Global Journal of Science Frontier Research
Volume 11 Issue 6 Version 1.0 September 2011
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN : 2249-4626 & Print ISSN: 0975-5896

Hypergeometric Forms of Well Known Partial Fraction Expansions of Some Meromorphic Functions

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Abstract - In this paper, we obtain hypergeometric forms of some meromorphic functions $\frac{1}{e^{z}-1}$, $\sec^2 z$, $\operatorname{cosec}^2 z$, $\tan z$, $\cot z$, $\operatorname{cosec} z$, $\sec z$, $\operatorname{sech}^2(z)$, $\operatorname{cosech}^2(z)$, $\tanh(z)$, $\coth(z)$, $\operatorname{cosech}(z)$, $\operatorname{sech}(z)$, $\frac{\pi}{8z^3}\frac{\sinh(2\pi z)+\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$, $\frac{\pi}{4z}\frac{\sinh(2\pi z)-\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$ and $\frac{\pi}{4z^2}\frac{\sinh(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$, from corresponding partial fraction expansions .

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2010 A.M.S. Subject Classifications: 33-Special Functions.

HYPERGEOMETRIC FORMS OF WELL KNOWN PARTIAL FRACTION EXPANSIONS OF SOME MEROMORPHIC FUNCTIONS

Strictly as per the compliance and regulations of:

Hypergeometric Forms of Well Known Partial Fraction Expansions of Some Meromorphic Functions

M.I. Qureshi[α], Izharul H. Khan[Ω], M.P.Chaudhary[β]

Abstract - In this paper, we obtain hypergeometric forms of some meromorphic functions $\frac{1}{e^{z}-1}$, $\sec^2 z$, $\operatorname{cosec}^2 z$, $\tan z$, $\cot z$, $\operatorname{cosec} z$, $\sec z$, $\operatorname{sech}^2(z)$, $\operatorname{cosech}^2(z)$, $\tanh(z)$, $\coth(z)$, $\operatorname{cosech}(z)$, $\operatorname{sech}(z)$, $\frac{\pi}{8z^3}\frac{\sinh(2\pi z)+\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$, $\frac{\pi}{4z}\frac{\sinh(2\pi z)-\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$ and $\frac{\pi}{4z^2}\frac{\sinh(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}$, from corresponding partial fraction expansions.

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I. Introduction

In the monumental work of Prudnikov et al.[8,Chapter 7] and other literature of Special functions, hypergeometric forms of following functions $\sin z$, $\cos z$, $\sin^2 z$, $\cos^2 z$, $\sinh(z)$, $\cosh(z)$, $\sinh^2 z$, $\cosh^2 z$, $\sin^{-1} z$, $(\sin^{-1} z)^2$, $\cos^{-1} z$, $\sec^{-1} z$, $\operatorname{cosec}^{-1} z$, $\tan^{-1} z$, $\cot^{-1} z$, $\frac{\sin^{-1} z}{\sqrt{1-z^2}}$, $\sinh^{-1} z$, $(\sinh^{-1} z)^2$, $\cosh^{-1} z$, $\operatorname{sech}^{-1} z$, $\operatorname{cosech}^{-1} z$, $\tanh^{-1} z$, $\coth^{-1} z$, $\frac{\sinh^{-1} z}{\sqrt{1+z^2}}$, $\log_a(1\pm z)$, $\ln(1\pm z)$, $e^{\pm z}$, $a^{\pm z}$, $(1\pm z)^{\pm a}$, $\sin(a\sin^{-1} z)$, $\cos(a\sin^{-1} z)$, $\frac{\cos(a\sin^{-1} z)}{\sqrt{1-z^2}}$, $\frac{\sin(a\sin^{-1} z)}{\sqrt{1-z^2}}$, associated composite functions and transcendental functions, are available.

In the Maclaurin's expansions of $\tan z$, $\cot z$, $\operatorname{cosec} z$, $\tanh(z)$, $\coth(z)$, $\operatorname{cosech}(z)$ and $\sec z$, $\operatorname{sech}(z)$, the coefficients of z^n are associated with Bernoulli numbers and Euler numbers [15] respectively.From Maclaurin's expansions, we are unable to obtain their corresponding hypergeometric forms.

Now we shall find the hypergeometric forms of $\tan z$, $\operatorname{cosec} z$, $\cot z$, $\sec z$ and other associated composite functions by means of corresponding partial fraction expansions obtained by Mittag-Leffler theorem or Fourier series method [5; pp.602-603 .

The Pochhammer's symbol or Shifted factorial (h_r) is defined by

$$(h)_r = \frac{\Gamma(h+r)}{\Gamma(h)} = \begin{cases} 1 & ;\ \text{if } r=0 \\ h(h+1)\cdots(h+r-1) & ;\ \text{if } r=1,2,3,\cdots \end{cases} . \tag{1}$$

where h $0,-1,-2,\cdots$ and the notation (Γ) stands for Gamma function.

Lemma: If a, p and n are suitably adjusted real or complex numbers such that associated Pochhammer's symbols are well-defined, then we have

$$(a+pn) = \frac{a\left(\frac{a+p}{p}\right)_n}{\left(\frac{a}{p}\right)_n} \tag{2}$$

Mittag - Leffler's expansion theorem[4;7;14;15;16]

(i) Suppose that the only singularities of f( ), except at infinity, in the finite plane are the simple poles at the points $z=a_1, z=a_2, z=a_3,\cdots$ arranged in order of increasing absolute value, that is:
$\cdots\cdots \geq |a_3| \geq |a_2| \geq |a_1| > 0$

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(ii) Let the residues of $f(z)$ at $a_1, a_2, a_3, \cdots$ be $b_1, b_2, b_3, \cdots$ respectively.

(iii) Let C_N be circles of radius R_N which do not pass through any poles and on which $|f(z)| < M$, where M is independent of N and $R_N \to \infty$ as $N \to \infty$.

When these conditions are satisfied then Mittag-Leffler's expansion theorem states that

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{z - a_n} + \frac{1}{a_n} \right\} \tag{3}$$

for all values of z except the poles.

In the literature of calculus of residues [2 to 7; 12 to 16], following partial fraction expansions are available.

[2, pp.296-297; 6, p.240(Q.No.3); 16, p.113]

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + 2z \sum_{n=1}^{\infty} \frac{1}{4n^2\pi^2 + z^2} \qquad ; z \neq 0, \pm 2i\pi, \pm 4i\pi, \qquad \cdots(4)$$

[15, p.187]

$$\sec^2 z = 4\left\{ \frac{1}{(\pi - 2z)^2} + \frac{1}{(\pi + 2z)^2} + \frac{1}{(3\pi - 2z)^2} + \frac{1}{(3\pi + 2z)^2} + \cdots \right\}$$

$$= 4 \sum_{n=-\infty}^{+\infty} \frac{1}{[(2n+1)\pi + 2z]^2} \qquad ; z \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \qquad \cdots(5)$$

[7, p.135; 16, p.113]

$$\operatorname{cosec}^2 z = \frac{1}{z^2} + \frac{1}{(z - \pi)^2} + \frac{1}{(z + \pi)^2} + \frac{1}{(z - 2\pi)^2} + \frac{1}{(z + 2\pi)^2} + \cdots$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{(z - n\pi)^2} \qquad ; z \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \qquad \cdots(6)$$

[2, p.296; 4, p.157(Q.No.36); 16, p.113]

$$\tan z = 8z\left\{ \frac{1}{\pi^2 - 4z^2} + \frac{1}{9\pi^2 - 4z^2} + \frac{1}{25\pi^2 - 4z^2} + \cdots \right\} \qquad ; z \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \cdots$$

$$= 2z \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2})^2\pi^2 - z^2} = -\sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{z - (n + \frac{1}{2})\pi} + \frac{1}{(n + \frac{1}{2})\pi} \right\} \tag{7}$$

[3, p.122(Q.No.8); 5, p.602; 12, p.310(Q.No.14)]

$$\cot z = \frac{1}{z} + 2z\left\{ \frac{1}{z^2 - \pi^2} + \frac{1}{z^2 - 4\pi^2} + \frac{1}{z^2 - 9\pi^2} + \cdots \right\}$$

$$= \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2} \qquad ; z \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \qquad \cdots(8)$$

[3, p.122(Q.No.9); 4, p.147; 7, pp.132-133]

$$\operatorname{cosec} z = \frac{1}{z} - 2z\left\{\frac{1}{z^2-\pi^2} - \frac{1}{z^2-4\pi^2} + \frac{1}{z^2-9\pi^2} - \cdots\right\}$$

$$= \frac{1}{z_n} + 2z\sum_{=1}^{\infty}\frac{(-1)^{n-1}}{n^2\pi^2-z^2} \qquad ; z \neq 0, \pm\pi, \pm2\pi, \pm3\pi, \qquad \cdots(9)$$

[4, p.156(Q.No.34); 5, p.603; 7, p.137(Q.No.18)]

$$\sec z = 4\pi\left\{\frac{1}{\pi^2-4z^2} - \frac{3}{9\pi^2-4z^2} + \frac{5}{25\pi^2-4z^2} - \cdots\right\}$$

$$= 2\pi\sum_{n=0}^{\infty}\frac{(-1)^n\,(n+\frac{1}{2})}{(n+\frac{1}{2})^2\pi^2-z^2} \qquad ; z \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \qquad \cdots(10)$$

[2, p.296; 15, p.187]

$$\tanh(z) = 8z\left\{\frac{1}{\pi^2+4z^2} + \frac{1}{9\pi^2+4z^2} + \frac{1}{25\pi^2+4z^2} + \cdots\right\}$$

$$= 2z\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2})^2\pi^2+z^2} \qquad ; z \neq \pm\frac{i\pi}{2}, \pm\frac{3i\pi}{2}, \pm\frac{5i\pi}{2}, \qquad \cdots(11)$$

[2, p.296; 7, p.134]

$$\coth(z) = \frac{1}{z} + 2z\left\{\frac{1}{z^2+\pi^2} + \frac{1}{z^2+4\pi^2} + \frac{1}{z^2+9\pi^2} + \cdots\right\}$$

$$= \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2+n^2\pi^2} \qquad ; z \neq 0, \pm i\pi, \pm2i\pi, \pm3i\pi, \qquad \cdots(12)$$

[2, p.296; 7, p.135]

$$\operatorname{cosech}(z) = \frac{1}{z} - 2z\left\{\frac{1}{z^2+\pi^2} - \frac{1}{z^2+4\pi^2} + \frac{1}{z^2+9\pi^2} - \cdots\right\}$$

$$= \frac{1}{z} + 2z\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2+z^2} \qquad ; z \neq 0, \pm i\pi, \pm2i\pi, \pm3i\pi, \qquad \cdots(13)$$

[14, p.175; 15, p.187]

$$\operatorname{sech}(z) = 4\pi\left\{\frac{1}{\pi^2+4z^2} - \frac{3}{9\pi^2+4z^2} + \frac{5}{25\pi^2+4z^2} - \cdots\right\}$$

$$= 2\pi\sum_{n=0}^{\infty}\frac{(-1)^n\,(n+\frac{1}{2})}{(n+\frac{1}{2})^2\pi^2+z^2} \qquad ; z \neq \pm\frac{i\pi}{2}, \pm\frac{3i\pi}{2}, \pm\frac{5i\pi}{2}, \qquad \cdots(14)$$

Ramanujan's partial fraction expansions [1, Part -IV, pp.380-381]

Ramanujan's systematic work on ordinary hypergeometric series is contained primarily in Chapters XII, XIII, XV of first notebook [9] and Chapters III, X and XI of second notebook [10]. Ramanujan evidently had an affinity for partial fraction expansions, which can be found in several places in his notebooks. The heaviest concentrations lie in Chapters 14 and 18 and in the unorganized pages at the end of the second notebook. See Berndt's books [part-II] and [part-III] for accounts of the material in Chapters 14 and 18, respectively. In this paper, we obtain the hypergeometric forms of three partial fraction decompositions in the unorganized pages of second notebook.

When $z \neq \frac{m}{2}(1 \pm i) \quad ; m = 0, \pm 1, \pm 2, \pm 3, \cdots$

then [1(Part-IV), pp.380-381, Entry 13; see also 7, p.137 (Q.No.20 i)]

$$\frac{\pi}{8z^3}\,\frac{\sinh(2\pi z)+\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)} = \frac{1}{8z^4} + \sum_{n=1}^{\infty} \frac{1}{4z^4+n^4} \tag{15}$$

[1(Part-IV), pp.380-381, Entry 14]

$$\frac{\pi}{4z}\,\frac{\sinh(2\pi z)-\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)} = \sum_{n=1}^{\infty} \frac{n^2}{4z^4+n^4} \tag{16}$$

[1(Part-IV), pp.380-381, Entry 15]

$$\frac{\pi}{4z^2}\,\frac{\sinh(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)} = \frac{1}{8z^3} + \sum_{n=1}^{\infty} \frac{n}{4z^4+n^4} + \frac{1}{2z}\sum_{n=1}^{\infty} \frac{1}{z^2+(z+n)^2} \tag{17}$$

II. Hypergeometric Forms of Some Partial Fraction Expansions

If we apply the Lemma (2) in real or complex linear factors of quadratic and biquadratic polynomials in n, associated with the denominators of partial fraction expansions (4) to (17), we get the following hypergeometric forms:

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \frac{2z}{(z^2+4\pi^2)}\,{}_3F_2\left[\begin{matrix} 1\,, & \frac{2\pi+iz}{2\pi}, & \frac{2\pi-iz}{2\pi}\,; \\ & \frac{4\pi+iz}{2\pi}\,, & \frac{4\pi-iz}{2\pi}\,; \end{matrix}\; 1\right] \;; z \neq 0, \pm 2i\pi, \quad \cdots(18)$$

$$\sec^2 z = \frac{4}{(2z+\pi)^2}\,{}_2H_2\left[\begin{matrix} \frac{\pi+2z}{2\pi}, & \frac{\pi+2z}{2\pi}\,; \\ \frac{3\pi+2z}{2\pi}\,, & \frac{3\pi+2z}{2\pi}\,; \end{matrix}\; 1\right] \quad ; z \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \quad \cdots(19)$$

$$\sec^2 z = \frac{4}{(2z-\pi)^2}\,{}_3F_2\left[\begin{matrix} 1\,, & \frac{\pi-2z}{2\pi}, & \frac{\pi-2z}{2\pi}\,; \\ & \frac{3\pi-2z}{2\pi}\,, & \frac{3\pi-2z}{2\pi}\,; \end{matrix}\; 1\right] + \frac{4}{(2z+\pi)^2}\,{}_3F_2\left[\begin{matrix} 1\,, & \frac{\pi+2z}{2\pi}, & \frac{\pi+2z}{2\pi}\,; \\ & \frac{3\pi+2z}{2\pi}\,, & \frac{3\pi+2z}{2\pi}\,; \end{matrix}\; 1\right] \tag{20}$$

$$\mathrm{cosec}^2 z = \frac{1}{z^2}\,{}_2H_2\left[\begin{matrix} -\frac{z}{\pi}, & -\frac{z}{\pi}\,; \\ \frac{\pi-z}{\pi}\,, & \frac{\pi-z}{\pi}\,; \end{matrix}\; 1\right] \quad ; z \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \quad \cdots(21)$$

$$\mathrm{cosec}^2 z = \frac{1}{(z+\pi)^2}\,{}_3F_2\left[\begin{matrix} 1\,, & \frac{\pi+z}{\pi}, & \frac{\pi+z}{\pi}\,; \\ & \frac{2\pi+z}{\pi}\,, & \frac{2\pi+z}{\pi}\,; \end{matrix}\; 1\right] + \frac{1}{z^2}\,{}_3F_2\left[\begin{matrix} 1\,, & -\frac{z}{\pi}, & -\frac{z}{\pi}\,; \\ & \frac{\pi-z}{\pi}\,, & \frac{\pi-z}{\pi}\,; \end{matrix}\; 1\right] \tag{22}$$

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$$\tan z = \frac{8z}{(\pi^2-4z^2)}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & \frac{\pi+2z}{2\pi}, & \frac{\pi-2z}{2\pi}\,; & \\ & & & 1 \\ & \frac{3\pi+2z}{2\pi}\,, & \frac{3\pi-2z}{2\pi}\,; & \end{array}\right] \quad ; z \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \qquad \cdots(23)$$

$$\cot z = \frac{1}{z} + \frac{2z}{(z^2-\pi^2)}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & \frac{\pi+z}{\pi}, & \frac{\pi-z}{\pi}\,; & \\ & & & 1 \\ & \frac{2\pi+z}{\pi}\,, & \frac{2\pi-z}{\pi}\,; & \end{array}\right] \quad ; z \neq 0, \pm\pi, \pm 2\pi, \qquad \cdots(24)$$

$$\operatorname{cosec} z = \frac{1}{z} + \frac{2z}{(\pi^2-z^2)}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & \frac{\pi+z}{\pi}, & \frac{\pi-z}{\pi}\,; & \\ & & & -1 \\ & \frac{2\pi+z}{\pi}\,, & \frac{2\pi-z}{\pi}\,; & \end{array}\right] \quad ; z \neq 0, \pm\pi, \pm 2\pi, \qquad \cdots(25)$$

$$\sec z = \frac{4\pi}{(\pi^2-4z^2)}\,{}_4F_3\left[\begin{array}{ccccc} 1\,, & \frac{3}{2}\,, & \frac{\pi+2z}{2\pi}\,, & \frac{\pi-2z}{2\pi}\,; & \\ & & & & -1 \\ & \frac{1}{2}\,, & \frac{3\pi+2z}{2\pi}\,, & \frac{3\pi-2z}{2\pi}\,; & \end{array}\right] \quad ; z \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \qquad \cdots(26)$$

By replacing by in (19) to (26) and using the identities $\sec(iz) = \operatorname{sech}(z)$, $\operatorname{cosec}(iz) = -i\operatorname{cosech}(z)$,$\tan(iz) = i\tanh(z)$,$\cot(iz) = -i\coth(z)$, we get the hypergeometric forms of corresponding hyperbolic functions.

$$\operatorname{sech}^2(z) = \frac{4}{(2iz+\pi)^2}\,{}_2H_2\left[\begin{array}{ccc} \frac{\pi+2iz}{2\pi}, & \frac{\pi+2iz}{2\pi}\,; & \\ & & 1 \\ \frac{3\pi+2iz}{2\pi}\,, & \frac{3\pi+2iz}{2\pi}\,; & \end{array}\right] \quad ; z \neq \pm\frac{i\pi}{2}, \pm\frac{3i\pi}{2}, \qquad \cdots(27)$$

$$\operatorname{sech}^2(z) = \frac{4}{(\pi-2iz)^2}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & \frac{\pi-2iz}{2\pi}, & \frac{\pi-2iz}{2\pi}\,; & \\ & & & 1 \\ & \frac{3\pi-2iz}{2\pi}\,, & \frac{3\pi-2iz}{2\pi}\,; & \end{array}\right] + \frac{4}{(\pi+2iz)^2}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & \frac{\pi+2iz}{2\pi}, & \frac{\pi+2iz}{2\pi}\,; & \\ & & & 1 \\ & \frac{3\pi+2iz}{2\pi}\,, & \frac{3\pi+2iz}{2\pi}\,; & \end{array}\right] \qquad (28)$$

$$\operatorname{cosech}^2(z) = \frac{1}{z^2}\,{}_2H_2\left[\begin{array}{ccc} -\frac{iz}{\pi}, & -\frac{iz}{\pi}\,; & \\ & & 1 \\ \frac{\pi-iz}{\pi}\,, & \frac{\pi-iz}{\pi}\,; & \end{array}\right] \quad ; z \neq 0, \pm i\pi, \pm 2i\pi, \pm 3i\pi, \qquad \cdots(29)$$

$$\operatorname{cosech}^2(z) = \frac{1}{z^2}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & -\frac{iz}{\pi}, & -\frac{iz}{\pi}\,; & \\ & & & 1 \\ & \frac{\pi-iz}{\pi}\,, & \frac{\pi-iz}{\pi}\,; & \end{array}\right] - \frac{1}{(\pi+iz)^2}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & \frac{\pi+iz}{\pi}, & \frac{\pi+iz}{\pi}\,; & \\ & & & 1 \\ & \frac{2\pi+iz}{\pi}\,, & \frac{2\pi+iz}{\pi}\,; & \end{array}\right] \qquad (30)$$

$$\tanh(z) = \frac{8z}{(\pi^2+4z^2)}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & \frac{\pi+2iz}{2\pi}, & \frac{\pi-2iz}{2\pi}\,; & \\ & & & 1 \\ & \frac{3\pi+2iz}{2\pi}\,, & \frac{3\pi-2iz}{2\pi}\,; & \end{array}\right] \quad ; z \neq \pm\frac{i\pi}{2}, \pm\frac{3i\pi}{2}, \qquad \cdots(31)$$

$$\coth(z) = \frac{1}{z} + \frac{2z}{(z^2+\pi^2)}\,{}_3F_2\left[\begin{array}{cccc} 1\,, & \frac{\pi+iz}{\pi}, & \frac{\pi-iz}{\pi}\,; & \\ & & & 1 \\ & \frac{2\pi+iz}{\pi}\,, & \frac{2\pi-iz}{\pi}\,; & \end{array}\right] \quad ; z \neq 0, \pm i\pi, \pm 2i\pi, \qquad \cdots(32)$$

$$\operatorname{cosech}(z)=\frac{1}{z}-\frac{2z}{(\pi^2+z^2)}\,{}_3F_2\left[\begin{matrix}1\;,\;\frac{\pi+iz}{\pi},\;\frac{\pi-iz}{\pi}\;;\\ \frac{2\pi+iz}{\pi}\;,\;\frac{2\pi-iz}{\pi}\;;\end{matrix}\;-1\right]\;;z\neq 0,\pm i\pi,\pm 2i\pi,\qquad\cdots(33)$$

$$\operatorname{sech}(z)=\frac{4\pi}{(\pi^2+4z^2)}\,{}_4F_3\left[\begin{matrix}1\;,\;\frac{3}{2}\;,\;\frac{\pi+2iz}{2\pi}\;,\;\frac{\pi-2iz}{2\pi}\;;\\ \frac{1}{2}\;,\;\frac{3\pi+2iz}{2\pi}\;,\;\frac{3\pi-2iz}{2\pi}\;;\end{matrix}\;-1\right]\;;z\neq\pm\frac{i\pi}{2},\pm\frac{3i\pi}{2},\qquad\cdots(34)$$

When $z\neq\frac{m}{2}(1\pm i)\quad;m=0,\pm1,\pm2,\pm3,\cdots$, then

$$\frac{\pi}{8z^3}\,\frac{\sinh(2\pi z)+\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}=\frac{1}{8z^4}+\frac{1}{(4z^4+1)}\times$$
$$\times\,{}_5F_4\left[\begin{matrix}1\;,\;-z+1+iz\;,\;-z+1-iz\;,\;z+1+iz\;,\;z+1-iz\;;\\ -z+2+iz\;,\;-z+2-iz\;,\;z+2+iz\;,\;z+2-iz\;;\end{matrix}\;1\right]\qquad(35)$$

$$\frac{\pi}{4z}\,\frac{\sinh(2\pi z)-\sin(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}=\frac{1}{(4z^4+1)}\times$$
$$\times\,{}_6F_5\left[\begin{matrix}2\;,\;2\;,\;-z+1+iz\;,\;-z+1-iz\;,\;z+1+iz\;,\;z+1-iz\;;\\ 1\;,\;-z+2+iz\;,\;-z+2-iz\;,\;z+2+iz\;,\;z+2-iz\;;\end{matrix}\;1\right]\qquad(36)$$

$$\frac{\pi}{4z^2}\,\frac{\sinh(2\pi z)}{\cosh(2\pi z)-\cos(2\pi z)}=\frac{1}{8z^3}+$$
$$+\frac{1}{(4z^4+1)}\,{}_5F_4\left[\begin{matrix}2\;,\;-z+1+iz\;,\;-z+1-iz\;,\;z+1+iz\;,\;z+1-iz\;;\\ -z+2+iz\;,\;-z+2-iz\;,\;z+2+iz\;,\;z+2-iz\;;\end{matrix}\;1\right]+$$
$$+\frac{1}{(4z^3+4z^2+2z)}\,{}_3F_2\left[\begin{matrix}1\;,\;z+1+iz\;,\;z+1-iz\;;\\ z+2+iz\;,\;z+2-iz\;;\end{matrix}\;1\right]\qquad(37)$$

Above hypergeometric forms are not available in the literature. It is to be noted that the hypergeometric series ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_6F_5$ are convergent.

III. Proofs

To derive (18), consider the following partial fraction expansion

$$\frac{1}{e^z-1}=\frac{1}{z}-\frac{1}{2}+2z\sum_{n=1}^{\infty}\frac{1}{z^2+4n^2\pi^2}\qquad;z\neq 0,\pm 2i\pi,\pm 4i\pi,\pm 6i\pi,\cdots$$

$$=\frac{1}{z}-\frac{1}{2}+2z\sum_{n=0}^{\infty}\frac{1}{[z+2i(n+1)\pi]\;[z-2i(n+1)\pi]}$$

$$=\frac{1}{z}-\frac{1}{2}+2z\sum_{n=0}^{\infty}\frac{1}{[(z+2i\pi)+(2i\pi)n]\;[(z-2i\pi)+(-2i\pi)n]}$$

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Now using the beautiful Lemma (2), we get

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+2z\sum_{n=0}^{\infty}\frac{\left(\frac{z+2i\pi}{2i\pi}\right)_n\left(\frac{z-2i\pi}{-2i\pi}\right)_n}{(z+2i\pi)\left(\frac{z+4i\pi}{2i\pi}\right)_n(z-2i\pi)\left(\frac{z-4i\pi}{-2i\pi}\right)_n}$$

$$= \frac{1}{z}-\frac{1}{2}+\frac{2z}{(z^2+4\pi^2)}\,{}_3F_2\left[\begin{matrix}1\,, & \frac{2i\pi+z}{2i\pi}, & \frac{2i\pi-z}{2i\pi}\,; \\ & \frac{4i\pi+z}{2i\pi}\,, & \frac{4i\pi-z}{2i\pi}\,;\end{matrix}\ 1\right]$$

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+\frac{2z}{(z^2+4\pi^2)}\,{}_3F_2\left[\begin{matrix}1\,, & \frac{2\pi+iz}{2\pi}, & \frac{2\pi-iz}{2\pi}\,; \\ & \frac{4\pi+iz}{2\pi}\,, & \frac{4\pi-iz}{2\pi}\,;\end{matrix}\ 1\right]$$

To derive (21) and (22), consider the following expansion

$$\operatorname{cosec}^2 z = \sum_{n=-\infty}^{+\infty}\frac{1}{(z-n\pi)^2} \qquad ; z\neq 0,\pm\pi,\pm 2\pi,\pm 3\pi,\cdots\cdots$$

$$= \sum_{n=-\infty}^{+\infty}\frac{1}{[z+(-\pi)n]\,[z+(-\pi)n]}$$

Now using the beautiful Lemma (2), we get

$$\operatorname{cosec}^2 z = \sum_{n=-\infty}^{+\infty}\frac{\left(\frac{z}{-\pi}\right)_n\left(\frac{z}{-\pi}\right)_n}{z\left(\frac{z-\pi}{-\pi}\right)_n z\left(\frac{z-\pi}{-\pi}\right)_n} = \frac{1}{z^2}\,{}_2H_2\left[\begin{matrix}-\frac{z}{\pi}, & -\frac{z}{\pi}\,; \\ \frac{\pi-z}{\pi}\,, & \frac{\pi-z}{\pi}\,;\end{matrix}\ 1\right]$$

which is the hypergeometric form (21).

Now replacing z by iz and using suitable circular-hyperbolic identity, we get (29).

Now again consider,

$$\operatorname{cosec}^2 z = \sum_{n=-\infty}^{+\infty}\frac{1}{(z-n\pi)^2} \qquad ; z\neq 0,\pm\pi,\pm 2\pi,\pm 3\pi,\cdots\cdots$$

$$= \sum_{n=-\infty}^{-1}\frac{1}{(z-n\pi)^2}+\sum_{n=0}^{\infty}\frac{1}{(z-n\pi)^2} = \sum_{n=1}^{\infty}\frac{1}{(z+n\pi)^2}+\sum_{n=0}^{\infty}\frac{1}{(z-n\pi)^2}$$

$$= \sum_{n=0}^{\infty}\frac{1}{[(z+\pi)+\pi n]^2}+\sum_{n=0}^{\infty}\frac{1}{[z+(-\pi)n]^2}$$

Now using the Lemma (2), we get

$$\operatorname{cosec}^2 z = \sum_{n=0}^{\infty}\frac{(1)_n\left(\frac{z+\pi}{\pi}\right)_n^2}{(z+\pi)^2\left(\frac{z+2\pi}{\pi}\right)_n^2 n!}+\sum_{n=0}^{\infty}\frac{(1)_n\left(\frac{z}{-\pi}\right)_n^2}{z^2\left(\frac{z-\pi}{-\pi}\right)_n^2 n!}$$

$$= \frac{1}{(z+\pi)^2}\,{}_3F_2\left[\begin{matrix}1\,, & \frac{\pi+z}{\pi}, & \frac{\pi+z}{\pi}\,; \\ & \frac{2\pi+z}{\pi}\,, & \frac{2\pi+z}{\pi}\,;\end{matrix}\ 1\right]+\frac{1}{z^2}\,{}_3F_2\left[\begin{matrix}1\,, & -\frac{z}{\pi}, & -\frac{z}{\pi}\,; \\ & \frac{\pi-z}{\pi}\,, & \frac{\pi-z}{\pi}\,;\end{matrix}\ 1\right]$$

which is the hypergeometric form (22).

Now replacing z by iz, we get (30).

Similarly, we can obtain the hypergeometric forms (19), (20), (23) to (28), (31) to (37) of remaining partial fraction expansions.

IV. Acknowledgement

The authors are thankful to Prof. R. Y. Denis and Prof. M. A. Pathan for their valuable suggestions during the preparation of this research paper. The corresponding author (MPC) is also thankful to the Library Staff and Scientists of CRM, Marseille, France, EUROPE for their cooperation during his academic visit in winter 2010.

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