuity of the family of one dimensional bar elements and to derive the strain matrix, stiffness matrix and then force vector. Use of the bar element makes it relatively easy to understand the method because it involves simple expressions. Then quantitative concepts for assessing and comparing effectiveness of these families of shape functions are given. Focus on the principles that should govern the selection of the trigonometric shape functions are discussed, and one dimensional elements are studied by employing these new shape functions obtained from trigonometric displacement functions to analyze the bars carrying the self-weight and the results have been compared with the exact solutions of classical methods of solid mechanics.

II. Isoparametric Concept and Coordinate Systems

The term isoparametric is derived from the use of the same shape functions (or interpolation functions) to define the element's geometric shape as are used to define the displacements within the element. Isoparametric element equations are formulated using a natural (or intrinsic) coordinate system T that is defined by element geometry and not by the element orientation in the global coordinate system. In other words, axial coordinate T is attached to the bar and remains directed along the axial length of the bar, regardless of how the bar is oriented in space [16]. The relationship between the natural coordinate system T and the global coordinate system X for each element of specific structure is called the *transformation mapping* and must be used in the element equation formulations. The coordinate systems are shown in fig. 1.

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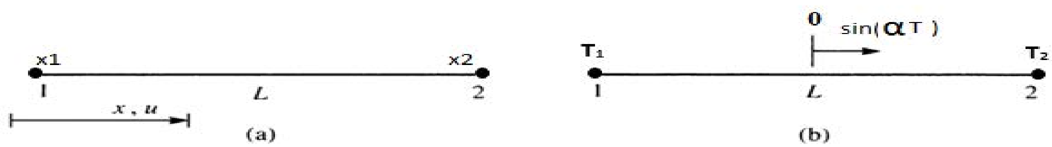

Figure 1 : Bar element in (a) a global coordinate system X and (b) a natural coordinate system T

The natural coordinate system **T** is a dimensionless quantity varying from T_1 to T_2 at node 1 and node 2 respectively. In natural coordinate system the position of any point inside the element is varying by Sin (αT). The natural coordinate system is attached to the element, with the origin located at its center, as shown in Fig. 1(b). The **T** axis needs not be parallel to the **x** axis, this is only for convenience.

For the special case consider a circle of unit radius shown in Fig.2, when the **T** and **x** axes are inside the circle and parallel to each other. The **T** and **x** axes having the origin located at the center of the element are coincided at the center of the circle ($X_c = \frac{x_1+x_2}{2}$). For the special case when $\alpha = \frac{\pi}{2}$ and the $-1 \leq T \leq 1$ and $-1 \leq x \leq 1$ the global and natural coordinates can be related by

$$X = X_c + \frac{L}{2}\sin(\frac{\pi}{2}T) \qquad (1)$$

Where X_c is the global coordinate of the element's centroid.

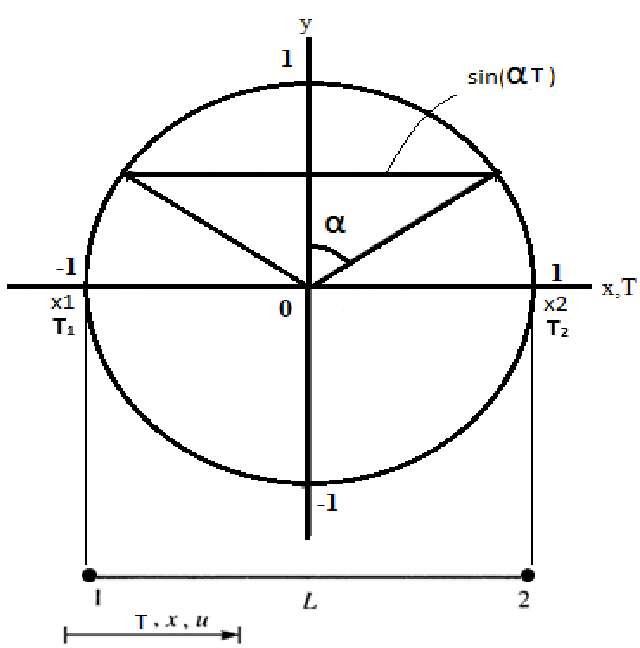

Figure 2 : Transformation mapping of global and natural coordinate system for bar element inside a circle

The displacement function within the bar which relates the displacement at any point inside the element to the nodal displacements is given by

$$U = \sum N_i U_i \qquad (2)$$

The function which relates the coordinate of any point within the element to the global coordinate is given by

$$X = \sum N_i X_i \qquad (3)$$

By using the equation (3) the shape functions have been used for coordinate transformation from natural coordinate system to the global Cartesian system and mapping of the parent element to required shape in global system successfully achieved. This formula is given by Taig [17].

In Eq. (3) the summation is over the number of nodes of the element. N is the shape function, U_i are the nodal displacements and X_i is the coordinates of nodal points of the element. The shape functions are to be expressed in natural coordinate system.

The equations (2) and (3) can be written in matrix form as

$$\{U\} = [N]\ \{U\}_e \qquad (4)$$

$$\{X\} = [N]\ \{X\}_e \qquad (5)$$

Where {U} is vector of displacement at any point, $\{U\}_e$ is vector of nodal displacements, $\{X\}_e$ is the vector of nodal coordinates and {X} is the vector of coordinate of any point in global system.

III. Interpolation Model and Shape Functions for Two Nodded Element

The quality of approximation achieved by Rayleigh–Ritz and FE approaches depends on the admissible assumed trial, field or shape functions. These functions can be chosen in many different ways. The most universally preferred method is the use of simple polynomials. It is also possible to use other functions such as trigonometric functions [18, 19]. While choosing the interpolation model and shape functions, the following considerations have to be taken into account[3, 20].

a) To ensure convergence to the correct result certain simple requirements must be satisfied as following criteria.

Criterion 1. The displacement shape functions chosen should be such that they do not permit straining of an element to occur when the nodal displacements are caused by a rigid body motion.

Criterion 2. The displacement shape functions have to be of such forms that if nodal displacements are compatible with a constant strain condition such constant strain will in fact be obtained.

Criterion 3. The displacement shape functions should be chosen such that the strains at the interface between elements are finite (even though they may be discontinuous).

b) The pattern of variation of the field variable resulting from the interpolation model should be independent of the local coordinate system.
c) The number of generalized coordinates should be equal to the number of nodal degrees of freedom of the element.

The interpolation model of the field variable (displacement model inside the element) in terms of nodal degrees of freedom is given by trigonometric sine function as

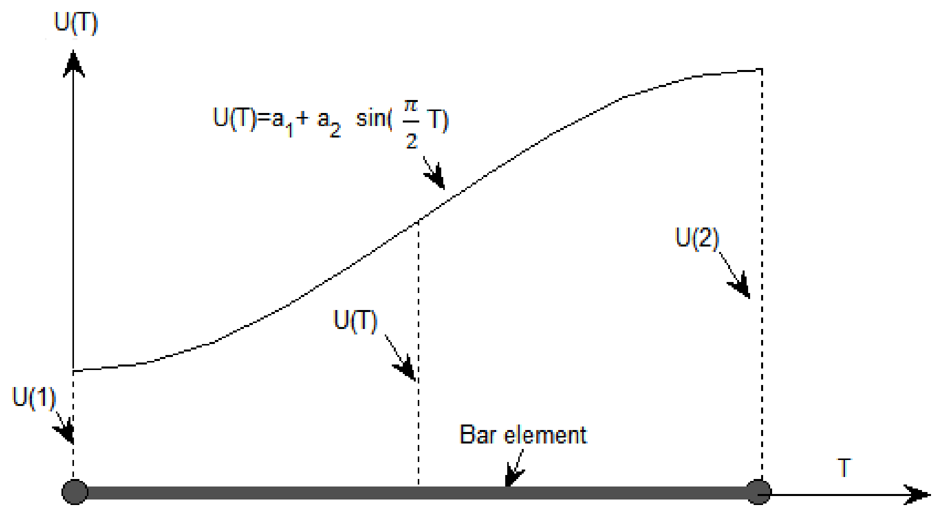

Figure 3 : Two nodded bar element and variation of displacement inside the element in natural coordinate system for $-1 \leq T \leq 1$

$$U(T) = a_1 + a_n \sin(\tfrac{\pi}{2}T) \quad \text{Where} \quad -1 \leq T \leq 1 \qquad (6)$$

Where a_1 and a_n, are the coefficients known as generalized coordinates and must be equal to the number of nodal unknowns M. In equation (6), the nodal values of the solution, also known as nodal degrees of freedom, are treated as unknowns in formulating the system or overall equations. To express the interpolation model in terms of the nodal degrees of freedom of a typical finite element **e** having *M* nodes, the values of the field variable at the nodes can be evaluated by substituting the nodal coordinates into the Eq. (6). The Eq. (6) can be expressed in general form of

$$\vec{U}_{(T)} = \vec{\eta}\vec{a} \qquad (7)$$

Where, $\vec{U}_{(T)} = U(T)$,

$$\vec{\eta}^T = \{1 \quad \sin(\tfrac{\pi}{2}T)\} \qquad (8)$$

$$U(1) = a_1 + a_2 \sin(-\frac{\pi}{2})$$

$$U(2) = a_1 + a_2 \sin(\frac{\pi}{2})$$

And, $\vec{a} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$

The evaluation of equation (7) at the various nodes of element **e** gives

$$\begin{Bmatrix} \vec{U}(at\ node\ 1) \\ \vec{U}(at\ node\ 2) \end{Bmatrix}^{(e)} = \vec{U}^{(e)} = \begin{bmatrix} \vec{\eta}^T(at\ node\ 1) \\ \vec{\eta}^T(at\ node\ 2) \end{bmatrix} \vec{a} \equiv \lfloor \eta \rfloor \vec{a}$$

$$\begin{Bmatrix} \vec{U}(1) \\ \vec{U}(2) \end{Bmatrix}^{(e)} = \vec{U}^{(e)} = \begin{bmatrix} 1 & \sin(-\frac{\pi}{2}) \\ 1 & \sin(\frac{\pi}{2}) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \equiv \lfloor \eta \rfloor \vec{a} \qquad (9)$$

Where $\vec{U}^{(e)}$ is the vector of nodal values of the field variable corresponding to element **e**, and the square matrix $\lfloor \eta \rfloor$ can be identified from Eq. (9). By inverting Eq. (9), we obtain

$$\vec{a} = \left[\underline{\eta}\right]^{-1} \vec{U}^{(e)} \tag{10}$$

Substituting the Eq. (10) Into Eq. (7) gives

$$\vec{U} = \vec{\eta}^T \vec{a} = \vec{\eta}^T \left[\underline{\eta}\right]^{-1} \vec{U}^{(e)} = [N]\vec{U}^{(e)} \tag{11}$$

$$\text{Thus } [N] = \vec{\eta}^T \left[\underline{\eta}\right]^{-1} \tag{12}$$

Where, $[N]$is the matrix of interpolation functions or shape functions.

Equation (11) expresses the interpolating function inside any finite element in terms of the nodal unknowns of that element, $\vec{U}^{(e)}$. A major limitation of trigonometric interpolation functions is that one has to invert the matrix $\left[\underline{\eta}\right]$ to find $\vec{U}$, and $\left[\underline{\eta}\right]^{-1}$ may become singular in some cases.

a) Two Nodded Bar Element With Trigonometric Shape Functions

There are two unknowns for two nodded bar element, therefore there must be only two shape functions N_1 and N_2 which are derived by following the foregoing procedure. The shape functions are given as

$$\begin{cases} N_1 = \dfrac{\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}T\right)}{\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)} \\ N_2 = \dfrac{\sin\left(\frac{\pi}{2}T\right) - \sin\left(-\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)} \end{cases} \tag{13}$$

Therefore, the shape functions are

$$\begin{cases} N_1 = \dfrac{1-\sin\left(\frac{\pi}{2}T\right)}{2} \\ N_2 = \dfrac{\sin\left(\frac{\pi}{2}T\right)+1}{2} \end{cases} \tag{14}$$

It must be noted that $-1 \le T \le 1$.

The variation of the resulting shape functions are shown in Fig. 4. The essential properties of shape functions are that they must be unity at one node and zero at the other nodes. It can be seen that by shifting the T to T_1 and T_2 we get

$$\begin{cases} At\ node\ 1\ where\ T = T_1 = -1 \\ N_1 = 1 \\ N_2 = 0 \end{cases} \quad \begin{cases} At\ node\ 2\ where\ T = T_2 = 1 \\ N_1 = 0 \\ N_2 = 1 \end{cases}$$

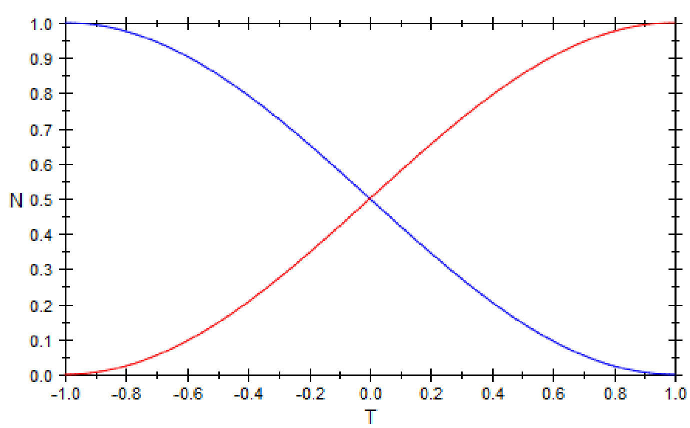

Figure 4 : Variation of shape functions for two nodded bar element

To have the C^0 continuity element the sum of the shape functions must be 1($i.e. \sum N_i = 1$) and the first derivative of the field variable must be zero ($i.e. \sum \frac{\partial N_i}{\partial T} = 0$). As there are two nodal unknowns U_1 and U_2 for node 1 and node 2 respectively, therefore in the natural coordinate system it can be written as

$$U = N_1 \vec{U}_1^e + N_2 \vec{U}_2^e \tag{15}$$

$$\begin{cases} N_1 + N_2 = 1 \\ \dfrac{1-\sin\left(\frac{\pi}{2}T\right)}{2} + \dfrac{\sin\left(\frac{\pi}{2}T\right)+1}{2} = 1 \end{cases}$$

And

$$\begin{cases} \dfrac{\partial N_i}{\partial T} = \dfrac{\partial N_1}{\partial T} + \dfrac{\partial N_2}{\partial T} = 0 \\ \dfrac{\partial N_i}{\partial T} = \dfrac{-\frac{\pi}{2}\cos(\frac{\pi}{2}T)}{2} + \dfrac{\frac{\pi}{2}\cos(\frac{\pi}{2}T)}{2} = 0 \end{cases}$$

It can be seen that the two essential requirements of the C^0 continuity element are satisfied.

It is of interest to mention that there is clear difference between the interpolation model of the element $\vec{U}_{(T)} = \vec{\eta}\vec{a}$ that applies to the entire element and expresses the variation of the field variable inside the element in terms of the generalized coordinates a_i and the shape function N_i that corresponds to the i^{th} nodal degree of freedom $\vec{U}_i^e$ and only the sum $\sum N_i \vec{U}_i^e$ represents the variation of the field variable inside the element in terms of the nodal degrees of freedom $\vec{U}_i^e$. In fact, the shape function corresponding to the i^{th} nodal degree of freedom N_i assumes a value of 1 at node *i* and 0 at all the other nodes of the element[20].

b) Mapping of the element in global coordinate system

The mapping of the parent element in global coordinate system can be done by using eq. (2) which can be written in matrix form as

$$\{\boldsymbol{X}\} = [N_1 \ \ N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{16}$$

The trigonometric shape functions in Eq. (14) map the T coordinate of any point in the element to the X coordinate. It is clear that by substituting T= -1 and T=1, we obtain $X=x_1$ and $X=x_2$ respectively.

c) Strain – displacement and stress - strain relationship

To formulate the element strain matrix [B] to evaluate the element stiffness matrix [K] the isopramatric formulation is used. The strain is defined in terms of the natural coordinate system T varying inside the element from the center of the element to -1 or 1. To determine the strain which is the first derivative of the displacement with respect to X the chain rule of the differentiation must be used. This is given as

$$\frac{du}{dT} = \frac{du}{dx}\frac{dx}{dT} \qquad where\ X = X_c + \frac{L}{2}\sin(\frac{\pi}{2}T) \tag{17}$$

Therefore

$$\frac{du}{dx} = \frac{\frac{du}{dT}}{\frac{dx}{dT}}$$

It is clear that $\frac{\partial U}{\partial T} = [\frac{-\frac{\pi}{2}\cos\left(\frac{\pi}{2}T\right)}{2} \ \ \frac{\frac{\pi}{2}\cos\left(\frac{\pi}{2}T\right)}{2}]\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ and $\frac{dx}{dT} = \frac{L}{2}\frac{\pi}{2}\cos(\frac{\pi}{2}T)$, therefore the Eq. (17) becomes

$$\epsilon = \frac{du}{dx} = \frac{1}{L}[-1 \ \ 1]\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}^{(e)} \tag{18}$$

Strain displacement relation is given as [3]

$$\epsilon = \sum B_i{}^e \vec{U}_i^e \tag{19}$$

Or in matrix form as

$$\{\epsilon\} = [B_i]^e\{U_i\}^e \tag{20}$$

Where, $\{\epsilon\}$ is strain at any point in the element, $\{U_i\}^e$ is displacement vector of nodal values of the element and $[B_i]^e$ is strain displacement matrix.

By comparing the Eq. (20) with expression given for the strain in Eq. (18) we have the strain displacement matrix of the bar as

$$[B] = \frac{1}{L}[-1 \ \ 1] \tag{21}$$

The essential necessity of liner interpolation functions is that the strain must be constant inside the element for with C^0- continuity. As it can be seen in Eq. (21) the strain is constant and is same as the stain matrix for bar element using the polynomial functions.

The stress strain relation is given by constitutive

$$\{\sigma\} = [D]\{\varepsilon\}^e = [D]\ [B]\{U_i\}^e \tag{22}$$

Where, $\{\sigma\}$ is the stress, $\{\varepsilon\}$ is the strain and $[D]$ is the matrix of constants of elasticity.

The stiffness matrix can be evaluated by using the following equation [16].

$$[k] = \iiint_0^v [B]^T[D][B]\,dV \tag{23}$$

The Eq. (23) can be written in the global coordinate system as

$$[K] = \int_0^L [B]^T[D][B]A\,dx \tag{24}$$

Where A is the cross section area of the bar

The eq. (24) is in terms of the global coordinate system and must be transformed to the natural coordinate system; because matrix [B] is, in general, a function of T. This general type of transformation is given by References [3, 16, and 21]. This can be done by following procedure

We know that $X = X_c + \frac{L}{2}\sin(\frac{\pi}{2}T)$, hence it can be written that

$$\frac{dx}{dT} = \frac{L}{2}\frac{\pi}{2}\cos(\frac{\pi}{2}T) \qquad therefore \qquad \frac{L}{2}\frac{\pi}{2}\cos(\frac{\pi}{2}T)\ dT = dx \tag{25}$$

By inserting Eq. (25) in Eq. (24), we can write the stiffness matrix in global coordinate system as

$$[\mathrm{k}]=\int_0^L [B]^T[D][B]A\,\frac{L}{2}\frac{\pi}{2}\cos(\frac{\pi}{2}T)\;dT \tag{26}$$

Or

$$[\mathrm{k}]=\int_0^L [B]^T[D][B]A\,|J|\;dT \tag{27}$$

Where, $|J| = \frac{dx}{dT} = \frac{\pi}{2}\frac{L}{2}\cos\left(\frac{\pi}{2}T\right)$ is the Jacobian determinant for one dimensional element with trigonometric displacement functions and relates an element length in the global coordinate system to an element length in the natural coordinate system. This is different from the Jacobian determinant for one dimensional element with polynomial displacement function given by $\frac{L}{2}$ but the concept is same.

By inserting the modulus of elasticity E= [D], Eq. (27) becomes

$$[\mathrm{k}]=\int_0^L [B]^T\,E\,[B]A\,\frac{L}{2}\frac{\pi}{2}\cos(\frac{\pi}{2}T)\;dT \tag{28}$$

By substituting the strain displacement matrix given in Eq. (21), the stiffness matrix can be evaluated as

$$[K] = \int_{-1}^{1}\frac{EA}{L^2}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\frac{L}{2}\frac{\pi}{2}\cos(\frac{\pi}{2}T)\,dT = \frac{EA\pi}{4L}\int_{-1}^{1}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\cos(\frac{\pi}{2}T)\,dT$$

Upon integrating we get the stiffness matrix as

$$[K] = \frac{EA}{L}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \tag{29}$$

It can be realized that the stiffness matrix is the same as that of given for the two nodded bar element evaluated employing the polynomial functions.

d) Derivation of the system equations in terms of the natural coordinate system

The body and surface forces in terms of the natural coordinate system can be evaluated by the following formulas

$$\{F\}^e = \iiint_V [N]^T[X_b]\,dV - \iint_S [N]^T[T_x]\,dS \tag{30}$$

Where, $\{F\}^e$ is the consistent load vector, X_b is the body force and T_x is the surface force or the traction force. Eq. (30), is in terms of the global coordinate system and by using the Jacobian determinant can be written in terms of natural coordinate system. For example for a bar having constant cross section it can be written as

$$\{F\}^e = \iiint_V [N]^T[X_b]A\,dx - \iint_S [N]^T[T_x]\,dx \tag{31}$$

By using the Eq. (25) in the natural coordinate system it can be written as

$$\{F\}^e = \iiint_V [N]^T[X_b]A\,\frac{L\pi}{4}\cos(\frac{\pi}{2}T)\;dT - \iint_S [N]^T[T_x]\,\frac{L\pi}{4}\cos(\frac{\pi}{2}T)\;dT \tag{32}$$

Where, the product of A and L represents the volume of the element and X_b the body force per unit volume, then $[X_b]A\frac{L\pi}{4}\cos(\frac{\pi}{2}T)$ is the total body force acting on the element and T_x is traction force-per-unit-length, $[T_x]\frac{L\pi}{4}\cos(\frac{\pi}{2}T)$ is now the total traction force.

The element equilibrium equation is

$$[K]\{U\}^e = \{F\}^e \tag{33}$$

The above equation of equilibrium is to be assembled for entire structure and boundary conditions are to be introduced. Then the solutions of equilibrium equations result into nodal displacements of all the nodal points. Once these basic unknowns are found, then displacement at any point may be obtained by Eq. (11), the strains may be assembled using the Eq. (12) and then stresses also can be found using the Eq. (22).

e) Shifting the domain from $-1 \le T \le 1$ to $0 \le T \le 1$

To shift the domain of the trigonometric function successfully from $-1 \le T \le 1$ to $0 \le T \le 1$ we consider a special case when the global coordinate system X and natural coordinate system T coincide and the centre of the circle shown in Fig.2 becomes the origin of the natural coordinate system T. It means that we consider only half of the element length shown in Fig.1. Therefore, the coordinates X and T can be related by

$$X = L\sin(\frac{\pi}{2}T) \tag{34}$$

The shape functions are given as

$$\begin{cases} N_1 = 1 - \sin\left(\frac{\pi}{2}T\right) \\ N_2 = \sin\left(\frac{\pi}{2}T\right) \end{cases} \tag{35}$$

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The variation of the resulting shape functions are shown in Fig. 5.

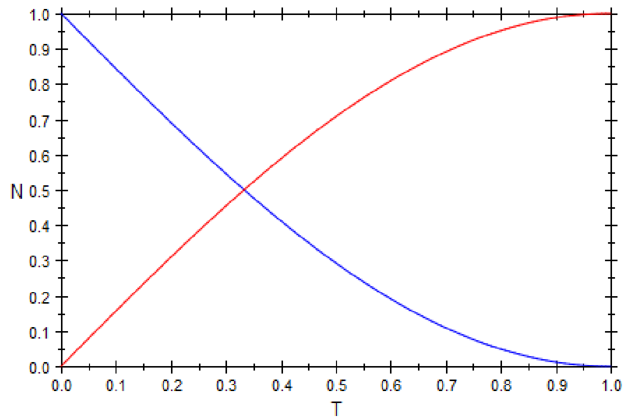

Figure 5 : Variation of shape functions for bar element in natural and global coordinate system

To relate the natural coordinate T (where, $0 \leq T \leq 1$) and global coordinate X (where, $0 \leq x \leq 1$) the Jacobian determinant given in Eq. (25) becomes

$$\frac{dx}{dT} = \frac{L\pi}{2}\cos(\frac{\pi}{2}T) \quad therefor \quad \frac{L\pi}{2}\cos(\frac{\pi}{2}T)\ dT = dx \tag{36}$$

The strain displacement matrix [B] will be same as given in Eq. (21) and the stiffness matrix [K] same as Eq. (29). The consistent forces will be as

$$\{F\}^e = \iiint_V [N]^T [X_b] A \frac{L\pi}{2}\cos(\frac{\pi}{2}T)\ dT - \iint_S [N]^T [T_x]\ \frac{L\pi}{2}\cos(\frac{L\pi}{2})\ dT \tag{37}$$

It must be noted that the limits of the integrations will be 0 to 1.

IV. Interpolation Model and Shape Functions for Three Nodded Element

To illustrate the concept of three nodded elements using the trigonometric functions, the element with three coordinates of nodes, x1, x2, and x3, in the global coordinates is shown in Fig. 6. Again the element is considered within a circle of unit radius and the third node is selected at the centre of the circle.

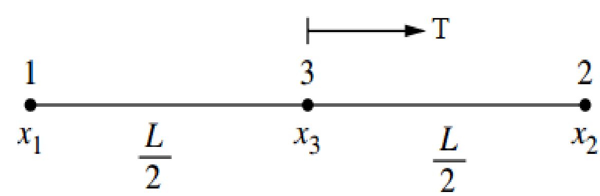

Figure 6 : Three nodded bar element in global coordinate system X

The interpolation model of the field variable (displacement model inside the element) in terms of nodal degrees of freedom is given by trigonometric function as

$$U(T) = a_1 + a_2 \sin\left(\frac{\pi}{2}T\right) + a_3(\sin(\frac{\pi}{2}\mathrm{T}))^2 \quad \text{Where} \quad \begin{cases} \mathrm{T} = -1 & at\ node\ one \\ \mathrm{T} = 1 & at\ node\ two \\ \mathrm{T} = 0 & at\ node\ three \end{cases} \tag{38}$$

Using the displacement field given in Eq. (38), the shape functions are given as

$$\begin{cases} N_1 = \dfrac{(\sin(\frac{\pi}{2}\mathrm{T}))^2 - sin(\frac{\pi}{2}T)}{2} \\ N_2 = \dfrac{(\sin(\frac{\pi}{2}\mathrm{T}))^2 + sin(\frac{\pi}{2}T)}{2} \\ N3 = 1 - (\sin(\frac{\pi}{2}\mathrm{T}))^2 \end{cases} \tag{39}$$

The variation of the resulting shape functions are shown in Fig. 7. The essential properties of shape functions are also satisfied as following

$$\begin{cases} At\ node\ 1\ where\ T = T_1 = -1 \\ N_1 = 1 \\ N_2 = 0 \\ N_3 = 0 \end{cases} \begin{cases} At\ node\ 2\ where\ T = T_2 = 1 \\ N_1 = 0 \\ N_2 = 1 \\ N_3 = 0 \end{cases} \begin{cases} At\ node\ 3\ where\ T = T_3 = 0 \\ N_1 = 0 \\ N_2 = 0 \\ N_3 = 1 \end{cases}$$

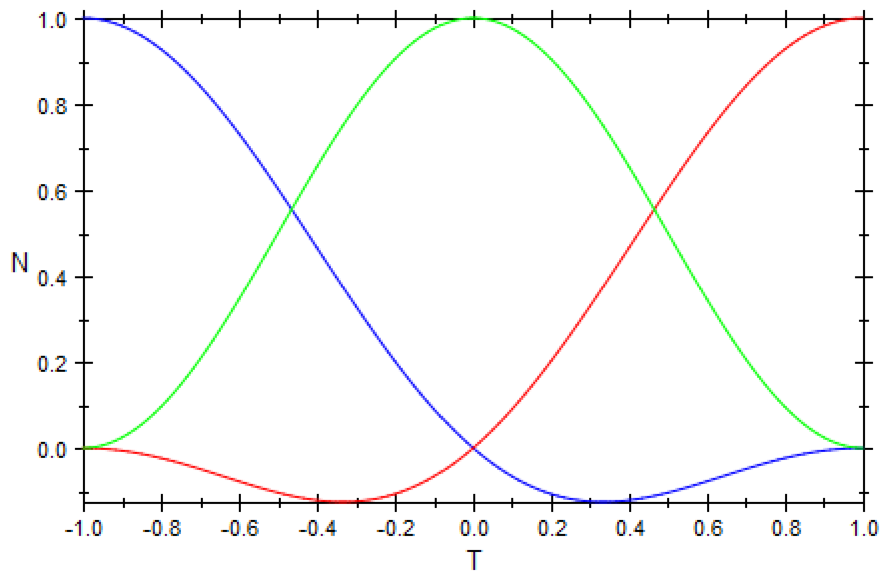

Figure 7 : Variation of shape functions for three nodded bar element

To have the C⁰ continuity for three nodded bar element here again $\sum N_i = 1$ and $\sum \frac{\partial N_i}{\partial T} = 0$. As there are three nodal unknowns U_1, U_2 and U_3 for node 1, 2 and node 3 respectively, therefore in the natural coordinate system it can be written as

$$U = N_1\vec{U}_1^e + N_2\vec{U}_2^e + +N_3\vec{U}_3^e \qquad (40)$$

$$\begin{cases} N_1 + N_2 + N_3 = 1 \\ \dfrac{(\sin(\frac{\pi}{2}T))^2 - \sin(\frac{\pi}{2}T)}{2} + \dfrac{(\sin(\frac{\pi}{2}T))^2 + \sin(\frac{\pi}{2}T)}{2} + 1 - (\sin(\frac{\pi}{2}T))^2 = 1 \end{cases}$$

And

$$\begin{cases} \dfrac{\partial N_i}{\partial T} = \dfrac{\partial N_1}{\partial T} + \dfrac{\partial N_2}{\partial T} + \dfrac{\partial N_3}{\partial T} = 0 \\ \dfrac{\partial N_i}{\partial T} = \dfrac{\pi}{2}cos\left(\dfrac{\pi}{2}T\right)sin\left(\dfrac{\pi}{2}T\right) - \dfrac{\pi}{4}cos\left(\dfrac{\pi}{2}T\right) + \dfrac{\pi}{2}cos\left(\dfrac{\pi}{2}T\right)sin\left(\dfrac{\pi}{2}T\right) + \dfrac{\pi}{4}cos\left(\dfrac{\pi}{2}T\right) - \pi cos\left(\dfrac{\pi}{2}T\right)sin\left(\dfrac{\pi}{2}T\right) = 0 \end{cases}$$

It can be seen that the two essential requirements of the C⁰ continuity element are satisfied.

a) *Strain – displacement and stress - strain relationship*

From our basic definition of axial strain we have

$$\{\epsilon\} = \frac{du}{dx} = \frac{\frac{du}{dT}}{\frac{dx}{dT}} = [B]\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}^{(e)} \qquad (41)$$

It has previously proven that $\frac{dx}{dT} = \frac{L\pi}{4}\cos(\frac{\pi}{2}T)$, this relationship holds for the three nodded one-dimensional elements as well as for the two-nodded constant strain bar element. Using this relationship and $\frac{\partial U}{\partial T}$ in Eq. (41), we obtain

$$\frac{du}{dx}=\left[\frac{4}{L\pi cos\left(\frac{\pi}{2}T\right)}\left(\left[\frac{\pi}{2}cos\left(\frac{\pi}{2}T\right)sin\left(\frac{\pi}{2}T\right)-\frac{\pi}{4}cos\left(\frac{\pi}{2}T\right),\quad \frac{\pi}{2}cos\left(\frac{\pi}{2}T\right)sin\left(\frac{\pi}{2}T\right)+\frac{\pi}{4}cos\left(\frac{\pi}{2}T\right),\right.\right.\right.$$
$$\left.\left.\left.-\pi cos\left(\frac{\pi}{2}T\right)sin\left(\frac{\pi}{2}T\right)\right]\right)\right]\begin{Bmatrix}U_1\\U_2\\U_3\end{Bmatrix}^{(e)}$$

Therefore

$$\frac{du}{dx}=\frac{1}{L}\left[2sin\left(\frac{\pi}{2}T\right)-1,\quad 2sin\left(\frac{\pi}{2}T\right)+1,\quad -4sin\left(\frac{\pi}{2}T\right)\right]\begin{Bmatrix}U_1\\U_2\\U_3\end{Bmatrix}^{(e)} \tag{42}$$

By comparing the expression given for the strain in Eq. (41) with Eq. (19), the strain-displacement matrix [B] for the three nodded bar is

$$[B]=\frac{1}{L}\left[2sin\left(\frac{\pi}{2}T\right)-1,\quad 2sin\left(\frac{\pi}{2}T\right)+1,\quad -4sin\left(\frac{\pi}{2}T\right)\right] \tag{43}$$

Substituting the expression for [B] into Eq. (27), the stiffness matrix is obtained as

$$[K]=\int_{-1}^{1}\frac{EA}{L^2}\begin{bmatrix}2sin\left(\frac{\pi}{2}T\right)-1\\2sin\left(\frac{\pi}{2}T\right)+1\\-4sin\left(\frac{\pi}{2}T\right)\end{bmatrix}\left[2sin\left(\frac{\pi}{2}T\right)-1,\quad 2sin\left(\frac{\pi}{2}T\right)+1,\quad -4sin\left(\frac{\pi}{2}T\right)\right]\frac{L\pi}{4}\cos\left(\frac{\pi}{2}T\right)dT$$

$$=\frac{EA}{L}\int_{-1}^{1}\begin{bmatrix}2sin\left(\frac{\pi}{2}T\right)-1\\2sin\left(\frac{\pi}{2}T\right)+1\\-4sin\left(\frac{\pi}{2}T\right)\end{bmatrix}\left[2sin\left(\frac{\pi}{2}T\right)-1,\quad 2sin\left(\frac{\pi}{2}T\right)+1,\quad -4sin\left(\frac{\pi}{2}T\right)\right]\frac{\pi}{4}\cos\left(\frac{\pi}{2}T\right)dT$$

$$=\frac{EA}{L}\int_{-1}^{1}\begin{bmatrix}(2sin\left(\frac{\pi}{2}T\right)-1)^2 & (2sin\left(\frac{\pi}{2}T\right))^2-1 & -8(sin\left(\frac{\pi}{2}T\right))^2+4sin\left(\frac{\pi}{2}T\right)\\(2sin\left(\frac{\pi}{2}T\right))^2-1 & (2sin\left(\frac{\pi}{2}T\right)+1)^2 & -8(sin\left(\frac{\pi}{2}T\right))^2-4sin\left(\frac{\pi}{2}T\right)\\-8(sin\left(\frac{\pi}{2}T\right))^2+4sin\left(\frac{\pi}{2}T\right) & -8(sin\left(\frac{\pi}{2}T\right))^2-4sin\left(\frac{\pi}{2}T\right) & (4sin\left(\frac{\pi}{2}T\right))^2\end{bmatrix}\frac{\pi}{4}\cos\left(\frac{\pi}{2}T\right)dT$$

Integrating the matrix the stiffness matrix for three nodded bar element becomes

$$[K]=\frac{EA}{L}\begin{bmatrix}2.333333 & 0.333333 & -2.666667\\0.333333 & 2.333333 & -2.666667\\-2.666667 & -2.666667 & 5.333333\end{bmatrix} \tag{44}$$

The stiffness matrix given in Eq. (27) is the same as that of given for the three nodded bar elements evaluated using polynomial functions.

Example 1. Analysis of bar of uniform cross section (A), Young's modulus of the material (E) due to self-weight (unit weight,ρ) when held as shown in Fig. 8. Self-weight acting in T direction.

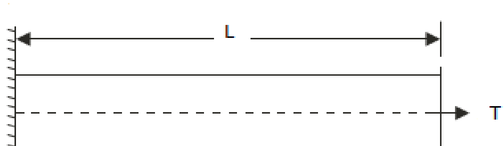

Figure 8 : Bar of constant cross section

By substituting the $\begin{Bmatrix}F_1\\F_2\end{Bmatrix}^{(e)}=\frac{\rho AL}{2}\begin{Bmatrix}1\\1\end{Bmatrix}^{(e)}$ obtained from Eq. (32), and boundary values of $\begin{Bmatrix}0\\U_2\end{Bmatrix}^{(e)}$ in Eq. (33), the extension of the bar evaluated is.

$$U_2=\frac{\rho L^2}{2E} \tag{45}$$

The Eq. (44) is the exact solution [22]. The strain may be evaluated using the Eq. (18) and stress also is found using the Eq. (22) as

$$\varepsilon = \frac{\rho L}{2E} \qquad (46)$$

$$\sigma = \frac{\rho L}{2} \qquad (47)$$

Equations (45) and (46) are the exact solutions for a bar having constant cross –section due to its own self weight.

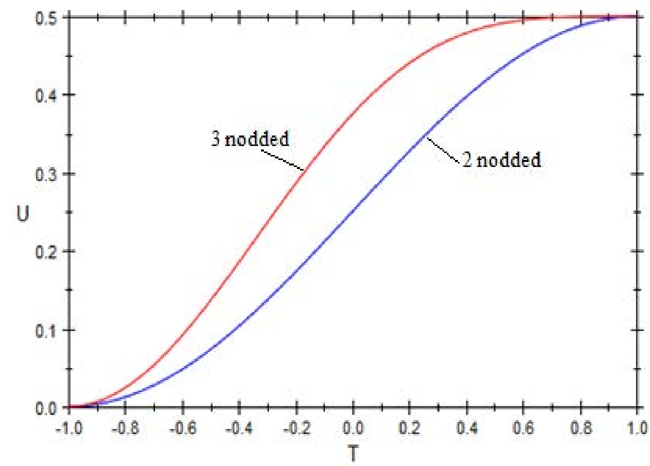

Figure 9 : Displacement of bar due to its self weigh using 2 and 3 nodded element with $-1 \leq T \leq 1$

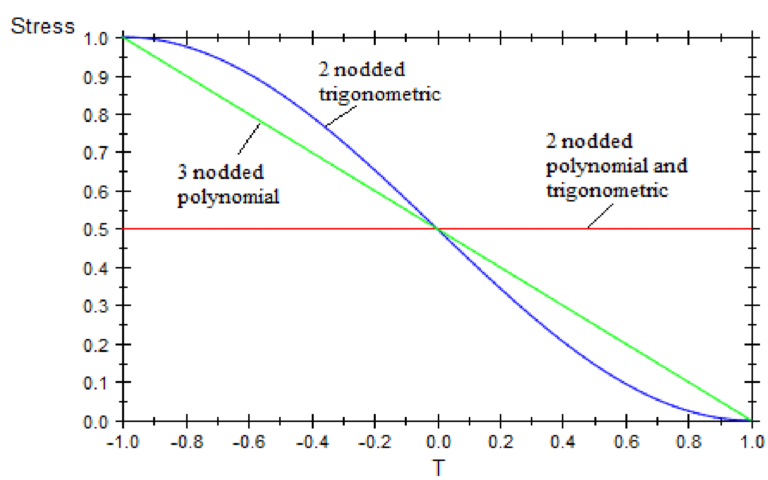

Figure 10 : Stress in the bar due to its self weigh using 2 and 3nodded element with $-1 \leq T \leq 1$

V. Conclusion

Using the trigonometric interpolation model, new family of C⁰- continuity elements are introduced. To obtain the constant stress and strain state in 2 nodded elements, trigonometric function is used instead of the polynomial Jacobian determinant to relate the natural and global coordinate system. The bar of uniform cross section is analyzed and results are compared with those of obtained using the polynomial functions.

References Références Referencias

1. Ivo Babusska, Uday Banerjee, John E. Osborn. On principles for the selection of shape functions for the Generalized Finite Element Method. Comput. Methods Appl. Mech. Engrg. 191:5595–5629, 2002.
2. I. Babusska, J. Osborn, Can a finite element method perform arbitrarily badly? Math. Comp. 69:443–462, 1999.
3. Zienkiwicz OC, Taylor RL. The Finite Element Method: Its Basis and Fundamentals, Sixth edition,published by ELSEVIER Butterworth-Heinemann; 2005.
4. W. Gui and I. Babuˇska. Theh, pand h-pversion of the finite element method in 1 dimension. Part 1: The error analysis of thep-version. Part 2: The error analysis of theh-andh-pversion. Part 3: The adaptiveh-pversion.Numerische Math., 48:557–683, 1986.
5. B. Guo and I. Babuˇska., The h-pversion of the finite element method. Part 1: The basic approximation results. Part 2: General results and applications. Comp. Mech., 1:21–41, 203– 226, 1986.
6. I. Babuˇska and B. Guo. Theh-pversion of the finite element method for domains with curved boundaries.SIAM J. Numer. Anal., 25:837–861, 1988.
7. I. Babuˇska and B.Q. Guo., Approximation properties of theh-pversion of the finite element method. Comp. Meth. Appl. Mech. Eng., 133:319–349, 1996.

8. L. Demkowicz, J. T. Oden, W. Rachowicz, and O. Hardy., Toward a universalh–padaptive finite element strategy. Part 1: Constrained approximation and data structure. Comp. Meth. Appl. Mech. Eng., 77:79–112, 1989.
9. W. Rachowicz, J.T. Oden, and L. Demkowicz., Toward a universalh–padaptive finite element strategy. Part 3: Design ofh–pmeshes.Comp. Meth. Appl. Mech. Eng., 77:181–211, 1989.
10. K.S. Bey and J.T. Oden.hp-version discontinuous Galerkin methods for hyperbolic conservation laws. Comp. Meth. Appl. Mech. Eng., 133:259–286, 1996.
11. C.E. Baumann and J. T. Oden., A discontinuoushpfinite element method for convection-diffusion problems., Comp. Meth. Appl. Mech. Eng., 175:311–341, 1999.
12. P. Monk., On thepandhpextension of Nedelecs curl-conforming elements.J. Comput. Appl. Math., 53:117–137, 1994.
13. L.K. Chilton and M. Suri., On the selection of a locking-freehpelement for elasticity problems. Int. J. Numer. Meth. Eng., 40:2045–2062, 1997.
14. L. Vardapetyan and L. Demkowicz. hp-Adaptive finite elements in electromagnetics. Comp. Meth. Appl. Mech. Eng., 169:331–344, 1999.
15. Irons, B. M., Engineering Applications of Numerical Integration in Stiffness Methods, Journal of the American Institute of Aeronautics and Astronautics, 411:2035–2037, 1966.
16. Daryl L. Logan. A First Course in the Finite Element Method, Fourth Edition, published by Thomson, 2007.
17. Taig I.C., Structural Analysis by the Matrix Displacement Method', Engl. Electric Aviation Report, 5017, 1961.
18. Milsted MG, Hutchinson JR. Use of trigonometric terms in the finite element method with application to vibration membranes. Journal of Sound and Vibration, 32:327–46, 1974.
19. Christian M, Explicit local buckling analysis of stiffened composite plates accounting for periodic boundary conditions and stiffener–plate interaction, Composite Structures, 91:249–65, 2009.
20. Singiresu S. Rao. The finite Element Method in Engineering, Fourth Edition, Published by Elsevier Science & Technology Books; 2004
21. Thomas, B. G., and Finney, R. L., Calculus and Analytic Geometry, Addison-Wesley, Reading, MA, 1984.

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