Solutions of Generalized Fredholm Type Integral Equations Pertaining To Product of Special Functions

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Abstract- **The object of this paper is to solve an integral equation whose kernel involves the product of the M-series and $\overline{H}$ -functions. The solution is derived by the application of Mellin-transform. The integral equation discussed here can be used to investigate a wide class of known (may be new also) integral equations, hitherto scattered in the literature. Special cases, involving generalized Riemann zeta function, g[τ,θ,ν,ϕ;z] function and generalized Wright function are considered.**

Key Words: Fredholm integral equations, $\overline{H}$ -function, generalized M-series, generalized Riemann zeta function, generalized Wright function, Mellin transform
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I. Introduction

In the last several years a large number of Fredholm type integral equations involving various special functions or polynomials as their kernels have been studied by many researchers notably Srivastava and Bushman (1992), Srivastava and Raina (1992), Goyal and Mukherjee (2002), Gupta and Agrawal (2009), and others.

In this paper, we obtain the solution of the following Fredholm type integral equation

$$\int_0^\infty y^{-1}\, u\left(\frac{x}{y}\right) f(y)\, dy = g(x)\,, \quad (x > 0) \tag{1}$$

where g is a prescribed function, f is an unknown function to be determined and the kernel u(x) is given by

$$u(x) = {}_k\overset{\alpha,\beta}{M}{}_\ell[a_1,\ldots,a_k;b_1,\ldots,b_\ell;\omega x^\rho]\,\overline{H}^{m',n'}_{p',q'}\left[zx^\sigma\left|\begin{matrix}(e'_j,E'_j;\alpha'_j)_{1,n'},(e'_j,E'_j)_{n'+1,p'}\\(f'_j,F'_j)_{1,m'},(f'_j,F'_j;\beta'_j)_{m'+1,q'}\end{matrix}\right.\right]$$
$$\times\overline{H}^{m,n}_{p,q}\left[t\,x^{-\lambda}\left|\begin{matrix}(e_j,E_j;\alpha_j)_{1,n},(e_j,E_j)_{n+1,p}\\(f_j,F_j)_{1,m},(f_j,F_j;\beta_j)_{m+1,1}\end{matrix}\right.\right]. \tag{2}$$

The generalized M-series is introduced by Sharma and Jain (2009) defined as

$${}_k\overset{\alpha,\beta}{M}{}_\ell(\omega) = {}_k\overset{\alpha,\beta}{M}{}_\ell(a_1,\ldots,a_k;b_1,\ldots,b_\ell;\omega)$$
$$= \sum_{r=0}^{\infty}\frac{(a_1)_r\ldots(a_k)_r}{(b_1)_r\ldots(b_\ell)_r}\,\frac{\omega^r}{\Gamma(\alpha r+\beta)},\ \omega,\alpha,\beta\in C,\ \mathrm{Re}(\alpha) > 0. \tag{3}$$

For convergence conditions and other details of the generalized M-series see Sharma and Jain (2009).

The $\overline{H}$ -function, a generalization of Fox H-function introduced by Inayat Hussain (1987) and studied by Bushman and Srivastava (1990) and others, is defined and represented in the following manner:

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$$\bar{H}^{m,n}_{p,q}[z]=\bar{H}^{m,n}_{p,q}\left[z\left|\begin{matrix}(e_j,E_j;\alpha_j)_{1,n},(e_j,E_j)_{n+1,p}\\(f_j,F_j)_{1,m},(f_j,F_j;\beta_j)_{m+1,q}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_L \bar{\phi}(\xi)z^{\xi}\,d\xi, \tag{4}$$

where

$$\bar{\phi}(\xi)=\frac{\prod_{j=1}^{m}\Gamma(f_j-F_j\xi)\prod_{j=1}^{n}\{\Gamma(1-e_j+E_j\xi)\}^{\alpha_j}}{\prod_{j=m+1}^{q}\{\Gamma(1-f_j+F_j\xi)\}^{\beta_j}\prod_{j=n+1}^{p}\Gamma(e_j-E_j\xi)} \tag{5}$$

and the contour L is the line from C – i∞ to C + i∞, suitably indented to keep the poles of $\Gamma(f_j-F_j\xi), j=1,2,...,m$ to the right of the path and the singularities of $\{\Gamma(1-e_j+E_j\xi)\}^{\alpha_j}$, j = 1,2,…,n to the left of the path.

For convergence conditions and other details of the $\bar{H}$-function see Inayat Hussain (1987), Bushman and Srivastava (1990) and Gupta et al. (2007).

The series representation of $\bar{H}$-function (1987) is as follows

$$\bar{H}^{m',n'}_{p',q'}[z]=\bar{H}^{m',n'}_{p',q'}\left[z\left|\begin{matrix}(e'_j,E'_j;\alpha'_j)_{1,n'},(e'_j,E'_j)_{n'+1,p'}\\(f'_j,F'_j)_{1,m'},(f'_j,F'_j;\beta'_j)_{m'+1,q'}\end{matrix}\right.\right]$$

$$=\sum_{g=1}^{m'}\sum_{h=0}^{\infty}\frac{(-1)^h\phi(\eta_{g,h})}{h!\,F'_g}z^{\eta_{g,h}},\ldots(6)$$

where

$$\phi(\eta_{g,h})=\frac{\prod_{\substack{j=1\\j\neq g}}^{m'}\Gamma(f'_j-F'_j\eta_{g,h})\prod_{j=1}^{n'}\{\Gamma(1-e'_j+E'_j\eta_{g,h})\}^{\alpha'_j}}{\prod_{j=m'+1}^{q'}\{\Gamma(1-f'_j+F'_j\eta_{g,h})\}^{\beta'_j}\prod_{j=n'+1}^{p'}\Gamma(e'_j-E'_j\eta_{g,h})}$$

and

$$\eta_{g,h}=\frac{f'_g+h}{F'_g}.$$

II. IMELLIN TRANSFORM OF U(X)

In order to solve the integral equation (1), we require the following result

Lemma 2.1. Let U(s) = M {u(x) ;s }, where u(x) is given by (2), then

$$U(s)=\sum_{g=1}^{m'}\sum_{h,r=0}^{\infty}(-1)^h\frac{(a_1)_r\ldots(a_k)_r}{(b_1)_r\ldots(b_\ell)_r}\frac{1}{\Gamma(\alpha r+\beta)}\frac{\phi(\eta_{g,h})}{h!\,F'_g}\omega^r z^{\eta_{g,h}}\lambda^{-1}$$

$$\times t^{\left(\frac{s+\rho r+\sigma\eta_{g,h}}{\lambda}\right)}\ \bar{\phi}\left(\frac{s+\rho r+\sigma\eta_{g,h}}{\lambda}\right) \tag{7}$$

provided that

$$\max_{1\le j\le n}\left[\mathrm{Re}\left\{\alpha_j\left(\frac{e_j-1}{E_j}\right)\right\}\right]<\mathrm{Re}\left(\frac{s+\rho r+\sigma\eta_{g,h}}{\lambda}\right)<\min_{1\le j\le m}\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right].$$

Proof: We have

$$U(s)=M\{u(x);s\}$$

$$=M\left\{{}_k M_\ell^{\alpha,\beta}[\omega x^\rho]\,\overline{H}_{p',q'}^{m',n'}[z\,x^\sigma]\,\overline{H}_{p,q}^{m,n}[t\,x^{-\lambda}]\right\}$$

$$=\int_0^\infty x^{s-1}\ {}_k M_\ell^{\alpha,\beta}[\omega x^\rho]\,\overline{H}_{p',q'}^{m',n'}[z\,x^\sigma]\,\overline{H}_{p,q}^{m,n}[t\,x^{-\lambda}]\,dx. \tag{8}$$

Expressing the M-series and one $\overline{H}$ -function in series form defined by (3) and (6) respectively and changing the order of summations and integration (which is permissible under the conditions stated), we get

$$U(s)=\sum_{g=1}^{m'}\sum_{r,h=0}^{\infty}(-1)^h\frac{(a_1)_r\ldots(a_k)_r}{(b_1)_r\ldots(b_\ell)_r}\frac{1}{\Gamma(\alpha r+\beta)}\frac{\phi(\eta_{g,h})}{h!\,F_g'}\omega^r z^{\eta_{g,h}}$$

$$\times M\left\{x^{\rho r+\sigma\eta_{g,h}}\ \overline{H}_{p,q}^{m,n}[t\,x^{-\lambda}]\right\}. \tag{9}$$

Now, applying the known formulae (Erdélyi (1954, p.307, eq.(8))

$$M\{x^\mu f(z\,x^{-\lambda});s\}=\lambda^{-1}z^{\frac{s+\mu}{\lambda}}\ F\left(-\frac{s+\mu}{\lambda}\right) \tag{10}$$

and [9, p.114, eq. (4.1)]

$$M\left\{\overline{H}_{p,q}^{m,n}[x];s\right\}=\bar{\phi}(-s). \tag{11}$$

provided that $\Omega> 0$ | arg (x) | < 1/2 $\Omega\pi$ and

$$-\min_{1\le j\le m}\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right]<\mathrm{Re}(s)<\min_{1\le j\le n}\left[\mathrm{Re}\left\{\alpha_j\left(\frac{1-e_j}{E_j}\right)\right\}\right]$$

in (9), we arrive at the desired result (7).

III. SOLUTION OF THE INTEGRAL EQUATION

In this section we will investigate the solution of the Fredholm type integral equation (1). The result is given in the form of the following theorem

Theorem 3.1. Let the Mellin transform F(s), G(s) and U(s) ≠ 0 of the functions f(x), g(x) and u(x) defined by (2) exist and are analytic in some infinite strip $s_1 < \mathrm{Re}(s) < s_2$ of complex s-plane. Also suppose that for a fixed C ∊ (s_1, s_2), $u^*(x)$ is defined by

$$u^*(x)=M^{-1}\{U^*(s);x\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\,U^*(s)\,ds, \tag{12}$$

where

$$U^*(s)=\left[\frac{\mu^{\xi}\Gamma(-s/\mu)}{\Gamma[-(s+\mu\xi)/\mu]}\sum_{g=1}^{m'}\sum_{h,r=0}^{\infty}(-1)^h\frac{(a_1)_r\ldots(a_k)_r}{(b_1)_r\ldots(b_\ell)_r}\right.$$

$$\times\frac{1}{\Gamma(\alpha r+\beta)}\frac{\phi(\eta_{g,h})}{h!F_g'}\omega^r z^{\eta_{g,h}}\lambda^{-1}t^{\left(\frac{s+\mu\xi+\gamma+\rho r+\sigma\eta_{g,h})}{\lambda}\right)}$$

$$\left.\times\bar{\phi}\left(\frac{s+\mu\xi+\gamma+\rho r+\sigma\eta_{g,h}}{\lambda}\right)\right]^{-1} \tag{13}$$

provided that the following conditions are satisfied

(i) $|\arg(t)|<\frac{1}{2}\Omega\pi$ and $\Omega>0$

(ii) $|\arg(t)|=\frac{1}{2}\Omega\pi$ and $\Omega\geq 0$

and (a) $\varsigma\neq 0$ and contour L is so chosen that $(C\varsigma+\in+1)<0$

(b) $\varsigma=0$ and $(\in+1)<0$,

where

$$\Omega=\sum_{j=1}^{m}F_j+\sum_{j=1}^{n}\alpha_jE_j-\sum_{j=m+1}^{q}F_j\beta_j-\sum_{j=n+1}^{p}E_j$$

$$\varsigma=\sum_{j=1}^{n}\alpha_jE_j+\sum_{j=n+1}^{p}E_j-\sum_{j=1}^{m}F_j-\sum_{j=m+1}^{q}F_j\beta_j$$

$$\in=\mathrm{Re}\left(\sum_{j=1}^{m}f_j+\sum_{j=m+1}^{q}F_j\beta_j-\sum_{j=1}^{n}e_j\alpha_j-\sum_{j=n+1}^{p}e_j\right)$$

$$+\frac{1}{2}\left(\sum_{j=1}^{n}\alpha_j-\sum_{j=m+1}^{q}\beta_j+p-m-n\right)$$

(iii) $\mu\neq 0$, ξ is a non-negative integer and $\min\{\rho,\sigma,\lambda\}>0$

(iv) $$\max_{1\leq j\leq n}\left[\mathrm{Re}\left\{\alpha_j\left(\frac{e_j-1}{E_j}\right)\right\}\right]<\mathrm{Re}\left(\frac{s+\mu\xi+\gamma+\rho r+\sigma\eta_{g,h}}{\lambda}\right)$$

$$<\min_{1\leq j\leq m}\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right].$$

Then, the solution of integral equation (1) is given by

$$f(x)=x^{-\mu\xi-\gamma}\int_0^{\infty}y^{-1}u^*\left(\frac{x}{y}\right)(y^{\mu+1}Dy)^{\xi}[y^{\gamma}g(y)]dy \tag{14}$$

provided that the integral on the right hand side of (14) exists.

Proof: Applying the Mellin transform on integral equation (1) and using the convolution theorem for Mellin transform (Erdélyi(1954)), we get

$$U(s)\,F(s)=G(s) \tag{15}$$

where U(s) is given by (7) and F(s) and G(s) are Mellin transforms of f(x) and g(x) respectively.

Now replacing s in (15) by s+ $\mu\xi+\gamma$ ($\mu\neq 0$and ξ is non-negative integer), we have

$$F(s+\mu\xi+\gamma)=U^*(s)\,\mu^{\xi}\left(-\frac{s+\mu\xi}{\mu}\right)_{\xi}G(s+\mu\xi+\gamma) \tag{16}$$

and using the formula [14]

$$M\{(x^{\xi+1}D_x)^n f(x);s\}=\xi^n\left(-\frac{s+\xi n}{\xi}\right)_n F(s+\xi n), \tag{17}$$

we get

$$F(s+\mu\xi+\gamma)=U^*(s)\,M\{(y^{\mu+1}D_y)^{\xi}[y^{\gamma}g(y)];s\}. \tag{18}$$

Further using the elementary result

$$M\{x^{\mu}f(x);s\}=F(s+\mu) \tag{19}$$

and well known convolution theorem for Mellin transform in (18), we get

$$M\{x^{\mu\xi+\gamma}f(x);s\}=M\left\{\int_0^{\infty}y^{-1}u^*\left(\frac{x}{y}\right)(y^{\mu+1}D_y)^{\xi}[y^{\gamma}g(y)]\,dy;s\right\}. \tag{20}$$

Finally, inverting both sides of (20) by using well known Mellin inversion theorem, we arrive at the required result (14).

IV. SPECIAL CASES

Corollary 4.1. Reducing generalized M-series to unity and one $\overline{H}$ -function to generalized Riemann zeta function $\phi(z,p',\eta')$ (Gupta and Soni (2005, p.159, eq.(5)) in the integral equation (1), it takes the following form

$$\int_0^{\infty}y^{-1}u_1\left(\frac{x}{y}\right)f(y)\,dy=g(x), \tag{21}$$

where

$$u_1(x)=\phi[z\,x^{\sigma},p',\eta']\,\overline{H}^{m,n}_{p,q}[t\,x^{-\lambda}] \tag{22}$$

has its solution given by

$$f(x)=x^{-\mu\xi-\gamma}\int_0^{\infty}y^{-1}u_1^*\left(\frac{x}{y}\right)(y^{\mu+1}D_y)^{\xi}[y^{\gamma}g(y)]\,dy, \tag{23}$$

provided that integral in (23) exists, $u_1^*(x)$ is the Mellin inverse transform of

$$U_1^*(s)=\left[\frac{\mu^{\xi}\Gamma(-s/\mu)}{\Gamma[-(s+\mu\xi)/\mu]}\sum_{h=0}^{\infty}\frac{1}{(\eta'+h)\,p'}z^h\lambda^{-1}t^{\left(\frac{s+\mu\xi+\gamma+\sigma h}{\lambda}\right)}\overline{\phi}\left(\frac{s+\mu\xi+\gamma+\sigma h}{\lambda}\right)\right]^{-1} \tag{24}$$

and the set of conditions of the theorem modified appropriately are satisfied.

Corollary 4.2. Reducing generalized M-series to unity and one $\overline{H}$ -function to $g[\tau,\theta,\nu,\phi;z]$ in the theorem, we easily observe that the integral equation

$$\int_0^{\infty}y^{-1}u_2\left(\frac{x}{y}\right)f(y)\,dy=g(x), \tag{25}$$

where

$$u_2(x)=g[\tau,\theta,\nu,\phi;zx^{\sigma}]\,\overline{H}^{m,n}_{p,q}[t\,x^{-\lambda}] \tag{26}$$

has its solution given by

$$f(x)=x^{-\mu\xi-\gamma}\int_0^\infty y^{-1}u_2^*\left(\frac{x}{y}\right)(y^{\mu+1}D_y)^\xi[y^\gamma g(y)]dy, \tag{27}$$

provided that integral in (27) exists, $u_2^*(x)$ is the Mellin inverse transform of

$$U_2^*(s)=\left[\frac{\mu^\xi\Gamma(-s/\mu)}{\Gamma[-(s+\mu\xi)/\mu]}\frac{K_{d-1}2^{-\phi-2}\Gamma(\phi+1)B\left(\frac{1}{2},\frac{1}{2}+\frac{\nu}{2}\right)}{\pi}\right.$$

$$\left.\times\sum_{h=0}^\infty\frac{\left(\tau-\frac{\nu}{2}\right)_h(\tau)_h(\theta+h)^{-(1+\phi)}}{\left(1+\frac{\nu}{2}\right)_h}\frac{z^h}{h!}\lambda^{-1}t^{\left(\frac{s+\mu\xi+\gamma+\sigma h}{\lambda}\right)}\bar{\phi}\left(\frac{s+\mu\xi+\gamma+\sigma h}{\lambda}\right)\right]^{-1} \tag{28}$$

and the conditions of validity follow easily from those mentioned with the theorem.

Corollary 4.3. Finally, if we reduce generalized M-series to unity and one $\overline{H}$ -function to generalized Wright hypergeometric function ${}_{p'}\overline{\Psi}_{q'}[z]$, then the integral equation

$$\int_0^\infty y^{-1}u_3\left(\frac{x}{y}\right)f(y)\,dy=g(x), \tag{29}$$

where

$$u_3(x)={}_{p'}\overline{\Psi}_{q'}[zx^\sigma]\;\overline{H}_{p,q}^{m,n}[t\,x^{-\lambda}] \tag{30}$$

has the solution given by

$$f(x)=x^{-\mu\xi-\gamma}\int_0^\infty y^{-1}u_3^*\left(\frac{x}{y}\right)(y^{\mu+1}D_y)^\xi[y^\gamma g(y)]dy, \tag{31}$$

provided that integral in (31) exists, $u_3^*(x)$ is the Mellin inverse transform of

$$U_3^*(s)=\left[\frac{\mu^\xi\Gamma(-s/\mu)}{\Gamma[-(s+\mu\xi)/\mu]}\sum_{h=0}^\infty\frac{\prod_{j=1}^{p'}\{\Gamma(e_j'+E_j'h)\}^{\alpha_j'}}{\prod_{j=1}^{q'}\{\Gamma(f_j'+F_j'h)\}^{\beta_j'}}\frac{z^h}{h!}\lambda^{-1}\right.$$

$$\left.\times\,t^{\left(\frac{s+\mu\xi+\gamma+\sigma h}{\lambda}\right)}\bar{\phi}\left(\frac{s+\mu\xi+\gamma+\sigma h}{\lambda}\right)\right]^{-1} \tag{32}$$

and the set of conditions of the theorem modified appropriately are satisfied.The importances of our results lie in their manifold generality. In view of the generality of the $\overline{H}$ -functions and the generalized M-series, on specializing the various parameters and variables, we can obtained from our results, several integral equations and solutions involving a remarkably wide variety of useful functions (or product of several functions), which are expressible in terms of H-functions, G-functions, Mittag-Leffler functions and their various special cases. Thus, the results presented in this paper would at once yield a very large number of results involving a large variety of special functions occurring in the problems of mathematical analysis, mathematical physics and engineering.

V. References

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