Inventory Production Control Model With Back-Order When Shortages Are Allowed

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Abstract—It is prohibited to have shortage of inventory since inventory cost is induced from the amount of product stored. This paper presents inventory control theory in production inventory problem when shortages are allowed and backorder takes place. Three assumptions are considered here on shortage and backorders and this leads to three models. The first: when demand is fixed and known, production is infinite and shortages are allowed although the cost of shortage is finite. Second when time (t) interval is fixed, replenishment is allowed and production rate is infinite. Third when production rate is finite. It makes economic sense from the applications that for any production where shortages are allowed, backorder must follows to avoid lost in sales.

Keywords—Inventory control, Backorder, Production, Shortage, Demand

I. INTRODUCTION

Due to the quest for efficiency accelerated by the so-called financial crisis, inventory control is a vital function in almost all kinds of productions. Inventory models majorly focused on minimizing the total inventory cost and to balance the economics of large orders or large production runs against the cost of holding inventory and the cost of going short. The method has been efficiently and successfully applied by some researchers in many areas of operation [2, 3, 5, 6, 7, 8 and 10]. Production and inventory planning and control procedures for a target firm depends on (i). Whether production is make-to-stock or make-to-order (which in turn depends on the relation between customer promise time and production lead time,) and (ii). Whether demand is for known production or anticipated production.

II. LITERATURE REVIEW

This paper introduce some typical papers involve in the topics with different subcategories, lost sales, backorders, shortages and deterioration as well as periodic review and continuous review. One critical factor playing major roles on the inventory theory is backorders. Much of the literatures on inventory models ignore backorders. Backorders means reordering and satisfying only part of the unsatisfied demand at a later stage when there is delay in meeting demands or inability to meet it at all. Most inventory models discuss two extreme situations when items are stock out. They are: (i). All demand within shortage period is backorder And (ii). All demand within shortage period is lost sales.

In real inventory systems, demands during the period of stock out can be partially captive. If demand is fully captive, the next replenishment will fulfill unsatisfied demands during the period of backorders. On the contrary, unsatisfied demands will be completely lost if demand cannot be fully captive, yet demand rate during the period of stock out is not a fix constant if to take backorders into consideration. The recent survey of [3, 4] and many other scholars have developed inventory models on related field, and initiated the concept of demands which will be changed through time cycle into model, also included backorder status, study on [9] optimal control of production inventory system with deteriorating items and dynamic cost, and a study on optimal control of production inventory system with deterioration items using Weibull distribution [1, 11]. Also there was an inventory model of replenishing the stock after a period of backorder [13], which is that deplete cycle always started from the period of backorder. A modification of the complete backorder assumptions and proposed the concept of partial backorders [12], which assumed the backorder ratio is a constant between 0 and 1. The assumption is that usually the time scale of backorder will become consumers’ main pondering factor to accept backorder. This paper looked into assumptions and models for production inventory of a single item when shortages are allowed and there is an order to meet exogenous demand at a minimum cost.

III. METHODOLOGY

Inventory control models assumed that demand from customer are known for planning period both at present and past period. It is prohibited to have shortage of inventory since inventory cost is induced from the amount of product storage. Three assumptions were considered.

Notations and Assumptions

To develop the proposed models, the following notations and assumptions are used in this paper.

\[ I(t) = \text{inventory level at time } t. \]

\[ \lambda = \text{Demand rate or the number of items required per unit time.} \]

\[ c_s = \text{Holding cost per unit time} \]

\[ c_r = \text{Shortage cost per unit item per time} \]

\[ c_q = \text{Production Set up cost per run} \]

\[ t_R = \text{interval between runs.} \]

\[ q = \text{Number of items produced per production run} \]

\[ q = \text{if a production is made at time interval } t, \text{ a quantity } q = R t \text{ must be produced in each run. Since the stock in small time } dt \text{ is } Rtdt, \text{ the stock in time period } t \text{ is} \]
\[ \int R \, dt = \frac{1}{2} R t^2 = \frac{1}{2} q t \]

**Assumptions I**

In this model, we assume that demand is fixed and known, production is infinite and shortages are allowed although the cost of shortage is finite, i.e.

1. The inventory system involves only one item.
2. Replenishment occurs instantaneously on ordering i.e. lead-time is zero.
3. Demand rate \( R(t) \) is deterministic and given by \( R(t) = \begin{cases} \text{constant}, & 0 < t < T \\ 0, & \text{otherwise} \end{cases} \)
4. Shortages are allowed and completely backlogged.
5. The planning period is of infinite length. The planning horizon is divided into sub-intervals of length \( T \) units. Orders are placed at time points \( t_1, t_2, \ldots \), the order quantity at each re-order point being just sufficient to bring the stock height to a certain maximum level \( S \)

\[ t = \frac{L}{q} t \]

Also,

\[ t_2 = \frac{q-I_t}{q} t \]

Total inventory during time \( t = \text{Area of } \Delta AOB = \frac{1}{2} l_t t_1 \]

Inventory holding cost during time \( t = \text{Area of } \Delta BCD = \frac{q-I_t}{2} t_2 \]

Shortage cost during time \( t = C_2 \frac{(q-I_t)}{2} t_2 \)

Total cost during time \( t = \frac{1}{2} C_1 l_t t_1 + \frac{1}{2} C_2 \frac{(q-I_t)}{2} t_2 + C_3 \)

Average total cost during time \( t \),

\[ = \frac{1}{2} C_1 l_t t_1 + \frac{1}{2} C_2 \frac{(q-I_t)}{2} t_2 + C_3 \]

\[ C(I_t, q) = \frac{C_1 l_t}{2q} + \frac{C_2 (q-I_t)}{2q} + \frac{C_3 R}{q} \]

**Assumption II.**

1. Fixed time interval \( t \).

When time interval \( t \) is fixed, it means inventory is to be replenished after every fixed time \( t \). All other assumptions in I above hold.

Total inventory holding cost during time \( t = \frac{1}{2} C_1 l_t t_1 \)

Total shortage cost during time \( t = \frac{1}{2} C_2 (q-I_t) t_2 \)

Set up cost \( C_3 \) and time interval \( t \) are both constant therefore, average set-up cost per unit time \( C_3 \) is also constant. It needs not to be considered. Total average cost per unit

\[ (C_1, l_t) = \frac{1}{t} \left[ \frac{1}{2} C_1 l_t t_1 + \frac{1}{2} C_2 (q-I_t) t_2 \right] \]

Or

\[ = \frac{C_1}{2q} l_t^2 + \frac{C_2}{2q} (q-I_t)^2 \]

\[ \left( C l_t \right) = 0 \]

Hence the minimum inventory level or order quantity given is
The minimum average cost per unit time from equation 2.0 is

\[
I_t = \frac{1}{2} \left( C_1 + C_2 \right) q^2 + \frac{1}{2} \left( q - \frac{C_1}{C_1 + C_2} q \right) - Rt
\]

Assumption III.

Finite production / planning rate.

The model here follows the assumptions in I except that production rate is finite. With this assumption, we found that inventory is zero at the beginning. It increases at a constant rate (K-R) for time \( t_1 \) until it reaches a level \( I_t \). No replenishment during time \( t_2 \), inventory decreases at the rate R until it reaches zero. Shortage start piling up at constant rate R during \( t_3 \) until this backlog reaches a level \( s \). Lastly, production start and backlog is filled at a constant rate K-R during \( t_4 \) till backlog become zero. This completes cycle.

The total time taken is \( t = t_1 + t_2 + t_3 + t_4 \).

Holding cost = \( \frac{1}{2} C_1 I_t (t_1 + t_2) \)

Shortage cost during time interval \( t = \frac{1}{2} C_2 s(t_3 + t_4) \)

Set up cost = \( C_3 \)

Hence, total average cost per unit time \( t \)

\[
I_t = \frac{q}{R} (K - R) - s
\]

From equation 3.1 & 3.2

\[
t_1 + t_2 = \frac{K}{R} = t
\]

And \( t_2 + t_3 = \frac{K}{R} + \frac{s}{R} \)

Hence, \( t = t_1 + t_2 + t_3 + t_4 \)

\[
\left( \frac{1}{K-R} + \frac{1}{R} \right) \left( \frac{K}{C_1} - \frac{R}{C_2} \right) = \frac{q}{R}
\]

3.10

Hence, equation 3.0 becomes,

\[
(q,s) = \frac{1}{2q-KR} \left[ C_1 \left( \frac{q^{K-R} - s}{q} \right) + C_2 s^2 \right] + \frac{R}{q} C_3
\]

3.11

Minimum lot size is

\[
\sqrt{\frac{2C_3(C_1+2)}{C+C}} \cdot \sqrt{\frac{KR}{K-R}}
\]

And \( \frac{\partial q}{\partial c(q,s)} = 0 \) implies

\[
\frac{(C_1+C_2)}{C+C} \cdot \sqrt{\frac{2C_3(C_1+C_2)}{K(K-R)} \cdot \frac{R}{K}}
\]

3.13

Substituting \( q_0 \) and \( S_0 \) into equation 3.5 above, we have the optimum shortage cost

\[
(q,s) = \sqrt{\frac{3}{C_1+C_2} \cdot \left( \frac{K-R}{K} \right)}
\]

3.14

Optimum time interval \( t_0 \) is

\[
-\sqrt{\frac{(C_1+C_2)}{K-R}}
\]

3.15

Optimum inventory level

\[
\left( 1 - \frac{R}{K} \right) - s
\]

\[
\sqrt{C_1+C_2} \cdot \sqrt{\frac{K(R)}{K}} \cdot \sqrt{\frac{2C_3R}{C_1}}
\]

3.16

IV. NUMERICAL APPLICATIONS

Example 1-

If a particular soap items has demand of 9000 units/year. The cost of one procurement is £100 and holding cost per unit is £2.40 per year. The replacement is instantaneous and the cost of shortage is also £5 per unit/year. We are required to determining the following:
i.) Economic lot size/Optimum lot size,
ii.) The number of orders per year,
iii.) The time between the orders,
iv.) The total cost per year if the cost of one unit is £1.

Solution
Step I.
Demand rate \( R = £9000 \) units/year,
Holding cost, \( C_1 = £2.40/\)unit/year,
Shortage cost \( C_2 = £5/\)unit/year,
Production set up cost per run = £100/procurement.

i.) From equation (1.3),
\[ L = \sqrt{\frac{2 \times \text{annual demand} \times \text{setup cost}}{\text{holding cost}}} \]
\[ = \sqrt{\frac{2 \times 9000 \times 100}{2.40}} \]
\[ = 1,053 \text{units/run} \]

ii.) The number of order per year \( = \frac{\text{annual demand}}{\text{order size}} \)
\[ = \frac{9000}{1053} \approx 8.55 \]
Recalled, equation (1.6)
Hence, the number of order per year is 8.55 or roughly 9 number of times ordered per year.

iii.) Time period between the order is as follows, From equation 1.6, \( \frac{1}{8.55} \approx 0.117 \)year
Here is approximately one month and thirteen days period between the order.

iv.) From equation (1.5), the total cost per year if the cost of one unit is £1.
\[ \text{Cost} = \frac{2 \times \text{setup cost} \times \text{order size}}{2 \times \text{holding cost} + \text{setup cost}} \]
\[ = \frac{2 \times 100 \times 1053}{2 \times 2.40 + 100} \]
\[ = £10710 \text{ per year} \]

Hence, the total cost per year if the cost of one unit is £1 is £10,710

Example 2-
Consider an inventory system with the following data in usual Notations:
\( R = 20 \) engines/ day
\[ = £12/ \text{month or} \frac{12}{30} = £0.4/\text{day}. \]

When \( t = \) fixed, we now want to check for the inventory level at the beginning of each month and the optimum cost per unit. Recall from equation (2.2)
Hence the optimum inventory level at the beginning of each month is 577 engines.
Also recall from equation 2.3
\[ (I_1) = \frac{2 \times C_1 + C_2 \times R \times t}{10} \]
\[ = \frac{2 \times 2.40 + 5 \times 0.4}{10} \]
\[ = 0.47 \text{unit/production run} \]

i.) The optimum inventory level at the beginning of each month is 577 engines.

ii.) Manufacturing time interval \( t_1 + t_4 \)
Recall from equation 3.3,
\[ t_1 + t_4 = \frac{q}{K} = \frac{3,410}{24,000} \]
Hence, the optimum inventory level at time \( t \) is 0.1421 year

iv.) Optimum time interval \( t_0 \) is given by
\[ t_0 = \frac{3,410}{1,200} = 0.2842 \text{ years} \]
This means that, the minimum time interval required is 103 days i.e 3 months and 8 days.

V. CONCLUSION

It can be deduced that when replenishment cost and demand rate per unit time \( R \) increase, order quantity \( q \), and relevant total cost \( C \) will increase. An increment of inventory holding cost per unit \( (h) \), backorder cost and penalty cost will lead to the phenomenon of increasing before diminishing. This idea can induce cost items in inventory depletion period having a trade–off relationship with cost

Example- 3.
A company has a demand of 12,000 units/year from an item and it can produce 2,000 such items per month. The cost of one set up is £400 and the holding cost/unit/month is £0.15. The shortage cost of one unit is £20 per year. Find the optimum lot size and the total cost per year, assuming the cost of one unit if £4. We can also find the maximum inventory manufacturing time and total time.

Given the following:
\( R = 12,000 \)
\[ K = 2000 \times 12 = 24,000/\text{units/year}, \]
\[ = 0.15 \times 12 = 1.8/\text{unit/year}, \]
\[ = £20/\text{year} \]
\[ = £400/\text{set-up} \]

Using equation 3.7
\[ \sqrt{\frac{2 \times 400 \times (1.8 + 20)}{20}} \]
\[ = 3,410 \text{ units} \]

The optimum lot size is 3,410 units.

The total cost per year is considered to be
\[ \sqrt{\frac{(q, s)}{\text{holding cost}} + \frac{3(K - R)}{C_1 + C_2}} \]
\[ (q, s) = 12000 \times 4 + \sqrt{\frac{3(2000 - 12000)}{0.15 + 20}} \]
\[ C_o(q, s) = 50,185/\text{per year} \]

The total cost per year is £50,185 when the cost of one item is £4

Using the equation 3.11, optimum inventory level at time \( t \) is
\[ \sqrt{\frac{2 \times 20 \times 400 \times 12000 \times 10}{20 + 1.8}} \]
\[ = 1,564 \text{unit/production run} \]

iii.) Manufacturing time interval \( t_1 + t_4 \)
Recall from equation 3.3,
\[ t_1 + t_4 = \frac{q}{K} = \frac{3,410}{24,000} \]
Hence, the optimum inventory level at time \( t \) is 0.1421 year

iv.) Optimum time interval \( t_0 \) is given by
\[ t_0 = \frac{3,410}{1,200} = 0.2842 \text{ years} \]
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items in backorders status. Also the decision about when an order should be placed will also be based on how low the inventory should be allowed to be depleted before the order arrives. The idea is to place an order early enough so that the expected number of units demanded during the replenishment lead time will not result in stock out every often

VI. CONTRIBUTION TO KNOWLEDGE

This research work contribute to knowledge in many areas of production or daily life activities where failure to meet up with demand/supply (activities) induced a nebulous cost and pay-price or (replenishment has to be done). Many industries can benefit from this through proper implementations/applications.

VII. REFERENCES

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