Differential Subordination and Superordination of Analytic Functions Defined By Cho - Kwon - Srivastava Operator

By Jamal M. Shenan
Alazhar University-Gaza, Palestine

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Differential Subordination and Superordination of Analytic Functions Defined By Cho-Kwon-Srivastava Operator

Jamal M. Shenan

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I. INTRODUCTION

Let \( H(U) \) be the class of functions analytic in \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \) and \( H[a,n] \) be the subclass of \( H(U) \) consisting of functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \), with \( H_0 \equiv H[0,1] \) and \( H \equiv H[1,1] \). Let \( A(p) \) denote the class of functions of the form

\[
f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots \}; z \in U),
\]

and let \( A(1) = A \). Let \( f \) and \( F \) be members of \( H(U) \). The function \( f(z) \) is said to be subordinate to \( F(z) \), or \( F(z) \) is said to be superordinate to \( f(z) \), if there exists a function \( w(z) \) analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) \( (z \in U) \), such that \( f(z) = F(w(z)) \). In such a case we write \( f(z) \prec F(z) \). In particular, if \( F \) is univalent, then \( f(z) \prec F(z) \) if and only if \( f(0) = F(0) \) and \( f(U) \subset F(U) \) (see [1] and [2]).

For two functions \( f(z) \) given by (1.1) and

\[
g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p},
\]

The hadamard product (or convolution) of \( f \) and \( g \) is defined by

\[
(f \ast g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g \ast f)(z).
\]

Saitoh [8] introduce a linear operator:

\[
L_p(a,c) : A_p \rightarrow A_p
\]

defined by

\[
L_p(a,c) = \phi_p(a,c;z) \ast f(z) \quad (z \in U),
\]

where

\[
\phi_p(a,c;z) = \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k},
\]

and \((a)_k\) is the Pochhammer symbol. In 2004, Cho, Kwon and Srivastava [4] introduced the linear operator \( L^\lambda_p(a,c) : A_p \rightarrow A_p \) analogous to \( L_p(a,c) \) defined by

\[
L^\lambda_p(a,c)f(z) = \phi^\lambda_p(a,c;z) \ast f(z) \quad (z \in U; a,c \in R \setminus Z_0; \lambda > -p),
\]
where $\phi_p^\alpha(a,c;z)$ is the function defined in terms of the Hadamard product (or convolution) by the following condition:

$$
\phi_p(a,c,z) \ast \phi_p(a,c;z) = \frac{z^p}{(1-z)^{2+p}}
$$

(1.5)

We can easily find from (1.4) and (1.5) and for the function $f(z) \in A_p$ that

$$
L_p^\alpha(a,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{1}{k!} a_k \cdot \frac{(\lambda + p)_k}{(c)_k} z^{k+p}
$$

(1.6)

It is easily verified from (1.6) that

$$
z \left( L_p^\alpha(a+1,c)f \right)^\prime(z) = aL_p^\alpha(a,c)f(z) - (a-p) L_p^\alpha(a+1,c)f(z)
$$

(1.7)

and

$$
z \left( L_p^\alpha(a,c)f \right)^\prime(z) = (\lambda + p) L_p^{\alpha+1}(a,c)f(z) - \lambda L_p^\alpha(a,c)f(z).
$$

(1.8)

To prove our results, we need the following definitions and lemmas.

Denote by $Q$ the set of all functions $q(z)$ that are analytic and injective on $\overline{U} / E(q)$ where

$$
E(q) = \{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \},
$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U / E(q)$.

Further let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

**Definition 1 ([6]).** Let $\Omega$ be a set in $C$, $q \in Q$ and $n$ be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$ consist of those functions $\psi : C^3 \times U \to C$ that satisfy the admissibility condition:

$$
\psi(r,s,t;\zeta) \notin \Omega
$$

whenever

$$
r = q(\zeta), \quad s = k \cdot q'(\zeta), \quad R \left\{ \frac{t}{s} + 1 \right\} \geq kR \left\{ q''(\zeta) + 1 \right\},
$$

where $z \in U$, $\zeta \in \partial U / E(q)$, and $k \geq n$. We write $[\Omega,q]$ as $\Psi[\Omega,q]$.

**Definition 2 ([7]).** Let $\Omega$ be a set in $C$, $q(z) \in H[a,n]$ with $q'(z) \neq 0$ The class of admissible functions $\Psi'_n[\Omega,q]$ consist of those functions $\psi : C^3 \times U \to C$ that satisfy the admissibility condition

$$
\psi(r,s,t;\zeta) \notin \Omega
$$

whenever

$$
r = q(z), \quad s = \frac{zq'(z)}{m}, \quad R \left\{ \frac{t}{s} + 1 \right\} \geq \frac{1}{m} R \left\{ q''(z) + 1 \right\},
$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$. In particular, we write $\Psi'_n[\Omega,q]$ as $\Psi[\Omega,q]$.

**Lemma 1 ([6]).** Let $\psi \in \Psi'_n[\Omega,q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$ satisfies

$$
\psi(p(z),zp'(z),z^2 p''(z);z) \in \Omega,
$$

then

$$
p(z) \prec q(z).
$$

**Lemma 2 ([7]).** Let $\psi \in \Psi'_n[\Omega,q]$ with $q(0) = a$. If $p(z) \in Q(a)$ and $\psi(p(z),zp'(z),z^2 p''(z);z)$ is univalent in $U$ then

$$\Omega \subset \\{ \psi(p(z),zp'(z),z^2 p''(z);z) : z \in U \},$$

implies

$$
\Omega = \mbox{conv}\{ \psi(p(z),zp'(z),z^2 p''(z);z) : z \in U \}.
$$
In the present investigation, the differential subordination result of Miller and Mocanu [6,7] is extended for analytic functions in the open unit disk, which are associated with Cho-Kwon-Srivastava operator $L_p^f(a,c)$, and we obtain certain other related results. A similar problem for analytic functions was investigated by Srivastava [5], Aouf and Seoudy [3], Aghalary et al. [1], Ali et al. [2]. Additionally, the corresponding differential superordination problem is investigated, and several sandwich-type result are obtained.

II. SUBORDINATION RESULTS INVOLVING THE CHO-KWON–SRIVASTAVA OPERATOR

Definition 3. Let $\Omega$ be a set in $C$ and $q(z) \in Q_0 \cap H[0, p]$. The class of admissible functions $\Phi_f[\Omega, q]$ consist of those functions $\phi : C^3 \times U \to C$ that satisfy the admissibility condition

$$\phi(u,v,w;z) \not\in \Omega,$$

whenever

$$u = q(\zeta), \quad v = \frac{k \zeta q'(\zeta) + (a - p)q(\zeta)}{a},$$

$$R \left\{ \frac{a(a-1)r - (a-p)(a-p-1)r}{av - (a-p)u - 2(a-p) + 1} \geq kR \left\{ \frac{\zeta q'(\zeta)}{q'(\zeta)} + 1 \right\}, \right.$$  

where $z \in U$, $\zeta \in \partial U / E(q)$, $p \in N$, $a \in R \setminus Z^+$ and $k \geq p$.

Theorem 1. Let $\phi \in \Phi_f[\Omega, q]$. If $f(z) \in A(p)$ satisfies

$$\phi \left\{ I_p^f(a+1,c)f(z), I_p^f(a,c)f(z), I_p^f(a-1,c)f(z); z \right\} \in \Omega; \quad (2.1)$$

then

$$I_p^f(a+1,c)f(z) < q(z) \quad (z \in U; a,c \in R \setminus Z^+; \lambda > -p; p \in N).$$

Proof. Define the analytic function $p(z)$ in $U$ by

$$p(z) = I_p^f(a+1,c)f(z) \quad \left( z \in U; a,c \in R \setminus Z^+; \lambda > -p; p \in N \right). \quad (2.2)$$

In view of the relation (1.7) from (2.2), we get

$$I_p^f(a,c)f(z) = \frac{zp'(z) + (a-p)p(z)}{a}. \quad (2.3)$$

Further computation show that

$$I_p^f(a+1,c)f(z) = \frac{z^2p''(z) + 2(a-p)zp'(z) + (a-p)(a-p-1)p(z)}{a(a-1)}. \quad (2.4)$$

Define the transformation from $C^3$ to $C$ by

$$u = r, \quad v = \frac{s + (a-p)r}{a}, \quad w = \frac{t + 2(a-p)s + (a-p)(a-p-1)r}{a(a-1)}.$$  

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi \left( r, \frac{s + (a-p)r}{a}, \frac{t + 2(a-p)s + (a-p)(a-p-1)r}{a(a-1)}; z \right). \quad (2.5)$$

The proof shall make use of Lemma 1. Using equation (2.2), (2.3) and (2.4), then from (2.5), we obtain

$$\psi(p(z),zp'(z),z^2p''(z); z) = \phi \left( I_p^f(a+1,c)f(z), I_p^f(a,c)f(z), I_p^f(a-1,c)f(z); z \right). \quad (2.6)$$
Hence (2.1) becomes
\[ \psi(p(z),zp'(z),z^2p''(z);z) \in \Omega. \]

The proof is completed if it can be shown that the admissibility condition for is equivalent to the admissibility condition for as given in Definition 1.

Note that
\[ \left\{ \frac{t}{s} + 1 \right\} = \left\{ \frac{a(a-1)\nu - (a-p)(a-p-1)\nu}{av - (a-p)u} - 2(\lambda + p) + 1 \right\}, \]

and hence \( \psi \in \Psi_p[\Omega,q] \). By Lemma 1,
\[ p(z) < q(z) \text{ or } I^\lambda_p(a+1,c)f(z) < q(z). \]

If \( \Omega \neq C \) is a simply connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case the class \( \Phi_i[h(U),q] \) is written as \( \Phi_i[h,q] \).

The following result is an immediate consequence of Theorem 1.

**Theorem 2.** Let \( \phi \in \Phi_i[h,q] \). If \( f(z) \in A(p) \) satisfies
\[ \phi\left( I^\lambda_p(a+1,c)f(z),I^\lambda_p(a,c)f(z),I^\lambda_p(a-1,c)f(z);z \right) \in h(z), \]
then
\[ I^\lambda_p(a+1,c)f(z) < q(z), \]
where \( \left( p \in N; a,c \in R \setminus Z_0^-; \lambda > -p; z \in U \right) \).

Our next result is an extension of Theorem 1 to the case where the behavior of \( q(z) \), on \( \partial U \) is not known.

**Corollary 1.** Let \( \Omega \subset C \) and let \( q(z) \) be univalent in \( U \), \( q(0) = 0 \). Let \( \phi \in \Phi_i[\Omega,q] \) for some \( \rho \in (0,1) \) where \( q_\rho(z) = q(\rho z) \). If \( f(z) \in A(p) \) and
\[ \phi\left( I^\lambda_p(a+1,c)f(z),I^\lambda_p(a,c)f(z),I^\lambda_p(a-1,c)f(z);z \right) \in \Omega, \]
then
\[ I^\lambda_p(a+1,c)f(z) < q(z) \]
\[ \left( p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U \right). \]

**Proof.** Theorem 1 yields \( I^\lambda_p(a+1,c)f(z) < q_\rho(z) \). The result is now deduced from \( q_\rho(z) < q(z) \).

If \( q(z) = Mz, M > 0 \), and in view of Definition 1, The class of admissible functions \( \Phi_i[\Omega,M] \), denoted by \( \Phi_i[\Omega,M] \) is described below.

**Definition 4.** Let \( \Omega \) be a set in \( C \) and \( M > 0 \). The class of admissible functions \( \Phi_i[\Omega,M] \) consist of those functions \( \phi: C^\lambda \times U \rightarrow C \) such that
\[ \phi\left( Me^{i\theta}, \frac{k + (a-p)}{a} Me^{i\theta}, \frac{L + [2(a-p-1)k + (a-p-1)(a-p-2)]Me^{i\theta}}{a(a-1)};z \right) \not\in \Omega \]
whenever \( z \in U, \theta \in R, R \{ Le^{i\theta} \} \geq (k-1)kM \) for all real \( \theta, p \in N, a \in R \setminus Z_0^- \) and \( k \geq p \).

**Corollary 2.** Let \( \phi \in \Phi_i[\Omega,M] \). If \( f(z) \in A(p) \) satisfies
\[ \phi\left( I^\lambda_p(a+1,c)f(z),I^\lambda_p(a,c)f(z),I^\lambda_p(a-1,c)f(z);z \right) \in \Omega, \]
then
\[ I^\lambda_p(a+1,c)f(z) < M. \]
\[ \left( p \in N; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U \right). \]

In the special case \( \Omega = h(U) = \{ w: |w| < M \} \), the class \( \Phi_i[\Omega,M] \) is simply denoted by \( \Phi_i[M] \), then the corollary (2.2) takes the following form.
Corollary 3. Let $\phi \in \Phi_{t,1}[M]$. If $f(z) \in A(p)$ satisfies
\[
\left| \phi \left( I_p^{\lambda}(a+1,c)f(z), I_p^{\lambda}(a,c)f(z), I_p^{\lambda}(a-1,c)f(z); z \right) \right| < M ,
\]
then
\[
\left| I_p^{\lambda}(a+1,c)f(z) \right| < M . \quad (p \in N; \lambda > -p; a,c \in R \setminus Z_0^+; z \in U)
\]

Now, we introduce a new class of admissible functions $\Phi_{t,1}[\Omega, q]$.

Definition 5. Let $\Omega$ be a set in $C$, $q \in Q_0 \cap H_0$. The class of admissible functions $\Phi_{t,1}[\Omega, q]$ consists of those functions $\phi: C^3 \times U \rightarrow C$ that satisfy the admissibility condition
\[
\phi(u,v,w;z) \notin \Omega
\]
whenever
\[
u = q(\zeta) , \quad v = \frac{k \zeta q' (\zeta) + (a-1)q(\zeta)}{a} ,
\]

\[
R \left\{ \frac{(a-1)(aw - (a-2)u)}{av - (a-1)u} - 2(a-p) + 3 \right\} \geq kR \left\{ \frac{\zeta q'' (\zeta)}{q' (\zeta)} + 1 \right\},
\]

where $z \in U$, $\zeta \in \partial U / E(q)$, $p \in N$, $a \in R \setminus Z_0^+$ and $k \geq p$.

Theorem 3. Let $\phi \in \Phi_{t,1}[\Omega, q]$. If $f(z) \in A(p)$ satisfies
\[
\left\{ \phi \left( \frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda}(a,c)f(z)}{z^{p-1}}, \frac{I_p^{\lambda}(a-1,c)f(z)}{z^{p-1}}; z \right); z \in U \right\} \subset \Omega ,
\]
then
\[
I_p^{\lambda}(a+1,c)f(z) \left( z^{p-1} \right) \sim q(z) . \quad (p \in N; \lambda > -p; a,c \in R \setminus Z_0^+; z \in U).
\]

Proof. Define the analytic function $p(z)$ in $U$ by
\[
p(z) = \frac{I_p^{\lambda}(a+1,c)f(z)}{z^{p-1}} \quad (2.11)
\]

In the view of relation (1.7) and from (2.11) we get,
\[
I_p^{\lambda}(a+1,c)f(z) = z(p'(z) + (a-1)p(z)) \left( z^{p-1} \right) .
\]

(2.12)

Further computation show that
\[
I_p^{\lambda}(a-1,c)f(z) = \left[ z^p p'(z) + 2(a-1)zp'(z) + (a-1)(a-2)p(z) \right] \left( z^{p-1} \right) .
\]

(2.13)

Define the transformation from $C^3$ to $C$ by
\[
u = r , \quad v = \frac{s + (a-1)r}{a} , \quad w = \frac{t + 2(a-1)s + (a-1)(a-2)r}{a(a-1)} .
\]

(2.14)
\[ \psi(r, s, t; z) = \phi(u, v, w; z) \]
\[ = \phi\left( r, \frac{s + (a - 1)r}{a}, \frac{t + 2(a - 1)s + (a - 1)(a - 2)r}{a(a - 1)}; z \right) \quad (2.15) \]

The proof shall make use of Lemma 1. Using equation (2.11), (2.12) and (2.13), from (2.15), we obtain

\[ \psi(p(z), zp'(z), z^2 p''(z); z) = \phi\left( \frac{I_p^\lambda (a + 1, c) f(z)}{z^{p-1}}, \frac{I_p^\lambda (a, c) f(z)}{z^{p-1}}, \frac{I_p^\lambda (a - 1, c) f(z)}{z^{p-1}}; z \right) \quad (2.16) \]

Hence (2.10) becomes

\[ \psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega. \]

The proof is completed if it can be shown that the admissibility condition for \( \phi \in \Phi_{I,1}[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.

Note that

\[ \left\{ \frac{t}{s} + 1 \right\} = \left\{ \frac{(a - 1)[aw - (a - 2)u]}{av - (a - 1)u} - 2(a - p) + 3 \right\}, \]

and hence \( \psi \in \Psi_p[\Omega, q] \). By Lemma 1, \( p(z) \prec q(z) \) or

\[ \frac{I_p^\lambda (a + 1, c) f(z)}{z^{p-1}} \prec q(z). \]

If \( \Omega \neq C \) is a simply connected domain, then \( \Omega = h(U) \), for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case the class \( \Phi_{I,1}[h(U), q] \) is written as \( \Phi_{I,1}[h, q] \).

The following result is an immediate consequence of Theorem 3.

**Theorem 4.** Let \( \phi \in \Phi_{I,1}[\Omega, q] \), if \( f(z) \in A(p) \) satisfies

\[ \phi\left( \frac{I_p^\lambda (a + 1, c) f(z)}{z^{p-1}}, \frac{I_p^\lambda (a, c) f(z)}{z^{p-1}}, \frac{I_p^\lambda (a - 1, c) f(z)}{z^{p-1}}; z \right) \prec h(z), \quad (2.17) \]

then

\[ \frac{I_p^\lambda (a + 1, c) f(z)}{z^{p-1}} \prec q(z). \]

\[ \left( p \in N; \lambda > -p; a, c \in R \setminus Z_0; z \in U \right). \]

If \( q(z) = Mz \), \( M > 0 \), the class of admissible functions \( \Phi_{I,1}[\Omega, q] \), denoted by \( \Phi_{I,1}[\Omega, M] \), is described below.

**Definition 6.** let \( \Omega \) be a set in \( C \) and \( M > 0 \). The class of admissible functions \( \Phi_{I,1}[\Omega, q] \), consists of those functions \( \phi : C^2 \times U \to C \) such that

\[ \phi\left( Me^{i\theta}, k + a - 1 \frac{k + a - 1}{a} Me^{i\theta}, L + (a - 1) \frac{2k + (a - 2)}{a(a - 1)} Me^{i\theta}; z \right) \notin \Omega, \quad (2.18) \]

whenever

\[ z \in U, \theta \in R, R \left\{ Le^{-i\theta} \right\} \geq (k - 1)k M \text{ for all real } \theta, p \in N \text{ and } a \in R \setminus Z_0, k \geq p. \]
Corollary 4. Let \( \phi \in \Phi_{I,1}[\Omega, M] \). If \( f(z) \in A(p) \) satisfies

\[
\phi \left( \frac{I^\lambda_p(a+1,c)f(z)}{z^{p-1}}, \frac{I^\lambda_p(a,c)f(z)}{z^{p-1}}, \frac{I^\lambda_p(a-1,c)f(z)}{z^{p-1}} \right) \in \Omega,
\]

then

\[
\left| \frac{I^\lambda_p(a+1,c)f(z)}{z^{p-1}} \right| < M \cdot \left( p \in N; \lambda > -p; a, c \in R \setminus Z_0^-; z \in U \right)
\]

In the special case \( \Omega = q(U) = \{ w : |w| < M \} \), the class \( \Phi_{I,1}[\Omega, M] \) is simply denoted by \( \Phi_{I,1}[M] \), then the previous Corollary 4 takes the following form.

Corollary 5. Let \( \phi \in \Phi_{I,1}[M] \). If \( f(z) \in A(p) \) satisfies

\[
\phi \left( \frac{I^\lambda_p(a+1,c)f(z)}{z^{p-1}}, \frac{I^\lambda_p(a,c)f(z)}{z^{p-1}}, \frac{I^\lambda_p(a-1,c)f(z)}{z^{p-1}} \right) < M,
\]

then

\[
\left| \frac{I^\lambda_p(a+1,c)f(z)}{z^{p-1}} \right| < M \cdot \left( p \in N; \lambda > -p; a, c \in R \setminus Z_0^-; z \in U \right)
\]

Next, we introduce a new class of admissible functions \( \Phi_{I,2}[\Omega, q] \).

Definition 7. Let \( \Omega \) be a set in \( C \), \( q(z) \in O_1 \cap H \). The class of admissible functions \( \Phi_{I,2}[\Omega, q] \) consists of those functions \( \phi : C^2 \times U \rightarrow C \) that satisfy the admissibility condition:

\[
\phi(u, v, w; z) \notin \Omega
\]

whenever

\[
u = q(\zeta), v = \frac{1}{a-1} \left\{ -1 + aq(\zeta) + \frac{k \zeta q'(\zeta)}{q(\zeta)} \right\},
\]

\[
R \left\{ \frac{(a-2)w - (a-1)v + 1}{(a-1)v - aw + 1} - 2 au + (a-1)w - 1 \right\} \geq R \left\{ \frac{\zeta q''(\zeta)}{q(\zeta)} + 1 \right\},
\]

where \( z \in U, \zeta \in \partial U / E(q), p \in N, a \in R \setminus Z_0^- \) and \( k \geq p \).

Theorem 5. Let \( \phi \in \Phi_{I,2}[\Omega, q] \) and \( I^\lambda_p(a+1,c)f(z) \neq 0 \). If \( f(z) \in A(p) \) satisfies

\[
\left\{ \phi \left( \frac{I^\lambda_p(a,c)f(z)}{I^\lambda_p(a+1,c)f(z)}, \frac{I^\lambda_p(a-1,c)f(z)}{I^\lambda_p(a+1,c)f(z)}, \frac{I^\lambda_p(a-2,c)f(z)}{I^\lambda_p(a+1,c)f(z)} \right); z \in U \right\} \subset \Omega, \quad (2.19)
\]

then

\[
\frac{I^\lambda_p(a,c)f(z)}{I^\lambda_p(a+1,c)f(z)} = q(z).
\]

\[
(p \in N; \lambda > -p; a, c \in R \setminus Z_0^-; z \in U).
\]

Proof. Define the analytic function \( p(z) \) in \( U \) by

\[
p(z) = \frac{I^\lambda_p(a,c)f(z)}{I^\lambda_p(a+1,c)f(z)}. \quad (2.20)
\]
Using (2.20), we get
\[
\frac{zp'(z)}{p(z)} = \frac{z (I^\lambda_p (a,c) f (z))'}{(I^\lambda_p (a,c) f (z))} - \frac{z (I^\lambda_p (a+1,c) f (z))'}{(I^\lambda_p (a+1,c) f (z))}.
\] (2.21)

By making use of the relation (1.7) in (2.21), we get
\[
\frac{I^\lambda_p (a-1,c) f (z)}{I^\lambda_p (a,c) f (z)} = \frac{1}{(a-1)} \left\{ -1 + ap(z) + \frac{zp'(z)}{p(z)} \right\}
\] (2.22)

Further computation show that
\[
\frac{I^\lambda_p (a-2,c) f (z)}{I^\lambda_p (a-1,c) f (z)} = \frac{1}{(a-1)} \left[ -2 + \frac{zp'(z)}{p(z)} + ap(z) + \frac{zp'(z) + z^2 p''(z)}{p(z)} - \left( \frac{zp'(z)}{p(z)} \right)^2 + ap'(z) \right] \frac{zp'(z) + ap(z) - 1}{p(z)}.
\] (2.23)

Define the transformation from \( C^3 \) to \( C \) by
\[
u = \frac{1}{(a-1)} \left\{ -1 + ar + \frac{s}{r} \right\}, \quad w = \frac{1}{(a-1)} \left\{ -2 + \frac{s}{r} + ar + \frac{t}{r} + s - \left( \frac{s}{r} \right)^2 + as \right\}
\] (2.24)

Let
\[
\psi(r,s,t;z) = \phi(u,v,w;z)
\]
\[
= \phi \left( r, \frac{1}{(a-1)} \left\{ -1 + ar + \frac{s}{r} \right\}, \frac{1}{(a-1)} \left\{ -2 + \frac{s}{r} + ar + \frac{t}{r} + s - \left( \frac{s}{r} \right)^2 + as \right\}; z \right)
\] (2.25)

The proof shall make use of Lemma 1. Using equation (2.20), (2.22) and (2.23), then (2.25), we obtain
\[
\psi(p(z),zp'(z),z^2 p''(z);z) = \phi \left( \frac{I^\lambda_p (a,c) f (z)}{I^\lambda_p (a,c) f (z)}, \frac{I^\lambda_p (a-1,c) f (z)}{I^\lambda_p (a,c) f (z)}, \frac{I^\lambda_p (a-2,c) f (z)}{I^\lambda_p (a-1,c) f (z)}; z \right)
\] (2.26)

Hence (2.19) becomes
\[
\psi(p(z),zp'(z),z^2 p''(z);z) \in \Omega.
\]

The proof is completed if it can be shown that the admissibility condition for \( \phi \in \Phi_{1,n} [\Omega,q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.

Note that
\[
\left\{ \frac{t}{s} \right\} + 1 = \left\{ \frac{v \left( (a-2)w - (a-1)v + 1 \right)}{(a-1)v - au + 1} - 2 au + (a-1)v + 1 \right\}.
\]
and hence \( \psi \in \Psi_p[\Omega, q] \). By Lemma 1, \( p(z) < q(z) \) or \( \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} < q(z) \).

If \( \Omega \neq C \) is a simply connected domain, then \( \Omega = h(U) \), for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case the class \( \Phi_{1,2}[h(U), q] \) is written as \( \Phi_{1,2}[h,q] \).

The following result is an immediate consequence of Theorem 5.

**Theorem 6.** Let \( \phi \in \Phi_{1,2}[h, q] \) and \( I_p^\lambda(a+c)f(z) \neq 0 \). If \( f(z) \in A(p) \) satisfies

\[
\phi \left( \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \right) < h \tag{2.27}
\]

then

\[
\frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} < q(z) \Rightarrow \left( p \in N ; \lambda > -p; a,c \in R \setminus \mathbb{Z}_0^+ ; z \in U \right).
\]

If \( q(z) = Mz, M > 0 \), the class of admissible functions \( \Phi_{1,2}[[\Omega, q] \), denoted by \( \Phi_{1,2}[[\Omega, M] \), is described below.

**Definition 8.** Let \( \Omega \) be a set in \( C \) and \( M > 0 \). The class of admissible functions \( \Phi_{1,2}[\Omega, M] \) consist of those functions \( \phi: C^3 \times U \rightarrow C \) such that

\[
\phi \left( Me^{i\theta}, \frac{1}{a-1}(k-1+aMe^{i\theta}), \frac{1}{a-1}(k-2+aMe^{i\theta}) \right) < \frac{Le^{-i\theta} + kM + akM^2 - k^2M}{M(k-1) + aM^2e^{i\theta}} \tag{2.28}
\]

whenever

\[z \in U, \theta \in R, \Re \{Le^{-i\theta}\} \geq (k-1)kM \text{ for all real } \theta, p \in N; a \in R \setminus \mathbb{Z}_0^+ \text{ and } k \geq p.\]

**Corollary 6.** Let \( \phi \in \Phi_{1,2}[[\Omega, M] \) and \( I_p^\lambda(a+c)f(z) \neq 0 \). If \( f(z) \in A(p) \) satisfies

\[
\phi \left( \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \right) \in \Omega,
\]

then

\[
\left| \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \right| < M \Rightarrow \left( p \in N ; \lambda > -p; a,c \in R \setminus \mathbb{Z}_0^+ ; z \in U \right)
\]

In the special case \( \Omega = q(U) = \{w : |w| < M\} \), the class \( \Phi_{1,2}[[\Omega, M] \) is simply denoted by \( \Phi_{1,2}[M] \), then Corollary 6 takes the following form.

**Corollary 7.** Let \( \phi \in \Phi_{1,2}[[M] \). If \( f(z) \in A(p) \) satisfies

\[
\phi \left( \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)}, \frac{I_p^\lambda(a-1,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \right) < M,
\]

then
\[
\left| \frac{I_p^\lambda(a,c)f(z)}{I_p^\lambda(a+1,c)f(z)} \right| < M , \quad (p \in \mathbb{N}; \lambda > -p; a,c \in R \setminus Z_0^-; z \in U)
\]

\section{Superordination of the Cho-Kwon-Srivastava Operator}

The dual problem of the differential subordination, that is, differential superordination of the operator \( I_p^\lambda(a,c)f(z) \) is investigated in this section. For this purpose the class of the admissible functions is given in the following definition.

**Definition 9.** Let \( \Omega \) be a set in \( q(z) \in H[0, p] \) with \( zq'(z) \neq 0 \). The class of admissible functions \( \Phi_p^\lambda[\Omega, q] \) consists of those functions \( \phi : C^3 \times \overline{U} \to C \) that satisfy the admissibility condition

\[
\phi(u,v,w;z) \notin \Omega
\]

whenever

\[
u = q(z), \quad \lambda = \frac{zq'(z) + m(a - p)q(z)}{ma}, \quad z \in U, \quad \zeta \in \partial U, \quad a \in R \setminus Z_0^-, z \in U \text{ and } m \geq p.
\]

**Theorem 7.** Let \( \phi \in \Phi_p^\lambda[\Omega, q] \). If \( f(z) \in A(p) \), \( I_p^\lambda(a,c)f(z) \in Q_0 \) and

\[
\phi(I_p^\lambda(a+1,c)f(z) , I_p^\lambda(a,c)f(z) , I_p^\lambda(a-1,c)f(z) ; z)
\]

is univalent in \( U \), then

\[
\Omega \subseteq \left\{ \phi(I_p^\lambda(a+1,c)f(z) , I_p^\lambda(a,c)f(z) , I_p^\lambda(a-1,c)f(z) ; z) ; z \in U \right\}
\]

implies

\[
q(z) < I_p^\lambda(a+1,c)f(z)
\]

\[
(z \in U; a,c \in R \setminus Z_0^-; \lambda > -p; p \in N).
\]

**Proof.** From (2.6) and (3.1), we have

\[
\Omega \subseteq \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\}.
\]

From (2.5), we see that the admissibility condition for \( \phi \in \Phi_p^\lambda[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 2. Hence \( \psi \in \Psi_p^\lambda[\Omega, q] \), and by

**Lemma 2.** \( q(z) < p(z) \) or \( q(z) < I_p^\lambda(a+1,c)f(z) \).

If \( \Omega \neq C \) is a simply connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \).

In this case the class \( \Phi_p^\lambda[h(U), q] \) is written as \( \Phi_p^\lambda[h, q] \).

The following result is an immediate consequence of Theorem 7.

**Theorem 8.** Let \( h(z) \) be analytic on \( U \) and \( \phi \in \Phi_p^\lambda[h, q] \). If \( f(z) \in A(p) \), \( I_p^\lambda(a,c)f(z) \in Q_0 \) and

\[
\phi(I_p^\lambda(a+1,c)f(z) , I_p^\lambda(a,c)f(z) , I_p^\lambda(a-1,c)f(z) ; z)
\]

is univalent in \( U \).
then
\[ h(z) < \phi \left( I_{\mu}^\lambda (a+1,c)f(z), I_{\mu}^\lambda (a,c)f(z), I_{\mu}^\lambda (a-1,c)f(z); z \right), \]

implies
\[ q(z) < I_{\mu}^\lambda (a+1,c)f(z). \]

Now, we introduce a new class of admissible functions \( \Phi_{I,1}[\Omega, q] \).

**Definition 9.** Let \( \Omega \) be a set in \( \mathbb{C} \), \( q(z) \in H_\Omega \) with \( zq'(z) \neq 0 \). The class of admissible functions \( \Phi_{I,1}[\Omega, q] \) consists of those functions \( \phi: \mathbb{C} \times \overline{U} \to \mathbb{C} \) that satisfy the admissibility condition:

\[ \phi(u,v,w; \zeta) \in \Omega \]

whenever
\[ u = q(z), \quad v = \frac{zq'(z) + m(a-1)q(z)}{ma}, \]

\[ R \left\{ \frac{a(a-1)v - (a-2)u}{av - (a-1)u} - 2(a-p) + 3 \right\} \leq \frac{1}{m} R \left\{ \frac{\zeta q''(\zeta) + q'(\zeta)}{q'(\zeta) + 1} \right\}, \]

where \( z \in U, \zeta \in \partial U \) and \( m \geq p \).

Now, we will give the dual result of Theorem 3 for differential superordination.

**Theorem 9.** Let \( \phi \in \Phi_{I,1}[\Omega, q] \). If \( f(z) \in A(p), \frac{I_{\mu}^\lambda (a+1,c)f(z)}{z^{p-1}} \in Q_0 \) and

\[ \phi \left( \frac{I_{\mu}^\lambda (a+1,c)f(z)}{z^{p-1}}, \frac{I_{\mu}^\lambda (a,c)f(z)}{z^{p-1}}, \frac{I_{\mu}^\lambda (a-1,c)f(z)}{z^{p-1}}; z \right) \]

is univalent in \( U \), then

\[ \Omega \subset \left\{ \phi \left( \frac{I_{\mu}^\lambda (a+1,c)f(z)}{z^{p-1}}, \frac{I_{\mu}^\lambda (a,c)f(z)}{z^{p-1}}, \frac{I_{\mu}^\lambda (a-1,c)f(z)}{z^{p-1}}; z \right): z \in U \right\} \quad (3.3) \]

implies

\[ q(z) < \frac{I_{\mu}^\lambda (a+1,c)f(z)}{z^{p-1}}. \]

\[ \left( z \in U; a,c \in R \setminus Z_0^-; \lambda > -p; p \in N \right). \]

**Proof.** From (2.16) and (3.3), we have

\[ \Omega \subset \{ \psi(p(z), zp'(z), z^2p''(z); z): z \in U \}. \]

From (2.12), we see that the admissibility condition for \( \phi \in \Phi_{I,1}[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 2. Hence \( \psi \in \Psi'[\Omega, q] \) and by

Lemma 2, \( q(z) < p(z) \) or \( q(z) < \frac{I_{\mu}^\lambda (a+1,c)f(z)}{z^{p-1}} \).

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case the class \( \Phi_{I,1}[h(U), q] \) is written as \( \Phi_{I,1}[h, q] \).

The following result is an immediate consequence of Theorem 9.
Theorem 10. Let \( q(z) \in H_0, h(z) \) be univalent in \( U \) and \( \phi \in \Phi_{1,1}[\Omega, q] \). If \( f(z) \in A(p) \),

\[
\frac{I_p^\lambda (a + 1, c)f(z)}{z^{p-1}} \in Q_0 \quad \text{and} \quad \phi \left( \frac{I_p^\lambda (a + 1, c)f(z)}{z^{p-1}}, \frac{I_p^\lambda (a, c)f(z)}{z^{p-1}}, \frac{I_p^\lambda (a - 1, c)f(z)}{z^{p-1}} ; z \right)
\]

is univalent in \( U \), then

\[
h(z) < \phi \left( \frac{I_p^\lambda (a + 1, c)f(z)}{z^{p-1}}, \frac{I_p^\lambda (a, c)f(z)}{z^{p-1}}, \frac{I_p^\lambda (a - 1, c)f(z)}{z^{p-1}} ; z \right)
\]

implies

\[
q(z) < \frac{I_p^\lambda (a + 1, c)f(z)}{z^{p-1}}.
\]

Finally, we introduce down a new class of admissible functions \( \Phi'_{1,2}[\Omega, q] \).

Definition 10. Let \( \Omega \) be a set in \( \mathbb{C} \), \( q(z) \neq 0 \), \( zq'(z) \neq 0 \) and \( q(z) \in H \). The class of admissible functions \( \Phi'_{1,2}[\Omega, q] \) consists of those functions \( \phi : \mathbb{C} \times \overline{U} \to \mathbb{C} \) that satisfy the admissibility condition

\[
\phi(u, v, w; z) \notin \Omega
\]

whenever

\[
u = q(z), \quad v = \frac{1}{a-1} \left\{ -1 + aq(z) + \frac{zq'(z)}{mq(z)} \right\},
\]

\[
R \left\{ \frac{(a-2)v - (a-1)v + 1}{(a-1)v - au + 1} + (a-1)v - 2av + 1 \right\} \leq \frac{1}{m} R \left\{ \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right\},
\]

where \( z \in U, \ \zeta \in \partial U \) and \( m \geq 1 \).

Now, we will give the dual result of theorem 5.

Theorem 11. Let \( \phi \in \Phi'_{1,2}[\Omega, q] \). If \( f(z) \in A(p) \),

\[
\frac{I_p^\lambda (a, c)f(z)}{I_p^\lambda (a + 1, c)f(z)} \in Q_1 \quad \text{and} \quad \phi \left( \frac{I_p^\lambda (a, c)f(z)}{I_p^\lambda (a + 1, c)f(z)}, \frac{I_p^\lambda (a - 1, c)f(z)}{I_p^\lambda (a, c)f(z)}, \frac{I_p^\lambda (a - 2, c)f(z)}{I_p^\lambda (a - 1, c)f(z)} ; z \right)
\]

is univalent in \( U \), then

\[
\Omega \subset \phi \left( \frac{I_p^\lambda (a, c)f(z)}{I_p^\lambda (a + 1, c)f(z)}, \frac{I_p^\lambda (a - 1, c)f(z)}{I_p^\lambda (a, c)f(z)}, \frac{I_p^\lambda (a - 2, c)f(z)}{I_p^\lambda (a - 1, c)f(z)} ; z \right)
\]

implies

\[
q(z) < \frac{I_p^\lambda (a, c)f(z)}{I_p^\lambda (a + 1, c)f(z)}.
\]
Proof. From (2.26) and (3.5), we have
\[ \Omega \subset \{ \psi(p(z), zp'(z), z^2 p''(z); z) : z \in U \} . \]

From (2.24), we see that the admissibility condition for \( \phi \in \Phi'_{1,\omega}[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 2. Hence \( \psi \in \Psi([\Omega, q]) \), and by

**Lemma 2.** \( q(z) \prec p(z) \) or \( q(z) \prec \frac{I^\lambda_p(a,c)f^{(z)}}{I^\lambda_p(a+1,c)f^{(z)}} \).

If \( \Omega \neq C \) is a simply connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case the class \( \Phi'_{1,\omega}[h(U), q] \) is written as \( \Phi'_{1,\omega}[h, q] \).

The following result is an immediate consequence of Theorem 11.

**Theorem 12.** Let \( h(z) \) be analytic in \( U \) and \( \phi \in \Phi'_{1,\omega}[h, q] \). If \( f(z) \in A(p) \), \( \frac{I^\lambda_p(a-1,c)f^{(z)}}{I^\lambda_p(a,c)f^{(z)}} \in \Omega_1 \), and

\[ \phi \left( \frac{I^\lambda_p(a,c)f(z)}{I^\lambda_p(a+1,c)f(z)}, \frac{I^\lambda_p(a-1,c)f(z)}{I^\lambda_p(a,c)f(z)}, \frac{I^\lambda_p(a-2,c)f(z)}{I^\lambda_p(a-1,c)f(z)} \right) \]

is univalent in \( U \), then

\[ h(z) \prec \phi \left( \frac{I^\lambda_p(a,c)f(z)}{I^\lambda_p(a+1,c)f(z)}, \frac{I^\lambda_p(a-1,c)f(z)}{I^\lambda_p(a,c)f(z)}, \frac{I^\lambda_p(a-2,c)f(z)}{I^\lambda_p(a-1,c)f(z)} \right) , \quad (3.6) \]

implies

\[ q(z) \prec \frac{I^\lambda_p(a-1,c)f(z)}{I^\lambda_p(a,c)f(z)} . \]

**References Références Referencias**

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