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Student's Learning Progression Through Instrumental Decoding of Mathematical Ideas

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STUDENTS LEARNING PROGRESSION THROUGH INSTRUMENTAL DECODING OF MATHEMATICAL IDEAS

Strictly as per the compliance and regulations of:

Student's Learning Progression through Instrumental Decoding of Mathematical Ideas

Stavroula Patsiomitou

Abstract- **The current study aims to focus on mathematical tasks for students' mathematical literacy and problem solving literacy. Excerpts are presented from dynamic hypothetical learning paths [DHLP] s and students' learning progression. The excerpts center around activities aimed to develop the students' geometrical thinking through the development of their ability to solve real-world problems. The students cooperated in class or worked individually to represent the images using their static or dynamic means and tools (e.g. compass and ruler, a computing environment, interactive boards, dynamic geometry software). My further aim was the students to utilize transformation processes for representations by instrumentally decoding their ideas on static and dynamic objects. An important role for the students' cognitive development was the design of propositions and theorems (e.g the Pythagorean Theorem), through Linking Visual Active Representations (LVAR). Especially for the latter option an essential role has played the dynamic geometry software, Geometer's Sketchpad. Furthermore, the paper provides examples that contain rich mathematical material; therefore, student's mathematical modeling through instrumental decoding of mathematical ideas is the means of reinforcing students' conceptual knowledge.**

Keywords: linking visual active representations, learning progression, 'dynamic' hypothetical learning path, teaching cycle.

I. Introduction

The current study aims to focus on mathematical tasks for students' development of geometrical thinking "in the process of developing and refining a learning progression to build a coherent [geometry] curriculum [connected with the other areas of mathematics] and the associated instructional materials" (Krajcik, Shin, Stevens & Short, 2009, p.27). For this, the paper describes excerpts from predicted [hypothetical] learning paths (/trajectories) "through which the learning might proceed. [The learning trajectories are hypothetical as it was not] knowable in advance" (Simon, 1995, p.135). Furthermore these learning paths are dynamic, as instructional DG (Dynamic Geometry) -- as The Geometer's Sketchpad (Jackiw, 1991) -- activities are incorporated. Therefore, they could be defined as Dynamic Hypothetical Learning Paths (DHLPs). I have initially been designed and modified the paths as a result of interactions with the students that participated, adding the destinations that were not known in advance to me (Simon, 1995, p.137).

The learning paths "are subsets of [a] learning progression […] as it requires developing and testing an entire series of learning [paths] that describe specifically how to move students toward conceptual understanding of the big idea[s] in [mathematics and particularly in geometry]" (Krajcik, Shin, Stevens & Short, 2009, p.27).

Furthermore, Simon (ibid.) developed the idea of a teaching cycle and created a diagram in order to represent the way that a learning trajectory is an ongoing modification of three components: "(a) the learning goal that defines the direction, (b) the learning activities and (c) the hypothetical learning process"(Simon, 1995, p. 136). Mathematics tasks are related to the teacher's mathematical and pedagogical knowledge. According to Simon (1995) "the ingredient necessary in order to initiate mathematics learning is *pedagogy*" (p. 115, italics in original manuscript). Furthermore, teacher's knowledge about effective mathematical pedagogy influences their instructional practices (e.g., Simon & Shifter, 1991; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989).

The DHLPs incorporated real-world problems or simulations of problems in the DGS environment that had been analyzed and designed in terms of (a) the students' van Hiele (vH) levels of thinking, starting from the lower vH levels to elicit higher vH levels, (b) their sequential conceptual content, and (c) the student's comprehension of the links between representations and mathematical meanings conceptually and procedurally.

Points of departure for the anticipation of the DHLPs were the questions:

- Do students understand the mathematical components of modeling when they see real-world environments' images [-representations]?
- What mental activities will the students develop when they participate in these learning [/ instructional] activities?
- What mathematical representations are most appropriate for student learning?
- How important is the role of a dynamic geometry program to reorganize students' mental representations?
- Does students' actual learning correspond with what was anticipated?

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- How effective is the teaching and learning process using linking visual active representations (LVARs) (e.g., Patsiomitou, 2008 a, b, c, d, 2009, 2010a, b, 2011, 2012a, b, 2013; Patsiomitou & Koleza, 2008, 2009; Patsiomitou & Emvalotis, 2009 a, b, c, d, Patsiomitou & Emvalotis 2010 a, b; Patsiomitou, Barkatsas, & Emvalotis, 2010) to overcome cognitive obstacles and develop understanding of the mathematical concepts?
- Which mathematical competencies are developed through the DHLP?
- What is the need for students' learning and understanding in upper-class curriculum processes for innovative learning when new practices and ideas (i.e., fractal activities) are incorporated?

The paper provides a link between "learning with actions" and the implementation of mathematics in educational and pedagogical contexts, answering the question of if we could incorporate real world mathematics into everyday school practices.
Mathematics is part of every day children's lives. The mathematics is obvious to them or not, sometimes is not perceived as they are implicit. On the other hand

> *"School courses and books have presented 'mathematics' as a series of apparently meaningless technical procedures. […]. Just as a phrase loses meaning or acquires an unintended meaning when removed from its context, so mathematics detached from its rich intellectual setting in the culture of our civilization and reduced to a series of techniques has been grossly distorted"* (Kline, 1990, p.15-16).

Moreover, the fractal approach that is presented here reflects Kaput's (1992) writing on the importance of technology in mathematics education, concerning the feasibility of innovative practices emanating from technological advents, which were otherwise impracticable. Ferrara, Pratt, & Robutti (2006) also, suggest that "what promotes change is the curricular project in which technology is inserted, and in particular, the didactic sequences planned by the teachers in order to introduce […] concepts, which use technology as a support." (p. 258).

The article does not intend to present the extended results of the research process but rather the theoretical perspective that underpins the teaching cycle (Simon, 1995) and the role of dynamic LVAR in students' cognitive development.

In the next sections, the article will begin with an articulation of the constructivist perspective that underpins the student learning process and my decision for the selection of activities. A review will be provided of mathematical competencies and the role of modeling processes in the DGS environment with the utilization of LVARs.

II. Theoretical Underpinning

a) *Student's cognitive development*

During the past several decades, researchers were concerned about the difficulties their students have faced when attempting geometry problems (e.g., Hoffer, 1981; Usiskin, 1982). This consistent result comes about through students' difficulty releasing their thoughts from a concrete frame (White & Mitchelmore, 2010, p. 206), and failure to develop the deductive reasoning (Peirce, 1998/1903) required. This prevents them from engaging in the abstract process (e.g., Skemp, 1986; White & Mitchelmore, 2010) that is required for the study of the conceptual structure of geometry.

According to Piaget (1937/1971), students' cognitive development depends on their biological maturity. That students' cognitive development depends on the teaching process was argued by Dina van Hiele-Geldof and Pierre van Hiele in their dissertations in 1957 (Fuys, Geddes & Tischler, 1988). Dina van Hiele-Geldof (Fuys, Geddes & Tischler, 1984) in her dissertation had the objective to investigate the improvement of learning performance by a change in the learning method. Central to this model, is the description of the five levels of thought development which are: Level 1 (recognition or visualization), Level 2 (analysis), Level 3 (ordering), Level 4 (deduction) and Level 5 (rigor).

Battista uses "constructivist constructs such as levels of abstraction to describe students' progression through the van Hiele levels" (Battista, 2011, p.515). He "has elaborated the original van Hiele levels to carefully trace students' progress in moving from informal intuitive conceptualizations of 2D geometric shapes to the formal property-based conceptual system used by mathematicians" (Battista, 2007, p.851).

He separated each phase in subphases (Battista, 2007). I briefly report Battista's first three levels elaboration, which are the most pertinent to secondary students, below:

Level 1 (Visual-Holistic Reasoning) is separated into sublevel 1.1. (prerecognition) and sublevel 1.2 (recognition). (p.851).

Level 2 (Analytic-Componential Reasoning) is separated into sublevel 2.1 (Visual-informal componential reasoning), sublevel 2.2 (Informal and insufficient-formal componential reasoning) sublevel 2.3 (Sufficient formal property-based reasoning). According to Battista (2007) "Students [acquire through instruction] a) an increasing ability and inclination to account for the spatial structure of shapes by analyzing their parts and how their parts are related and b) an increasing ability to understand and apply formal geometric concepts in analyzing relationships between parts of shapes". (pp.851-852).

Level 3 (Relational –Inferential Property-Based Reasoning) into sublevel 3.1 (Empirical relations), sublevel 3.2 (Componential analysis), sublevel 3.3

(Logical inference) and sublevel 3.4 (Hierarchical shape, classification based on logical inference). According to Battista (2007) "Students explicitly interrelate and make inferences about geometric properties of shapes. […] The verbally-stated properties themselves are interiorized so that they can be meaningfully decomposed, analyzed, and applied to various shapes". (pp. 852-853).

Researchers have shown that students "often fail in the construction of a geometric configuration which is essential for the solution of the underlying geometric problem" (Schumann & Green, 1994, p.204). This happens because students at the lower levels "identify, describe, and reason about shapes and other geometric configurations according to their appearance as visual wholes" (Battista, 2007, p.851). According to van Hiele (1986) "when after some time, the concepts are sufficiently clear, pupils can begin to describe them. With this the properties possessed by the geometric figures that have been dealt with are successively mentioned and so become explicit. The figure becomes the representative of all these properties: It gets what we call the "symbol character". In this stage the comprehension of the figure means the knowledge of all these properties as a unity. […].When the symbol character of many geometric figures have become sufficiently clear to the pupils, the possibility is born that they also get a signal character". This means that the symbols can be anticipated. […]. When this orientation has been sufficiently developed, when the figures sufficiently act as signals, then, for the fisrt time geometry can be practiced as a logical topic" (p. 168).

Many researchers (e.g., Guitierrez & Jaime, 1998; Govender & De Villiers, 2002, 2004; Patsiomitou, 2008, 2012a, b, 2013; Patsiomitou & Emvalotis, 2010 a, b) describe student's processes of constructing definitions and justification at every van Hiele level as they develop geometrical thought. This evolution of students' formulation of definitions, justification, and reasoning was adopted by this study as the characteristic that would indicate their movement through several van Hiele levels. For definitions, I adopted Govender and De Villiers' (2004) clarification (see Patsiomitou, 2013). In addition, dynamic perceptual definition (e.g. Patsiomitou, 2013, p.806) is the term for the process by which the student informally 'defines' a geometrical object by using the tools of the software.

b) The development of student's mathematical competencies

Another point of view suggests that the development of student's geometrical thinking results from the development of their competencies in mathematical thinking and reasoning, argumentation, modeling etc. Therefore, if the teaching process of students is aimed to develop their competencies, then it leads to the development of their geometrical thinking. Many researchers (e.g, Burkhardt, 1981; Pierce & Stacey, 2009) have highlighted the idea of solving problems in the real world as essential to understanding and learning mathematics, as well as "a key ability for citizens [who are prepared to make] judgments and decisions" (Stacey, 2012, p.3).

According to De Corte, Verschaffel & Greer (2000), the implementation of the mathematics to solve real world problems can be useful "as a complex process involving a number of phases: understanding the situation described; constructing a mathematical model that describes the essence of those elements and relations embedded in the situation that are relevant; working through the mathematical model to identify what follows from it; interpreting the outcome of the computational work to arrive at a solution to the practical situation that gave rise to the mathematical model; evaluating that interpreted outcome in relation to the original situation; and communicating the interpreted results".(p.1).

Through the solution of the real world problems, students will be assessed regarding their competency for horizontal and vertical mathematization (Jupri, Drijvers, & van den Heuvel-Panhuizen, 2012). "The difficulty in horizontal mathematization concerns students' difficulty in going from the world of real phenomena to the world of symbols and vice versa. The difficulty in vertical mathematization concerns students' difficulty in dealing with the process of moving within the symbolic world (Treffers, 1987; Van den Heuvel-Panhuizen, 2003)" (Reported in http://igitur-archive.library.uu.nl/math/2013-0304-200631/12102012.pdf).

As I previously mentioned, my further aims, were the student's mathematical literacy and problem-solving literacy. The latter PISA (Programme for International Student Assessment) definition of mathematical literacy is as follows (OECD, 2010):

> *"Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens." (p. 4)*

It is very important for the students to develop their modeling competency in order to transform real-world problems from the three-dimensional world to the two-dimensional world of the paper and pencil [or DG] environment. Additionally, it is important for them to be able to process in an abstract way.

Epigrammatically, the students, through the problems that will be presented below, will be assessed with regard to the development of the following competencies (OECD, 2006), that have been analyzed from Niss (1999) and his colleagues but similar formulations can be found in the work of many others (e.g., Neubrand et al. 2001):

Briefly, these competencies can be described as an individual student's ability to (e.g., Niss, 1999, 2003; Neubrand et al. 2001):

Mathematical thinking and reasoning:… mastering mathematical modes of thought; posing questions characteristic of mathematics; knowing the kind of answers that mathematics offers, distinguishing among different kinds of statements; understanding and handling the extent and limits of mathematical concepts; generalizing results to larger classes of objects.

Mathematical reasoning and argumentation: …knowing what proofs are; knowing how proofs differ from other forms of mathematical reasoning; following and assessing chains of arguments; having a feel for heuristics; creating and expressing mathematical arguments; devising formal and informal mathematical arguments, and transforming heuristic arguments to valid proofs, i.e. proving statements.

Mathematical communication: … being able to communicate, in, with, and about mathematics; expressing oneself in a variety of ways in oral, written, and other visual form; understanding someone else's work.

Modelling competency:… being able to analyse and build mathematical models concerning other subjects or practice areas; structuring the field to be modeled; translating reality into mathematical structures; interpreting mathematical models in terms of context or reality; working with models; validating models; reflecting, analyzing, and offering critiques of models or solutions; reflecting on the modeling process; communicating about the model and its results; monitoring and controlling the entire modeling process.

Problem posing and handling competency:… problem identifying, posing, specifying; solving different kinds of mathematical problems.

Representation competency:… being able to handle different representations of mathematical entities; decoding, encoding, translating, distinguishing between, and interpreting different forms of representations of mathematical objects and situations as well as understanding the relationship among different representations; choosing and switching between representations.

Symbol and formalism competency:… decoding and interpreting symbolic and formal mathematical language, and understanding its relations to natural language; understanding the nature and rules of formal mathematical systems (both syntax and semantics); translating from natural language to formal/symbolic language; handling and manipulating statements and expressions containing symbols and formulae.

Communicating in, with, and about mathematics competency: …. understanding others' written, visual or oral 'texts', in a variety of linguistic registers, about matters having a mathematical content; expressing oneself, at different levels of theoretical and technical precision, in oral, visual or written form, about such matters.

*Aids and tools competency:…*being able to make use of and relate to the aids and tools of mathematics, including technology when appropriate.

The visualization competency and the competency of students to develop recursive processes conceptually and structurally is [also] very important for the solution of problems with fractal constructions.

In a paper of PME conference (Patsiomitou, 2011) I had also distinguished the kinds of apprehension when selecting software objects. Competence in the DGS environment depends on the competence of the cognitive analysis which students bring to bear when decoding the utilization of software tools, based on Duval's (1995) semiotic analysis of students' apprehension of a geometric figure. During the development of a construction in a DGS environment, I believe that the student has to develop three kinds of apprehension when selecting software objects which accord with the types of cognitive apprehension outlined by Duval (1995, pp.145-147) namely perceptual, sequential, discursive, and operative apprehension. In concrete terms, the competence of instrumental decoding in the software's constructions depends on (Patsiomitou, 2011, p.363):

a) the sequential apprehension of the tools selection (i.e. s/he has to select point C and segment AB and then the command (fig. 1) meaning that s/he has to follow a predetermined order); b) the verbal apprehension of the tools selection which means the student has to verbalize this process, (i.e. s/he says "I am going to select point C and the segment AB") and c) a place way type of elements operation on the figure (i.e. when s/he transforms the orientation of the elements to apply the command selecting point B and the opposite side AC, for example in fig.4) due to his/her perceptual apprehension (fig. 2, 4). Then s/he has constructed the operative apprehension of the figure's elements for the construction, meaning the competence to operate the construction.

c) Is learning and knowledge development a cognitive process? The role of teacher in learning process

Van Hiele theory has its roots in constructivist theories. Cognitive constructivism is connected with the work of Piaget's (1937/1971) and his views as 'constructivist'. Bruner's (1961, 1966) proposal of discovery learning [as 'constructionist"] is based on prior knowledge and the understanding of a concept, which [through discovery] grows and deepens. According to Bruner (1986) "learning is a social process in which children grow into the intellectual life of those around them" (Clements & Battista, 1990, p.6).

The sociocultural approach has its roots in Vygotsky (1987) who focuses on the acquisition of mathematical understanding as a product of social interactions. Von Glasersfeld (1995) a radical constructivist is differentiated from the work of Piaget as he argues that "knowledge [does not represent an independent world, instead] represents something that [...] we can do in our experiental world" (p.6).

Building on the concepts mentioned above, the concept of social constructivism is a complex process, while being interactive, constructivist and sociocultural (e.g., Yackel, Cobb, Wood, Wheatley & Merkel 1990; Cobb, Yackel & Wood 1992; Yackel, Rasmussen & King 2001; Yackel & Rasmussen 2002; Jaworski, 2003). According to sociocultural and interactive approaches, learning is a part of the culture (Steffe & Gale, 1995) in which the students construct knowledge through their participation in social practices (e.g social class environment) (Cobb & Bauersfeld, 1995, p.4). "A social-constructivist perspective sees discussion, negotiation and argumentation in inquiry and investigation practices to underpin knowledge growth in mathematics, in teaching mathematics and in mathematics teacher education" (e.g., Cobb & Bowers, 1999; Lampert, 1998; Wood, 1999 in Jaworski, 2003, p . 17).

Besides, learning is an individual constructive process while knowledge is actively constructed by the student; it depends on the individual's personal work and negotiation of mathematical ideas (e.g., Jaworski, 2003). From the perspective of constructivist theories the process of mathematical knowledge and understanding arises as students try to solve math problems during the classroom (Cobb, Yackel, & Wood, 1992; Simon & Shifter, 1991) and is instigated when students confront problematic situations. Knowing therefore is not taken passively by students but in an active way. Learning thus is characterized in Bauersfeld's interactionism view "by the subjective reconstruction of societal means and models through negotiation of meaning in social intervention" (Bauersfeld, 1992, p.39).

Vygotsky (1987) argues that "the child begins to perceive the world not only through his eyes [visually] but also through speech" (p. 32). According to Vygotsky (1987), learning is a complex interplay between scientific and spontaneous use of language.

For this, learning is an internalization of social relations and understanding is a result of common negotiation of concepts created by students while interacting with other students in the class (or group) during the mathematical discussions developed (Bartolini Bussi, 1996).

For the current study, I used the strategy of "thinking aloud" (Hayes & Flower, 1980; Smith & Wedman, 1988) in class or group discussions because I strongly believe "that this [action] influence[s] students' own use of language". [...]. "Language is important for cognitive development and learning; without it, an individual lacks [an] efficient system for storing certain types of information that are needed for thinking, reasoning, and concept development"(Westwood, 2004, p.141).

Sfard also defines "learning as the process of changing one's discursive ways in a certain well-defined manner" (Sfard, 2001, p.3). According to Sfard (2001) "thinking is a special case of the activity of communicating" [...]"A person who thinks can be seen as communicating with himself/herself, [...] whether the thinking is in words, in images or other form of symbols, [..] as our thinking is [an interactive] dialogical endeavour [through which] we argue..." (p.3); with his/her participation the student in a mathematical discussion s/he "learns to think mathematically" (Sfard, ibid., p. 4). Under this approach, the development of thought occurs through dialogue that develops the subject within himself/herself internally (intrapersonally) or in a group in which s/he participates. Moreover, learning is expanding the capacity for dialectical skills and solving problems that could not previously be solved. Furthermore "putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned" (Sfard, 2001, p.1). Consequently, learning is first and foremost the modification / transformation of the ways we think and how we exchange this thought. Moreover, learning is the capacity of dialectical skills and of problem-solving that could not be solved before.

Goos and her colleagues carried out a series of studies --based on sociocultural perspective-- to investigate the teacher's role, the students' discussion in small groups and the use of technology as a tool that mediates teaching and learning interactions (e.g. Goos, 2004, Goos, Galbraith, Renshaw, & Geiger, 2003). If we take the role of teacher seriously as concerns the realisation and planning of activities then, every activity should be based on geometry exactly as Goldenberg (1999) purports it to be –a fundamental principle. The current study leads us, as Goldenberg (ibid.), writes "to select idea-editors that have supported the connections. Tools like Geometer's Sketchpad present geometric structures in an environment that emphasizes the continuous nature of Euclidean space, and thus serve as an excellent bridge between geometry and [the other field of mathematics, as well as] analysis." This is very important for the teaching practice because the construction of the meaning can not only be depended or is located in the tool per se, nor uniquely pinpointed in the interaction of student and tool, but it lies in the schemes of use (e.g., Trouche, 2004) of the tool itself.

Simon (1995) has developed a view of the teacher's role that includes both the psychological and the social aspects. He supports that "a teacher is directed by his conceptual goals for his students, goals that are constantly being modified" (p.135). I adopted Simon's view for my role as teacher-researcher for the current study. In the next sections it will be articulated the "rational for choosing the particular instructional design; thus I make my design decisions based on my guess of how learning might proceed" (Simon, 1995, p.135).

d) What are Learning Progressions? What is 'The Teaching Cycle'?

Duschl, Schweingruber, & Shouse (2007) define learning progressions as "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (Duschl, Schweingruber, & Shouse, 2007, p.214). A learning progression is also committed to the "notion of learning as an ongoing developmental progression. It is designed to help children continually build on, and revise their knowledge and abilities, starting from the initial conceptions about how the world works and curiosity about what they see around them" (National Research Council (NRC), 2010, p.2). Learning progressions has among others students' assessment as important component that aids to "measure student understanding of the key concepts or practices and can track their developmental progress over time" (Corcoran, Mocher & Rogat, 2009, p.15).

The learning progression in mathematics (or other disciplines, e.g language, science) has been built upon the concept of the Assessment Triangle (Pellegrino, Chudowsky & Glaser, 2001) which "explicate three key elements underlying any assessment: (1) a model of student cognition and learning in the domain, (2) a set of beliefs about the kinds of observations that provide evidence of students competencies and (3) an interpretation process for making sense of evidence" (National Research Council, 2001, p.44). As Smith, Wiser, Anderson, & Krajcik (2006) argue the progress through learning progressions depends on instruction as well as the theory of van Hiele supports.

Krajcik, Shin, Stevens & Short, (2009, p.28), have created an illustration of the difference between learning progressions and learning trajectories. As they support "sets of learning trajectories of instructional sequences that describe specific ways of supporting student's learning [constitute] a learning progression" (p.28). Besides Battista (2011) supports that a learning progression differs from a learning trajectory because it has not been designed "to test a curriculum, based on a fixed sequence of learning tasks in that curriculum. [Instead] it is focusing on a formative assessment system that applies to many curricula […] based on many assessment tasks, not those in a fixed sequence" (p. 513).

In the current study, the vH [learning] progression describes the development of students' thinking through a vH instructional path, which was kept for a year and then repeated the next year. It focuses on describing the evolution of the students' mathematical learning, cognitive structures, and reasoning and has been "developed from examining how students' ideas develop by the analysis of various assessment tasks" (Stevens, Shin & Krajcik, 2009, p.2). I was the activities' instructor as well as student's teacher. The activities are "linked to [students'] curriculum and are being tested by collecting their work from classroom in which the curriculum is being used to specify the expected achievement levels; [the current] study tend to be longitudinal, with changes in student thinking and ability tracked over the course of […] a year." (Corcoran, Mocher & Rogat, 2009, p.33). According to Neubrand (2004) mathematical achievement can be structured by three "types of mathematical activities," employing only the techniques of modeling and problem-solving activities that use mathematical tools and procedures, call for connections, and utilize mathematical conceptions.

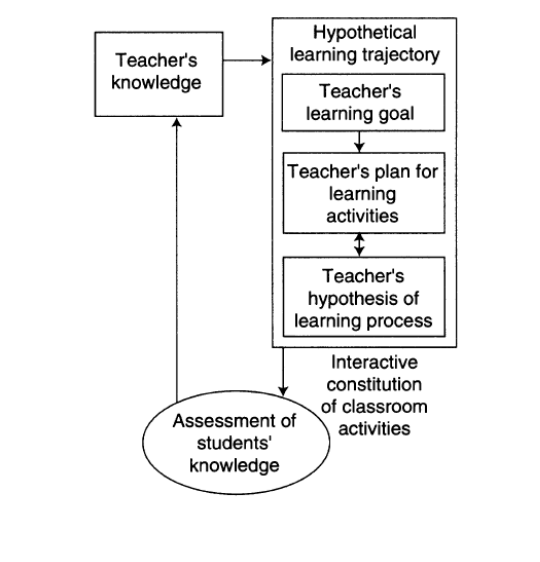

Figure 1 : Simon's (1995, p.136) *Mathematics Teaching Cycle.*

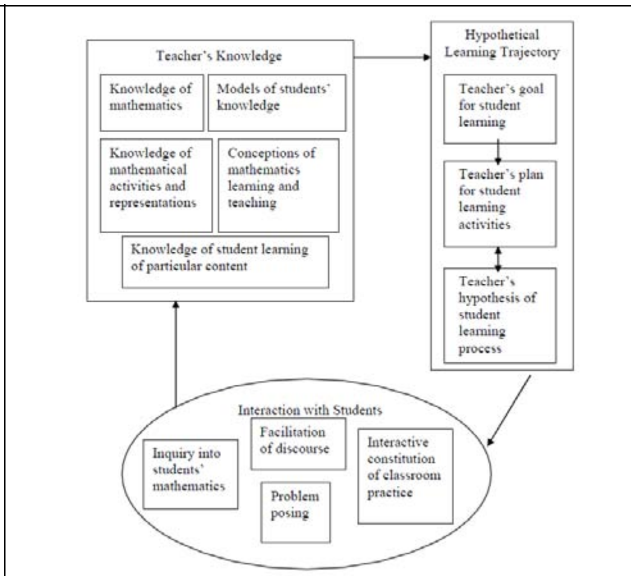

Figure 2 : An adaptation of Simon's (1995) *Mathematics Teaching Cycle*, created by McGraw (2002, p.10).

Simon (1995) created a diagram (Figure 1) in order to represent the way that a leaning trajectory is an ongoing modification of three components: (a) the learning goal, (b) the learning activities and (c) the hypothetical learning process. McGraw (2002, p.10) in her Ph.D. thesis created an adaptation on Simon's (1995) teaching cycle that is represented in Figure 2, aiming to include the actual discussions with students that "occurred within the 'interaction with students', which influenced the "teacher's knowledge".

What has not been examined is the use of technology in the teaching cycle which plays an important role in the development of discussions, as well as students' vH level.

e) The modeling process in a DGS environment –What are LVARs?

From a representational view of learning mathematics the DHLP is supported theoretically by the concept of representation. According to Vergnaud (1987) "representation is an important element in the theory of teaching and learning of mathematics [... especially since they] play an important role in understanding the real world. Representations are provided to the students in different forms (Gagatsis & Spirou, 2000) (e.g., real-world situations, images or diagrams, oral or written symbols).

Vergnaud (1998) claims that "[n] either Piaget nor Vygotsky realized how much cognitive development depends on situations and on the specific conceptualizations that are required to deal with them" (p. 181). Piaget focused on the subjectivity of representation and Vygotsky on a social process of gaining control over external "sign" forms. Children have difficulty to perceive the signs of the meanings in the images of the real world. They perceive them as a whole image especially at the lower van Hiele levels. When students move to upper van Hiele levels they increase their ability to transform the visual image or drawing (Parsysz, 1988) they perceive, into a figure with concrete properties.

For most researchers, representations can help students to reorganize and translate their ideas using symbols. They are also useful as communication tools (Kaput, 1991) and can function as tools for understanding of concepts, since they help with the communication of ideas and provide a social environment for the development of mathematical discussion. The knowledge of supporting instruments, which are external representational systems for planning activities, allows us to choose between technological tools. The [external] representations facilitate the provision of information about the problem, capture the structure of the problem, and support visual reasoning. On the other hand, the external representations (e.g., formulations or figures) that students construct serve as an indicator of their internal representations, constituting their level of understanding and the developmental level of their geometric thinking.

The use of a computing environment as dynamic geometry facilitates the teaching and learning of Euclidean geometry and helps students overcome the difficulties in translation between representations through automatic translation or "dyna-linking"(Ainsworth, 1999, p. 133) since "visually encode causal, functional, structural, and semantic properties and relationships of a represented world – either abstract or concrete (Glasgow, Narayanan & Chandrasekran, 1995; Peterson, 1996; Card, MacKinlay & Shneiderman, 1999; Cheng, 2002)" (Sedig & Sumner, 2006, p. 2).

Dynamic geometry software has been used broadly in research regarding the teaching and learning process of geometry over the past several decades (see for example the articles written in Educational Studies in Mathematics and International Journal of Computers for Mathematical Learning) (Leung & Or, 2007, p. 177). Such research with dynamic geometry has verified that the software is useful in provoking cognitive conflicts (e.g, Hadas, Hershkowitz, & Schwarz, 2000; Giraldo, Belfort & Carvalho, 2004), developing students' deductive reasoning (e.g, Hollebrands & Smith, 2009), and developing students' geometrical thinking (e.g, Yousef, 1997; Almeqdadi, 2000; Sinclair, 2001; Patsiomitou, 2008a, 2010a, 2012a,b, 2013), according to the theory of van Hiele. Üstün & Ubuz (2004) consider that "the Geometer's Sketchpad is an important vehicle of technological chance in geometry classroom. […] The shapes are first created and then they are explored, manipulated and transformed to ideal concept". Olkun, Sinoplu & Deryakulu (2005) argued that "the Geometer's Sketchpad is a suitable dynamic environment in which students can explore geometry according to their van Hiele levels" (p.3). The diagrams that are provided to the students in a DGS environment are important spatiovisual representations that facilitate understanding of the problem's information as well as the conceptualization of the problem's structure. In other words the 'dynamic' diagrams support visual reasoning, which aids translation from visual to verbal representations and the construction of meaning.

In Geometer's Sketchpad v4 DGS environment, LVAR (e.g, Patsiomitou, 2008a, b,c,d, 2009, 2010a, b, 2011, 2012a,b, and 2013) are interpreted as a [real world's or not] problem modeling process. The definition of LVAR (Patsiomitou, 2012, p. 76) is given below:

Linking Visual Active Representations are the successive building steps in a dynamic representation of a problem, the steps that are repeated in different problems or steps reversing a procedure in the same phase or between different phases of a hypothetical learning path. LVARs reveal an increasing structural complexity by conceptually and structurally linking the transformational steps taken by the user (teacher or student) as a result of the interaction techniques provided by the software to externalize the transformational steps s/he has visualized mentally (or exist in his/her mind) or organized as a result of his/her development of thinking and understanding of geometrical concepts.

Real world images (or digital images) "are potential representations […and] offer the heuristic part of learning" as they "denote something" (Kadunz & Straesser, 2004, p.241, 242). What is important is how the students perceive these potential representations of the environment (natural images or digital), how they use and communicate with each other and how they manage their mental mathematical structures in order to represent the objects. Mogeta, Olivero & Jones (1999) in their report "Providing the Motivation to Prove in a Dynamic Geometry Environment" argue that "setting problem solving within these environments requires a careful design of activities, which need to take into account the interaction between three elements: the dynamic software, as an instance of the milieu, a problem, and a situation, through which the devolution of the problem takes place (Brousseau, 1986)". Most importantly, the diagrams that the students are obliged to translate and the relations that link the objects in the diagram will provide researchers and teachers insights to see their abilities and their weaknesses with respect to the mathematical knowledge that they have structured as a result of the teaching process in class. For this, the verification of students' mistakes and cognitive obstacles during the construction of diagrams will lead us to the reinforcement of the teaching of mathematics in the context of real-world problems.

Vergnaud proposed an approach for investigation in mathematics education, which includes the steps presented in the Figure 3 (Vergnaud, 1988, p.149 reported in Long, 2011, p.123).

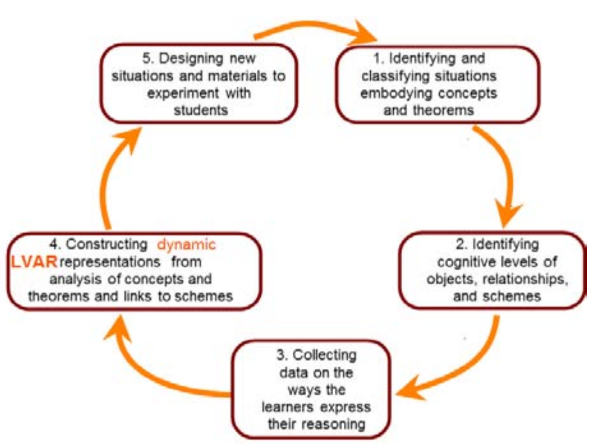

Figure 3 : An adaptation for the current study on Vergnaud's (1988, p.149) approach.

Gonzalez & Herbst (2009) have defined the dynamic diagram as "a diagram made with DGS and that has the potential to be changed in some way by dragging one or more of its parts" (p.154).

According to Mariotti (2000, p.36) "the dragging test, externally oriented at first, is aimed at testing perceptually the correctness of the drawing; as soon as it becomes part of interpersonal activities […] it changes its function and becomes a sign referring to a meaning, the meaning of the theoretical correctness of the figure." Hollebrands (2007) also supported that the students in her study "used reactive or proactive strategies when dragging, either in response to or in anticipation of the effects on dragging" (cited in Gonzalez and Herbst, 2009, p.158-159). Building on Mariotti's considerations and Hollebrands distinction about dragging strategies, in a previous study (Patsiomitou, 2011) I introduced the notions of theoretical dragging (i.e., the student aims to transform a drawing into a figure on screen, meaning s/he intentionally transforms a drawing to acquire additional properties) and experimental dragging (i.e., the student investigates whether the figure (or drawing) has certain properties or whether the modification of the drawing in the picture plane through dragging leads to the construction of another figure or drawing).

Students execute on screen constructions using software's tools and primitive geometrical objects in an effort to decode their mental representations into software actions. This sense of how the student's competence at instrumental decoding affects the development of their ability in constructing meanings, may lead to an understanding of how the tools the students use, play a fundamental role as a non linguistic warrant. The construction of a figure on screen in a DGS environment is a result of a complex process on the student's part. The student has first to transform the verbal or written formulation ("construct a parallelogram" for example) into a mental image, which is to say an internal representation recalling a prototype image (e.g., Hershkovitz, 1990) that s/he has shaped from a textbook or other authority, before transforming it into an external representation, namely an on-screen construction. This process requires the student to decode their actions using software primitives, functions etc. In order to accomplish a construction in the software the student must acquire the competence for instrumental decoding meaning the competence to transform his/her mental images to actions in the software, using the software's interaction techniques. Furthermore, dynamic reinvention of knowledge (Patsiomitou, 2012b, p. 57) is the kind of knowledge the students could reinvent by interacting with the artefacts made in a DGS environment, "knowledge for which they themselves are responsible" (Gravemeijer & Terwel, 2000, p.786).

For the representation of student's argumentation I used a pseudo-Toulmin's model (Patsiomitou, 2012b, p.57) --based on Toulmin's model (1958)-- in which: (1) the data could be an element or an object of the dynamic diagram, and (2) a warrant could be a tool or a command that guarantees the result which is the claim (or the resulted formulation). So, students perceive the properties of the rotated segment and during instrumental genesis. According to Schumann (2004) the diagram below "presents an outline of methods and ways of working with DGS in the context of geometry teaching in lower and middle secondary schools; modeling in DGS is supported by all other methods and options"

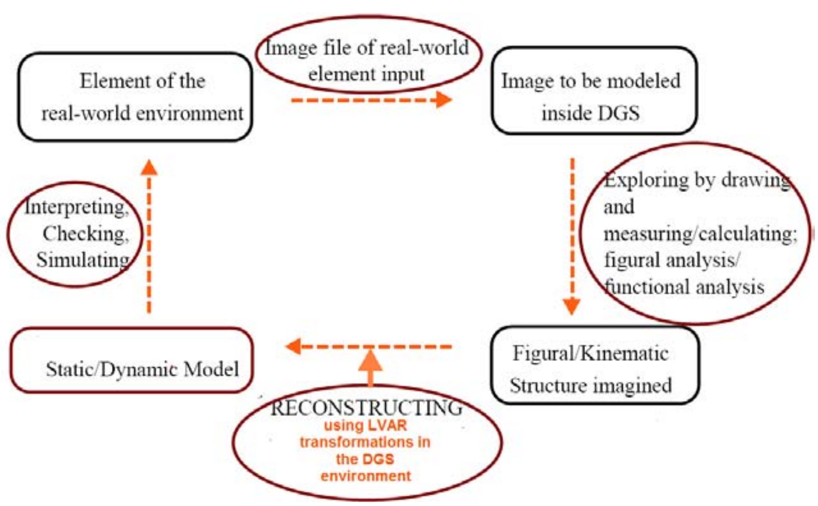

Figure 4 : An adaptation for the current study of Schuman's (2004, p. 7) modeling process using LVAR

In the figure 4, the factor "reconstructing using LVAR transformations" is the warrant (W) for the claim (static /dynamic model). This means that LVAR transformations guarantee the interpretation of the dynamic model to students' mind into external [verbal or iconic] representations.

In the next section a description will be presented of the DHLPs. As it has been told previously the DHLPs are empirically tested parts of the learning progression. The students applied their mathematical knowledge (pre-existing or not) to solve real-world problems. The modeled problems have been presented in a dynamic geometry environment or the students had to manipulate the images of the real world in their minds in order "to bridge connections between the pure world of mathematics --with fixed solutions and "perfect" forms-- and the more messy, ambiguous, or subjective world of experience" (Sinclair & Jackiw, 2007). In every situation, the experience with a real or simulated object played a major role for the construction of students' knowledge.

III. A Van Hiele Learning Progression for Secondary Students Using LVAR in Mathematics

a) Methodology of the learning progression

The current teaching experiments (Cobb & Steffe, 1983) are evolving as students' van Hiele learning progression analyzes non-routine, real-world problems in addition to student assignments from the problem-solving process. It is constituted from (a) a learning trajectory in quadrilaterals (b) a learning trajectory in fractals. The trajectory in quadrilaterals follows the structure of the DHLP created in my PhD thesis. The difference is in the objects, which in this case have been selected from the real world [e. g objects in museums, mainly archaeological, in Greece].

The teaching experiment involved 81 students aged 13-14, equally separated into three classrooms. Every sub-class included the same number of boys and girls and the same number of high- or low-achievement students at the beginning of the year. The study investigated (a) ways to foster students learning by hypothesizing what the students might learn (e.g. develop real-world problem representations, reasoning and problem-solving, making decisions and receiving feedback about their ideas and strategies) working individually or collaboratively (b) ways in which students develop abstracting processes through building linking visual active representations and (c) ways to develop students' van Hiele level.

I was the teacher and the instructor of the activities. I developed the instructional activities based on an analysis of the results of my PhD thesis, with regard to students' evolution of understanding on instrumental decoding when they construct quadrilaterals. We worked as a whole class, trying to develop a form of practice compatible with social constructivism (e.g., Wood & Yackel, 1990). I was actively involved with the children, encouraging small group cooperation both in and outside of class, without intently to show the process to complete the activity. I started the activity with a question; after the answers were given, I continued with sequential questions to clarify the explanations or to help students with the cognitive conflicts. Then, I asked the students to complete the task in the paper-pencil environment and collected their work to see the level of understanding from the correct answers. After the evaluation of the students' work, I continued with follow-up activities in the DGS environment to help the children reconstruct the solution methods. After the intervention with GSP activities, the paper-pencil work was repeated to see the difference in the students' learning and understanding of the concepts. Indicative of students' wrong representations will be presented and a short report made of their mistakes and misconceptions.

> *"The situations that children find problematic take a variety of forms and can include resolving obstacles or contradictions that arise when they attempt to make sense of a situation in terms of their current concepts and procedures, accounting for a surprising outcome (particularly when two alternative procedures lead to the same result), verbalizing their mathematical thinking, explaining or justifying a solution, resolving conflicting points of view, developing a framework that accommodates alternative solution methods, and formulating an explanation to clarify another child's solution attempt" (Cobb & Steffe, 1991, p.395)*

The complete study includes:

- ....a detailed procedural analysis of the situations, the involved problems, in addition the problems' conceptual analysis, instrumental decoding and learning targets (e.g., different solving strategies, formulas or figure's decomposition). This includes the recognition and demonstration of transformations (e. g, recognition and drawing of symmetry lines or demonstration of reflections, translations and rotations) using multiple contexts (e.g., graphpapers, a computing environment). Furthermore, is described the recognition and utilization of properties that belong to a class of figures (or a subclass) and description of the characteristics of shapes and their relationships.
- ....an example of a theorem's LVAR modeling process (e.g an LVAR modeling for the Pythagorean theorem).

Students' uploading of assignments was facilitated through the free open-source Learning Management System Moodle (Modular Object-Oriented Dynamic Learning Environment) (Dougiamas, PhD thesis, described at http://www.moodle.org.nz/).

i. Presentation and analysis of problems

For the design of activities I always had in mind: "What would the individual have to know in order to be capable of doing this task without undertaking any learning, but given only some instructions?" (Battista, 2011, p. 515).

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Case A: The problem was presented modeled in the dynamic environment. In the modeled dynamic representation, emphasis was given to the features associated with mathematics (e.g., the modeling of a kite can be done by constructing a rhomboid that emphasizes the verticality of the diagonals, etc.), rather than to other characteristics (e.g., the material, color, etc.). The students were able to experiment with the software tools on the digital image and to visualize the properties of the shapes that they were not able to perceive in the static environment.

Case B: The problem was not presented modeled in the dynamic environment, but the students were prompted to manage the image as if it was perceived in the natural environment. The students had to construct a simulation of the problem in a static, digital, or other physical means as a model of the natural environment. They also had to manage the (digital or not) image to gain intuition about the properties of the shape.

According to Johnson-Laird (1983) the human beings understand the world through the representations of the world they create in their minds. Johnson-Laird (1983) argues

> *"to understand a physical system or a natural phenomenon one needs to have a mental model of this system that will allow […] the person who will build it to explain it and to predict about it" (p. 430).*

In essence, the image conversion of the natural environment in the dynamic environment is a result of a complex process on the student's part. The student has first to transform the verbal or written formulation ("construct a parallelogram" for example) into a mental image, which is to say an internal representation recalling a prototype image (e.g., Hershkovitz, 1990) that s/he has shaped from a textbook or other authority, before transforming it into an external representation, namely an on-screen construction. The student needs to explore the shape of the natural environment (e.g., properties of shapes such as its symmetry lines, etc.) and then construct the scale model. The digital image plays a supporting role in understanding the properties of shape but also can bring to the surface students' cognitive obstacles and, consequently, lead to errors. These errors are mainly due to their vH level. As a result, students may not have the capacity to recognize the figure's properties, and, generally, to develop the solution with deductive reasoning. Especially for the fractal activities, the experimental teaching was carried out on 18 students at different school levels, including activities (on different software pages with linking representations) with increasing degree of difficulty depending on the age-related level of students. No student that participated had previously processed the software, or any other related software. As it was verified henceforth at many points of process the students were led to conclusions and formulations of definitions that had not been made known during their course of mathematics.

ii. Student's mathematical knowledge

In secondary high school, the students are taught the kinds of quadrilaterals, which they are asked to memorize. Most of students are able to recall only the basic relations regarding perpendicularity and parallelism of the sides of quadrilaterals. Furthermore, students construct parallelograms in static means using their traditional tools (compass or ruler), which only fulfill the visual criteria. In Greece, dynamic geometry is rarely used in high schools to facilitate the teaching and learning of geometry. As it is concluded the teaching of reflective symmetry (or symmetry by axis) and symmetry by centre in a DGS environment is not correlated with the notion of symmetry and particularly the students do not examine the notion of symmetry in relation to quadrilaterals. Furthermore, the students' difficulties in constructing a figure are due to their ignorance of the different thought processes involved in dynamic rather than static means. The knowledge of a figure's symmetry is essential for students. I distinguished a few types of obstacles due to student lack of competence in instrumental decoding (:i.e. this is to say an instrumental obstacle). In the current study, I have devoted enough time for the students to understand the meanings (for example, the notion of symmetry by axis and symmetry by center) through the dynamic geometry software. The kinds of transformations on which the activities are focused are reflection--which corresponds to symmetry by axis in static means, rotation--which corresponds to symmetry by center, and translation. The dynamic geometry system helped students to instrumentally decode the properties of figures, as we will see in the description of the activities.

IV. Description of Activities

a) Situation First (Visual-Holistic Reasoning -- Visual-informal componential reasoning): Recognizing quadrilaterals and symmetry in real world.

The aim of the activity was the recognition of quadrilaterals and the investigation of the symmetry lines of quadrilaterals in a real-world context. Our actions included three phases: a tour in the museum, the teaching in the class (including training in my e-class: the operation of the e-class to facilitate posting and downloading of material), and finally, the realization of the activity for the students within a predetermined time. Briefly, the students had to construct a shape using the figures' properties, in terms of its sides and angles. The description of the activity consisted of the following parts:

1. Recognize the kinds of shapes that you observe in the decorative pattern of the image below (see Figure 5).

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2. What kinds of symmetry do you recognize?
3. Construct the axes of symmetry and the center of symmetry in every image.
4. Construct the same image using your ruler and compass.
5. Construct the same motif using a dynamic geometry software.

Figure 5 : Traditional Greek embroidery

The problem is important for the development of students' ability to translate among different representational systems. An essential combination of visual skills, representational competency, and mathematical reasoning is required to solve this problem. A further aim of the activity was the creating of a diagram representing a real-world object in which the students can connect the abstraction, the art, and the timelessness of beauty.

i. Phase One: Visit to the Museum of Greek Folk Art

Guides [students of History and Archaeology at the University of Athens] took the students to different areas of the museum exhibits. The exhibits included local folk-costumes of various parts of Greece (e.g. mountainous Epirus, Thessaly, and the Aegean). These acquired meaning through the detailed presentation of the guides, who aimed to underline the particular characteristics of the local folk-pieces. The students were impressed by the embroidered women's costumes. Some had geometrical recurring motifs and expressed the inner desire of every woman [every bride] to have good fortune, happiness, and longevity.

Then, the students subsequently had to capture a part of the entire plan on paper.

> *"On the ground floor of the museum, visitors will see elegant examples of traditional embroidery from the whole of Greece. They include polychrome and white embroideries-laces and gold embroideries intended to meet the needs of dress, house and church. Particular interest attaches to pleated embroidered chemises of Crete, the relics of a female dress type with Renaissance roots that is found in other islands in the Archipelago during the period of Frankish rule (17th – 18th c.).Their hems are embroidered with alternating representations of gorgons, double-headed eagles, flower-vases, fantastic birds, and etc.*

(Excerpt from the text written in the description of the Folk Art museum website available at http://www.melt.gr).

I asked them questions such as: "What shapes can you 'see'?" "What kind of symmetry do you recognize in the decorative pattern?" The dominant feature of the costumes' geometric motifs [converted into images for the students' work] was the symmetry of its parts. As we know, the relations between depicted objects in a picture or additional information concerning the objects (e.g. colors or other symbols that convey a certain message) and their style allow us to place it in context. However, in a picture, the data could hinder students' ability to 'see'(/meaning perceive) the geometrical shapes/figures. For example in Figure 5, the symmetries in the pattern are apparent (central symmetry or axial symmetry). Additionally, it is also clear [to teachers] the symmetries of the shapes that form the overall motif. However, this is not true for students.

From the work of students resulted in the following conclusions: Regarding the functionality of the e-class, there was no particular difficulty with the operation of asynchronous learning by students. As to the concepts found that: students were not aware of the concepts of central and axial symmetry, did not understand the differences between quadrilaterals and for this reason they didn't 'see' the usefulness of such an activity, as some even use rice paper to replicate the project. In other words, it was found that students were not 'seeing' mathematics to the real environment and faced more difficulty in manufacturing patterns [see Figure 6]. Thus, I utilized this alternative way of teaching when I understood that students faced problems in understanding the concepts.

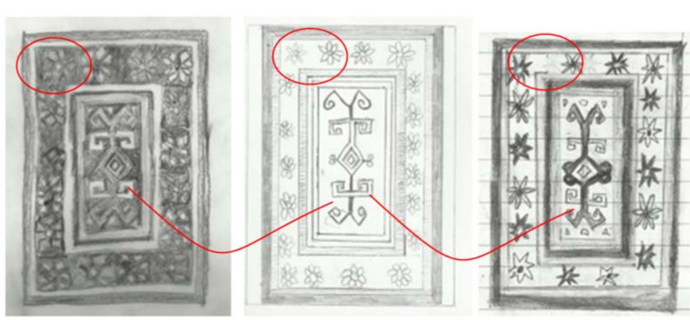

Figure 6 : Indicative of students' [pre-]constructions in the paper-pencil environment.

ii. Phase Two: Classroom [dynamic] organization

Firstly, the students recognized the parallel lines and parallelograms in the Figure 5. The students constructed the parallelograms using the "copy-paste" tools of the software or joined four line segments so they produced rectangular figures. Students make mechanical use of the software, which makes it impossible for them to understand the logic underlying the command options. It was my intention to familiarize the students with the software, "'step by step', in parallel with the corresponding theory" (Mariotti, 2000, p. 41).

In order to construct a parallel line using the software, one has to select two objects: a straight object (for example a line) and the point from which the line parallel to the initial line will be drawn. Most students at van Hiele level 1 were unable to understand the sequential apprehension of the tools selection, because they were unable to understand the logic of the sequence of actions or unable to link this logic with the theory of geometry.

Figure 7 : Snapshots of the copy-paste input process in the DGS environment

For example a student (van Hiele level 1 at the pre-test) faced an instrumental obstacle which depended on the sequential apprehension of the objects to be used for the construction. The student tried to construct a parallel line by selecting the line alone and then the menu command, which is to say the student followed an irrational sequence of actions. At this point, s/he faced an instrumental obstacle and commended in an informal way on the non-activation of the software's command. Subsequently, student's interaction with the software, led to a cognitive conflict which helped him/her to apprehend the sequence of actions. Therefore, is the construction that leads students to "shape" inadequate or alternative definitions regarding parallelograms. The definitions followed the introduction of the parallelism and dragging tools of the software.

Then, the students participated in the process of introducing the concepts of symmetry through dynamic geometry software. In the lesson that followed, transformations were introduced to students using the GSP software tool. For example, we focused on the transformations we have to apply to a triangle to construct a parallelogram. In the images below, a mode-A LVAR construction in the software presented the translation of a triangle, the use of a coordinate plane, the rotation of the triangles (and the rotation angle that remained stable at every point in the triangles), the reflection of the triangles (including the visualization of the similarities and differences on the coordinates of the transformed images).

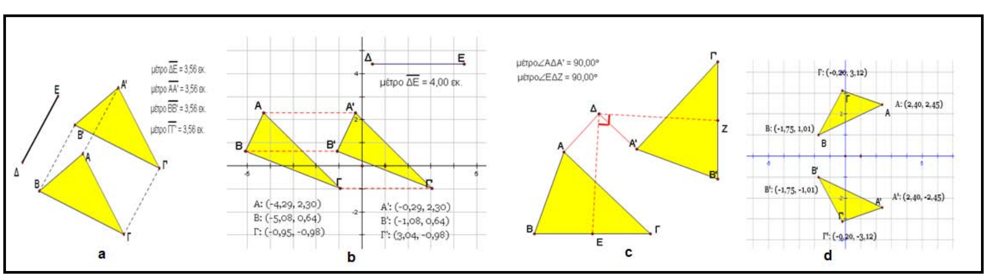

Figure 8 : Transformations and use of coordinate planes in the DGS environment (Patsiomitou, 2010b, in Greek)

The process of implementing the GSP software was simple, and students participated with great interest, answering my questions. Furthermore, the students demonstrated the capability to recognize the constructed parallelograms (e.g AA′B′B, AA′Γ′Γ in the figures 8 a, b). They globally recognized the axes of symmetry of the image and the (sub) axes of symmetry in the subfigures (Figure 8).

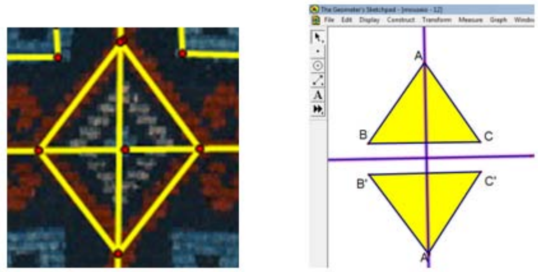

Figure 9 : Simulation of the object in the DGS environment

Then the students' task was to construct a rhombus based on the figure's symmetry (Figure 9). They dynamically reinvented that a single diagonal can divide the rhombus into two congruent isosceles triangles. Therefore, two congruent isosceles triangles can be together to form the shape of a rhombus. In Figure 10, the importance of the software's tools [e.g., the reflection tool] for the students' modification, or change their way of thinking, is represented. The most important indicator was that students tried to construct the symmetry by center of an arbitrary point on screen by using the reflection tool.

The utilization of the reflection tool during the previous phase led students, through instrumental genesis, to construct a utilization scheme for the tool. In this case, the students used the reflection tool by economy (Rabardel, 1995), despite having the option to use the rotation tool, in order to avoid the efforts required to use a less familiar one (Docq & Daele, 2001, p.200).

This action led to an instrumental obstacle as the result of the students' cognitive conflict with regard to the meanings of symmetry by center / axial symmetry.

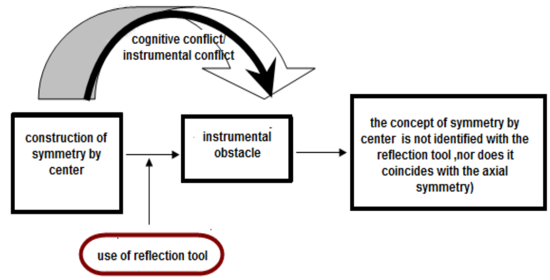

Figure 10 : A Pseudo-Toulmin model for the representation of the instrumental conflict

iii. Phase Three: Students' meta[-constructions]

The image below illustrates three indicative representations of students after the interaction with the DGS representations. The students dynamically reinvented meanings and the difference between terms [i.e. the meaning "axis of symmetry as perpendicular bisector of the segment which joins the original point and its reflected point", the difference between axial symmetry and symmetry center].

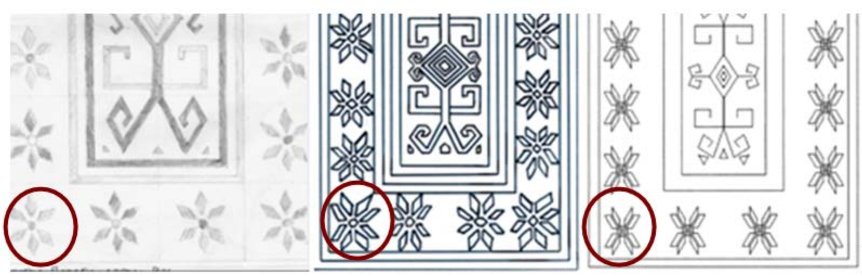

Figure 11 : Indicative of students' [meta-] representations in the paper–pencil environment

Students visualized the figure's symmetry, identified the symmetry axes of the quadrilaterals, but their representations [still] indicate that they recognized the shapes holistically. For instance, they globally recognized the sub figures of the flowers in the pattern, which were different for every student.

b) Situation Second (Visual-informal componential reasoning): Quadrilaterals' transformations

i. Phase One: Visit to the museum of Ancient Agora-Stoa of Attalos

Our actions included among others the visit to the museum of Ancient Agora. "The museum of the Athenian Agora is housed in the reconstructed Stoa of Attalos, […] Ever since, all finds from the area, together with the excavation archives, have been housed in the building, making the Stoa a significant research center for scholars and students of archaeology from all over the world"

(Excerpt from the text written in the description of the Athenian Agora website available at http://www.ascsa.edu.gr/pdf/uploads/CATALOGUE_6_2012_C_Layout_1_FINAL.pdf)

The archaeologists also discovered amazing mosaics in an ancient room. The next activity included students' construction of the mosaic, using their rulers and compasses, and the construction of the same motif using dynamic geometry software. In the representation of the mosaic the students tried to discover the angle of rotation of the parallelograms but they [still] confused the shape of parallelograms with the shape of rhombus because of their orientation. They tried to construct the successive parallelograms, but they failed to find the angle of rotation.

ii. Phase Second: The utilization of trace tool - reflection tool

The trace tool in correlation with the reflection tool proved essential for students' understanding of concepts; it helped the students to develop argumentation with regard to the equal distances of the points (original-, reflected point-image), and the identification of the axis of symmetry as a perpendicular bisector. The construction of the figure was completed using the reflection tool of the software. Furthermore, the students discovered the axes of symmetry of the shape. They also considered what the center of symmetry is and in what angle the parallelogram can be reproduced. Moreover, students identified that a perpendicular constructed from the symmetry centre crosses the figure. Then they (1) recognized that the interior figure was a rhombus (2) if a line is perpendicular to one of the two parallel lines, it is perpendicular to the other (3) a perpendicular constructed from the symmetry centre of a rectangle to a side of the rectangle, crosses the midpoints of this side and its opposite side (4) the lines through the midpoints of two opposite sides of a rectangle dissects the rectangle into four rectangles that are congruent to each other.

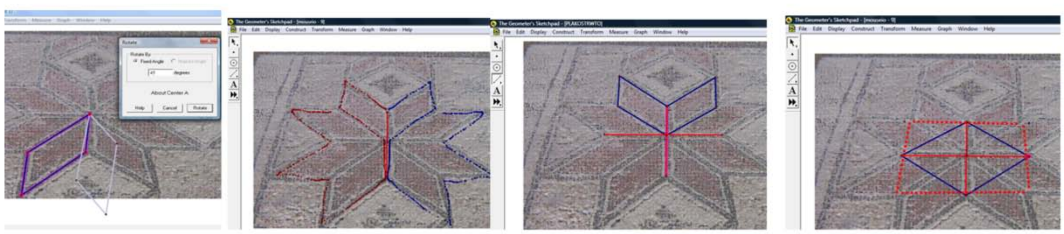

Figure 12 : Snapshots of the process in the DGS environment of a mosaic in the Ancient Agora of Attalos

The pseudo –Toulmin model below represents the process and the tools that led the students to construct the meanings.

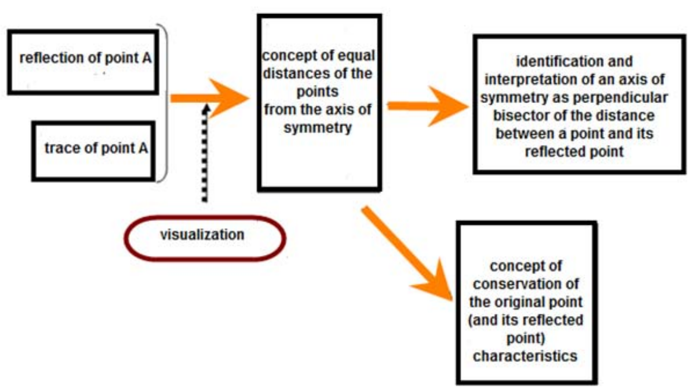

Figure 13 : A Pseudo-Toulmin model for the representation of the construction of the meaning

Through visualization of the object, students are initially led to use empirical methods (such as trace of points) in order to confirm that symmetrical objects lie at an equal distance from the axis of symmetry. Consequently, the students discovered a few properties of the figure, meaning the symbol character of the figure. Therefore, we had a theoretical construct which derived through instrumental decoding of the dynamic diagram.

c) Situation Third (Informal and insufficient-formal componential reasoning): Quadrilateral's properties in real context.

i. Phase One: Visit to our school courtyard

The modeling of the courtyard area of the school through static or dynamic means on the part of the students is an example of a real-world context modeling problem. The organization of the activity consisted of three phases: (1) the processing of the courtyard pattern of the school during instruction, (2) measuring the tiles forming the courtyard's shape, and (3) modeling by students during fixed intervals. This problem is important for developing the capacity of students to convert between different representational systems.

Figure 14 : The courtyard area of the school

The conceptual frame mentioned was the areas of surfaces, the surface measurement units, conversion between different surface measurement units, and the areas of shapes. Students were required to consider the type of triangle formed by the diagonal of the square and then to justify the measurement of the formed angle. The measurement of the surface could occur in many ways (e.g., measuring the tiles forming the shape) in which the students had to observe the shape that each of them had and determine the area. The students used their geometric instruments (e.g., ruler, compass) to measure the dimensions of the tiles. The aim was to construct a representation of the pattern to scale.

Skilful combination of visuospatial ability and representational capacity is required, as well as the capacity for mathematical thinking. The questions that I posed to students (e.g., "Is the inner quadrilateral a square or a rhombus?") focused on the recognition of the kind of quadrilaterals representing the exterior and the interior shapes on the floor and to calculate the area of the surface covered by the red and the white tiles.

Students' responses led to extensive dialogue/debate among them and gave me feedback. This example illustrates Cobb, Yackel & Wood (1992) claim that

> *"students will inevitably construct the correct internal representation from the materials presented implies that their learning is triggered by the mathematical relationships they are to construct before they have constructed them. (Cobb, 1987; Gravemeijer, 1991; von Glasersfeld, 1978). How then, if students can only make sense of their worlds in terms of their internal representations, is it possible for them to recognize mathematical relationships that are developmentally more advanced than their internal representations? (p. 5).*

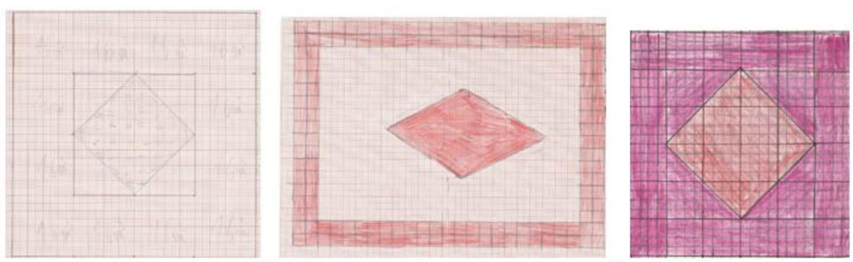

Figure 15 : Students' indicative constructions before the interaction with the GSP activities

The images (Figure 15) are indicative of students' representations in the paper –pencil environment. The students faced many difficulties connected with their conceptual obstacles in regards to the meanings of quadrilaterals. For example, many students perceived a rhombus rather than a square in the internal figure, due to the figure's orientation. Other students had constructed, in the middle, a rhombus whose edges coincided with the midpoints of the external figure.

ii. Phase Two: [Students'] 'dynamic' actions

The figure 16 represents a draft of the instructional design of the activity, using the software's tools. In the lesson that followed the in-class simulation, transformations of a lattice/grid were introduced to students using the GSP tool. While investigating the problem, the students used the rotation tool to rotate a segment by 90 degrees. Through this, they constructed the meaning of the right isosceles triangle. Moreover, the lines through the midpoints of two opposite sides of a square dissect the square into four squares that are congruent to each other. These properties are elements of the object being built with the GSP tool.

The manipulation of the dynamic objects in the software led the students to construct the properties of the square, while the transformations of the dynamic objects led to acquire the symbol character.

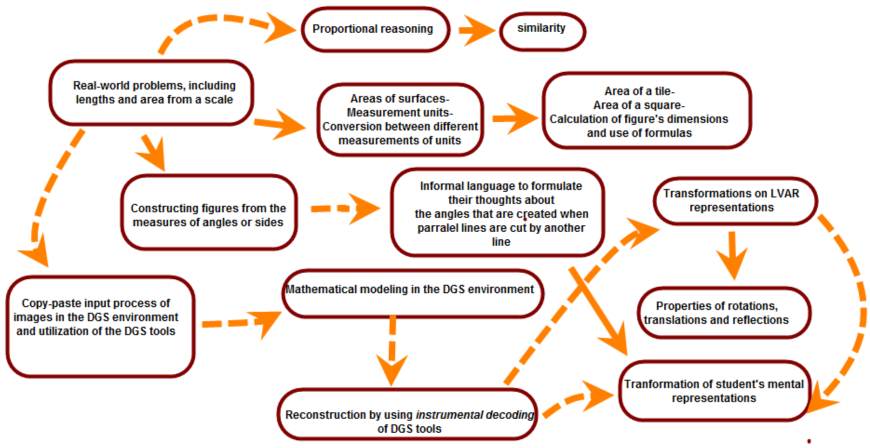

Figure 16 : A draft of the instructional design of the activity

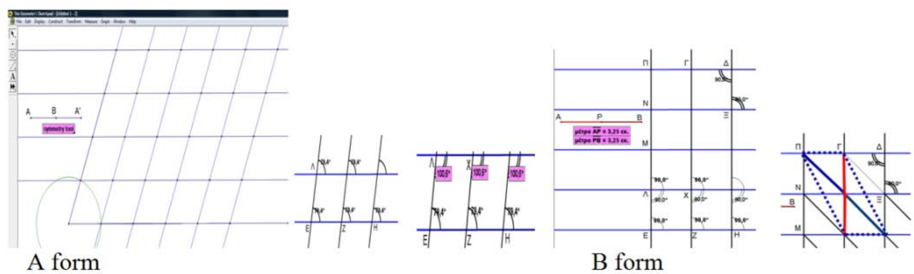

Figure 17 : Snapshots of the transformations of a lattice/grid in the DGS (Patsiomitou, 2010b, in Greek)

The modeling of the problem in the DGS environment of the lattice structure is of form A, while in the real environment, the grid is of B form. This renders essential the investigation of students' capabilities to imagine the right figure or to construct the analogous mental representation. If this obstacle is overcome, then the students are able to move on to the next process.

To facilitate the students, I created a custom tool 'symmetry' (see for example Patsiomitou, 2012b, p. 68). The custom tool 'symmetry' could be used to construct the symmetry by center of an arbitrary point on screen. The grid's construction in the DGS environment can be created by using the transformation of translation of congruent segments horizontally and vertically or with the use of the "symmetry" custom tool (Patsiomitou, 2012b, p. 68). My aim was for students to formulate the relations and the conditions under which a figure is shaped as a square, and establish whether these conditions are still valid generally. The students had to examine the different cases of shapes arising from the use of dragging. The experimental sequential dragging (and then the theoretical one) until the angles become 90 degrees leads to forming squares. Moreover, the diagonals' constructions shape isosceles and right triangles.

The students then used the custom tool 'symmetry' to reverse the process. It is important that the students were able to connect mentally the reversed representations and to follow their successive structure. In this way, the transformations evoked in the initial representation were reversed through mental operations following a concrete order. This is better explained in the next situation.

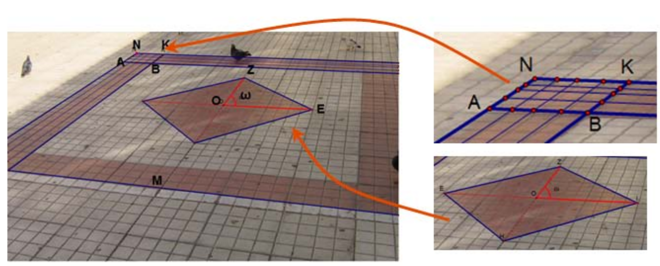

Figure 18 : Snapshots of the image's input in the Geometer's Sketchpad dynamic geometry system

iii. *Phase Three: Students' representations of the courtyard*

The students used graph paper and static or digital material of their choice (e.g., cardboard or dynamic geometry software) to represent the construction (Figure 19). The application was of particular interest, and students were able to calculate the figure's area in several ways: (1) using as a unit the area of a tile and then that of a square (whose side was made of four sides of a tile), (2) calculating the area of the shape of a tile and from this, the area of the whole shape, and (3) calculating the dimensions of the shapes (squares) and then using geometrical formulas for the total area of the figure.

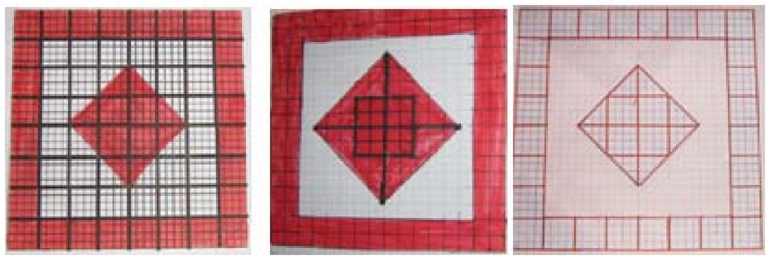

Figure 19 : Students' indicative constructions after the interaction with the GSP activities

This means that the students have developed thinking processes and applied skills, developing a mathematical model to interpret the realistic problem. They connected visual and formal criteria which is crucial for the transition from the lower levels, to the upper levels. Therefore, the information regarding the figure's properties were transformed to a signal.

d) *Situation Fourth (Sufficient formal property-based reasoning): Quadrilaterals' structural analysis.*

i. *Phase One: Museum of the Ancient Agora -Stoa of Attalos.*

"The Agora museum is housed in the Stoa of Attalos, a reconstructed building of around 150 B.C. The characteristic feature of the museum is that the exhibits are all closely connected with the Athenian Democracy, as the Agora was the focus of the city's public life". (Excerpt from the text written in the description of the Agora museum website available at http://odysseus.culture.gr/h/1/eh152.jsp?obj_id=3290).

Figure 20 : Snapshot of the Ancient Greek Geometric Pottery in the GSP dynamic geometry system.

The students' activity included the representation of the figures of Greek geometric pottery. First, the students recognized the shapes, and then they used their geometric instruments to construct the figure, but they faced difficulties to analyze the figure into subfigures.

ii. *Phase Two: The visualization of parallelogram's diagonals in real world images*

Simulations of a scissor lift or Centre Pompidou's designs (Figure 21) in the GSP have been introduced to students, in order to focus on and interrelate the meaning of a parallelogram with the bisection of its diagonals. This means that the parallelogram's symbol character was completed with its primary properties. In the activity aforementioned, the students recognized the parallelogram on the screen from the structure of its bisected diagonals.

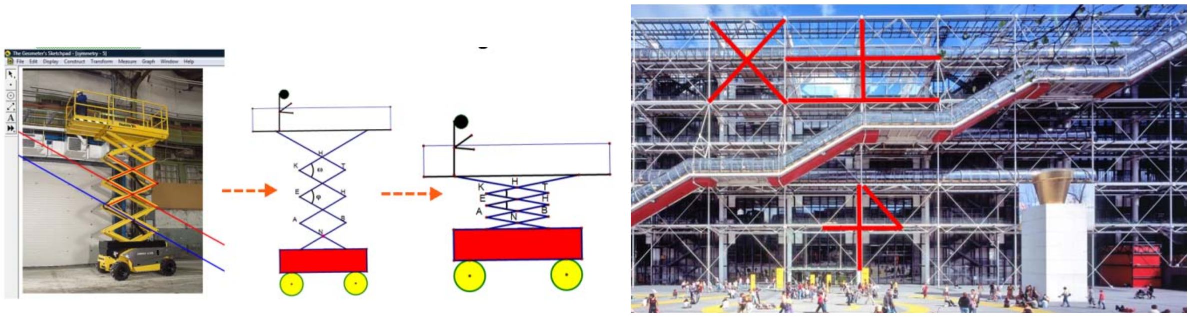

Figure 21 : The diagonals' visualization in real world images

iii. *Phase Three: The instrumental decoding of the reverse process.*

The instrumental decoding of the reverse process (i.e., the construction of a square) was more difficult for the students. The next step was the analysis of the relationships between parts of pottery's figures in the DGS environment. For example, the quadrilateral constructed from the connection of the points that intersect the diameters on the circle is a square (Figure 22). The students were not able to justify why the shapes were squares. They also changed the orientation of the diagonals in the DGS environment, applying the experimental dragging tool.

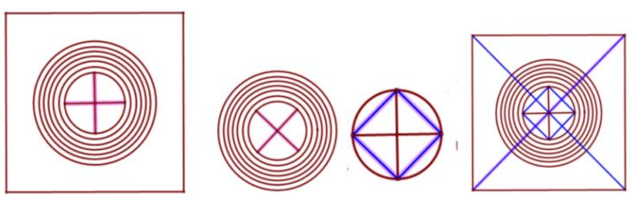

Figure 22 : Snapshots of the diagonals' transformations in the DGS environment

My questions helped them to identify the figure's properties and to analyze the figure into subfigures. They had to reverse the process, meaning they had to replace the figure with its properties. In other words, they had to construct the square's signal character. The sequence of questions led students to think of figure similarity. (For example, "Are squares similar figures?", "Are rectangles similar figures? Explain your answer"). Moreover, the students had to connect the meaning of the symmetry by center with the meaning of the segment's midpoint.

The students constructed the figure by taking into account the structure of its diagonals. They constructed two perpendicular lines intersecting at O, constructed a circle with center O and connected the four points where the circle cut the lines. It is crucial for the students to recall the properties of the figure's diagonals that were investigated in the previous phases of the research process by mentally linking the reverse representations in this procedure.

The students also used the custom tool 'symmetry' to reverse the process. The utilization of the custom tool 'symmetry' twice with the second application point at the symmetry center O, will lead to the construction of two segments that have the same midpoint. Consequently, the meaning of "diagonals are dichotomized" can be constructed by the students through the use of the custom tool. Dragging the construction from a point-vertex, the properties remain stable, meaning point O remain the midpoint of both the segments. The students are able to recognize that: "if the diagonals of a quadrilateral have the same midpoint then the quadrilateral is a parallelogram or if the diagonals of a quadrilateral bisect each other then quadrilateral is a parallelogram". Subsequently, the students are able -by using the custom tool "symmetry" to transform an iconic representation into a verbal one through mental transformations.

"This is a very complex process since the students must have both conceptual and procedural competence, meaning the competence to instrumentally decode their mental representations of a set of properties with actions through the use of tools. This means, for example, to interpret the congruency with the circle tool and simultaneously bisect with the custom tool. Furthermore, for them to construct the hierarchical categorization and definition of figures through their symmetrical properties and in accordance to their understanding." (Patsiomitou, 2012b, p. 71).

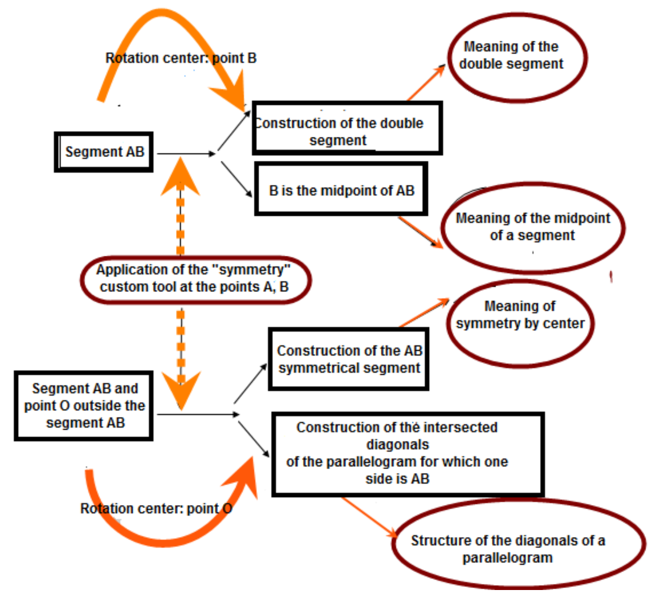

Figure 23 : The utilization of the custom tool 'symmetry'

The process is described in the pseudo-Toulmin's model above (Figure 24). The diagram expresses the way in which students in cooperation constructed the square, using the Sketchpad tools. Through the construction, they extended the structure of the intersected diagonals, including the meaning of the perpendicularity and the congruency: "[a square's diagonals] are perpendicular and congruent segments intersected in a [common] midpoint."

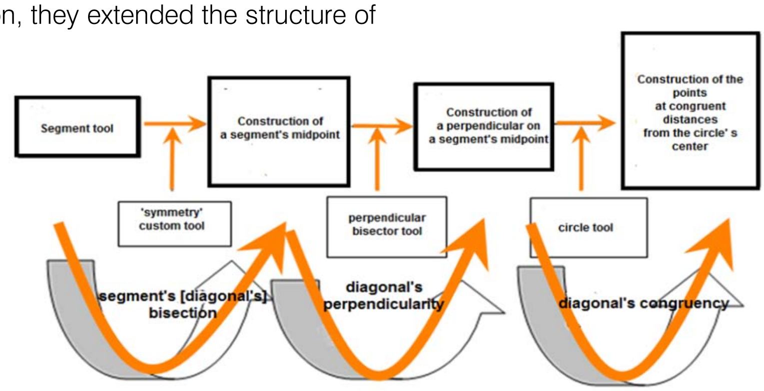

Figure 24 : The instrumental decoding of the reverse process

In the Figure 25 (Patsiomitou, 2012b, p.72) we are able to observe the linking representations of the diagonals of different types of parallelograms. Dragging theoretically the endpoint of the diagonals of the parallelogram in order these to acquire the property of the perpendicularity leads to the structure of the rhombus diagonals (or a square's diagonals). Dragging theoretically the endpoint of the diagonals of the parallelogram in order these to acquire the property of the congruency leads to the structure of the rectangle's diagonals. The construction of two arbitrary diameters in a circle (i.e. the diagonals are not perpendicular to one another) leads to the structure of the diagonals of a rectangle. The construction of two diameters perpendicular to one another in a circle leads to the structure of the square's diagonals. In this way conceptually and procedurally linking representations are created.[…]. Subsequently, this learning path can lead to the development of an abstract way of thought through the development of linking representations in student's mind. .

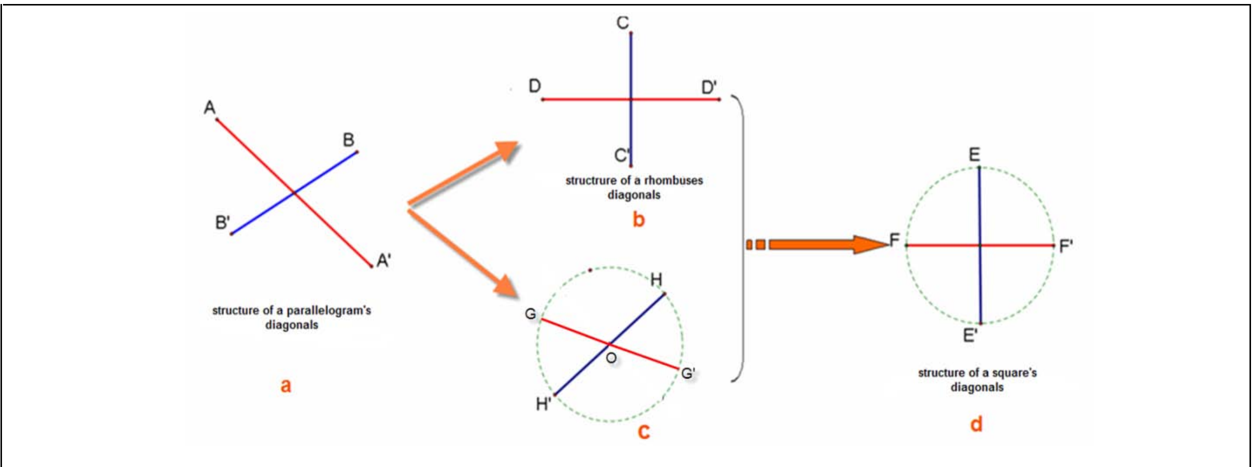

Figure 25 : Linking Visual Active Representations of parallelogram's diagonals (Patsiomitou, 2012b, p. 72)

e) *Situation Fifth (Relational –Inferential Property- Based Reasoning- Empirical relations). A structural process*

i. *Phase One: Building structures*

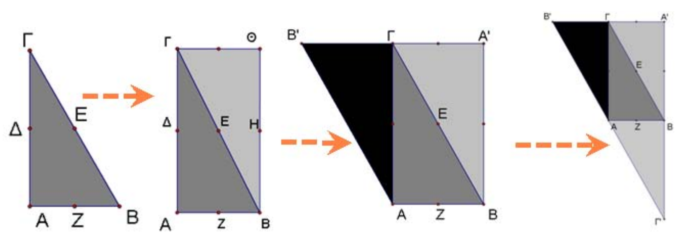

Figure 26 : A structuring process (Patsiomitou, 2010b, in Greek)

The next activity was the transformation of a geometrical figure that was used as a building unit for the construction and a means by which the students could construct the meanings of theorems inductively and experimentally, which were included in their class curriculum. For example, in the figure 26 the rotation of a right triangle leads to the construction of a rectangle, then to the construction of a trapezoid, and, finally, to the construction of a right triangle whose sides are double from the sides of the initial triangle.

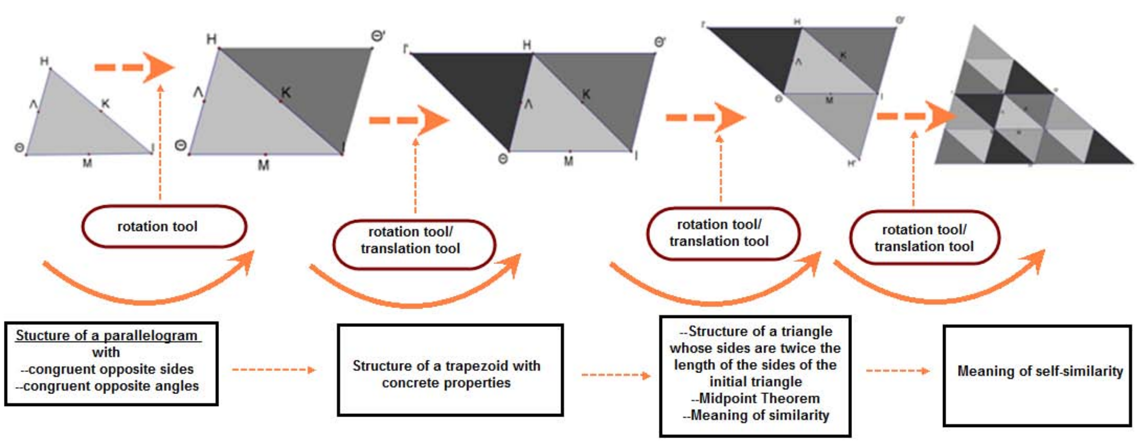

Figure 27 : The structuring process of the meaning of self-similarity (Patsiomitou, 2010b, in Greek)

In the figure 27 the rotation of a triangle leads to the construction of a parallelogram, then to the construction of a trapezoid, then to the construction of a triangle whose sides are double from the sides of the initial triangle, etc. Most important is the development of students' correlation of the properties (for example, "How is the meaning of a right and isosceles triangle linked with the meaning of a rectangle, and how is this consequently linked to the meaning of a square?", "What are the similarities and differences of the properties of a square and a rhombus, etc., as a result of the different structuring of its figure?", "How do the similarity of the building block's figures affect the similarity of the sequential figures?").

The students organized their thoughts for the sequential steps of the construction (for example, "What should be the property that must have a right triangle to be the building unit for the construction of an equilateral triangle?" or "What are the properties the sequential figures have?" The study of the building block's properties helped the students to organize the properties of the figure evoked from the initial figure. This process is in accordance with what Freudenthal (1973, 1983) has told that the teaching and didactic process must focus at the understanding of the structuring process and not the learning of ready-made structures. Moreover he argued that students could discover mathematics when they work with contexts and confront interactive and reflective activities.

ii. Phase Two: The Pythagorean Theorem through LVAR representations

In their calculations, the students had to use the Pythagorean Theorem. For this, the next activity was aimed to increase understanding of the application of the theorem in the class.

The teaching process consisted of three items:

- First, the visual proof of the Pythagorean Theorem with the utilization of linking visual active representations that I created using the Geometer's Sketchpad.
- Second, the meaning of the Pythagorean Theorem, and generalizations of the concept.
- Third, the extension of the Pythagorean Theorem to fractal structures (e.g., the construction of Pythagorean trees), such as successive calculations, the areas of squares, etc.

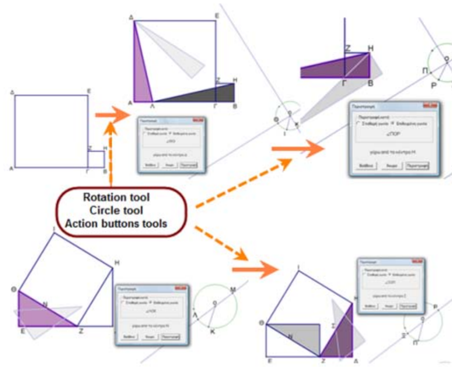

Figure 28 : Visual Pythagorean Theorem's proofs (Patsiomitou, 2010b, in Greek)

Students participated in the process of introducing and exploring the Pythagorean Theorem. The instrumental orchestration process (e.g., Trouche, 2004) included a laptop computer and an interactive board. Initially, the students were guided to explore the Pythagorean Theorem visually. Then, I asked the students to construct the shape using their paper and pencil environment as an assignment in class to determine how they perceived the objects.

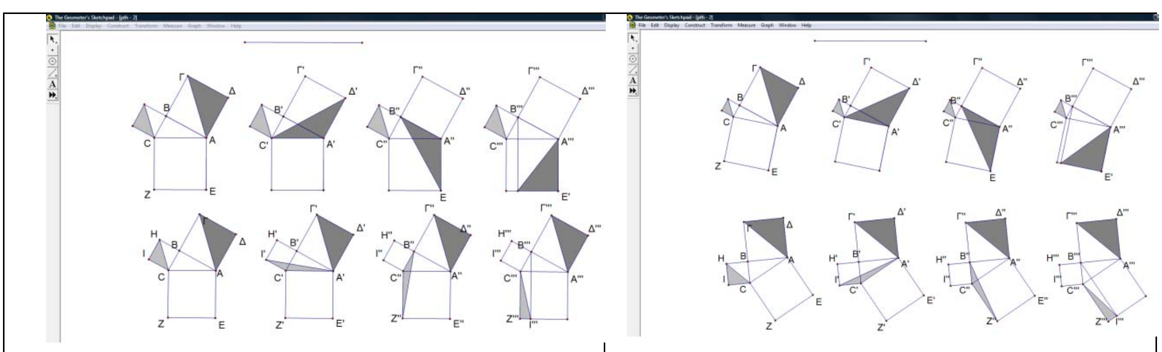

Figure 29 : LVAR translations of Euclid's proof (Patsiomitou, 2010b, in Greek)

Then the students conducted a visual proof of the Pythagorean Theorem with LVAR representations (e.g., Patsiomitou, 2010a, b) that I was created using the Geometer Sketchpad software. It captured the interest of the students. The combination of the interaction techniques of the software (e.g., navigation links, buttons to hide/show objects) for the production of visual mathematical representations (VMR) (Visual Mathematical Representations) (Sedig & Sumner, 2006) can lead students to develop conjectures, analyze the problem, and synthesize the solution.

In the images of figure 29, we can see the linking representations of the Euclidean proof of the Pythagorean Theorem. The successive phases of the constructional steps have been achieved using transformational processes like the use of the translation command (Figures 29). By dragging a point of the original configuration or the translated images, the students can observe the processes that emerged previously being modified simultaneously. Students are able to directly assume or infer the properties and the interrelationships between figures from properties indicated on the diagram by conventional marks (for example the equality of angles, or the angles measurements). In the first row, four linking [translated] representations led the students to understand that the half square is transformed to the half rectangle. The same is visually demonstrated in the second row for the other square. The important point from the LVAR constructions is that the students can transform the shapes simultaneously and see the same theorem from a different orientation. Additionally, an important point, segment, or shape is highlighted as the students develop their explanation orally.

I explained to the students that this method provides a visual confirmation of the Pythagorean Theorem and pointed out the need for proofs. The challenge is the interaction of students with LVARs to help them develop their level of geometric thinking. A pupil can develop his/her level of knowledge by proceeding through increasingly complex, sophisticated and integrated figures and visualizations to a more complex linked representation of problem, and thereby moving instantaneously between two successive Linking Visual Active Representations only by means of mental consideration, without returning to previous representations to reorganize his/her thoughts (e.g., Patsiomitou, 2008a, 2010a; Patsiomitou & Koleza, 2008). A student voluntarily presented the other students with the dynamic objects and the transformation of the shapes, which was a part of the process. If someone failed to provide the correct answer, the other students tried to help, expressing their point of view.

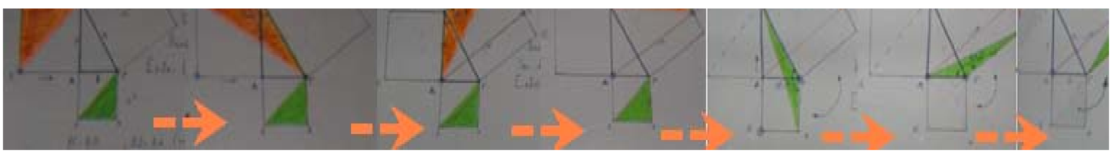

Figure 30 : Student's construction of the Euclid's proof linking representations (see [2, 4])

The action involved the modeling of the Pythagorean Theorem using a cardboard. The students' understanding emerged because of their interaction with the LVAR dynamic representations within the community of practice in the mathematics classroom.

iii. Phase Three: Transformations and calculations of geometrical objects

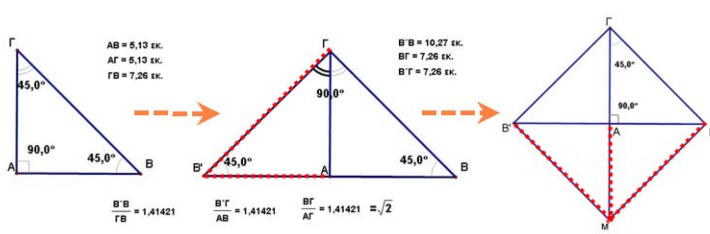

Figure 31 : Transformations and calculations of a right and isosceles triangle

LVAR played a significant role in developing students' deductive reasoning is clear from the fact that students demonstrated a shift from visual to formal proof. For example, in the figure 31 the rotation of a right and isosceles triangle leads to the construction of another right and isosceles triangle with concrete properties. Recognition of structures leads students to schematize and manipulate mental objects. Therefore, construction through sequential transformations can lead to the organization of the figure's properties. This process is more than the synthesis and extension of properties.

iv. Phase Four: Selfsimilarity and calculations

The image (Figure 32) represents a work carved in stone from the famous workshop in Tinos Island. "Works carved in stone by popular craftsmen are used

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either as independent ornamental structures (fountains, grave monuments) or as architectural features of practical and decorative purpose". (Excerpt from the text written in the description of the Folk Art museum website available at http://www.melt.gr)

Concretely, the image mentioned above is a snapshot of the image's input in the Geometer's Sketchpad dynamic geometry system. The repetition of the square's construction inside the figure leads to a fractal construction.

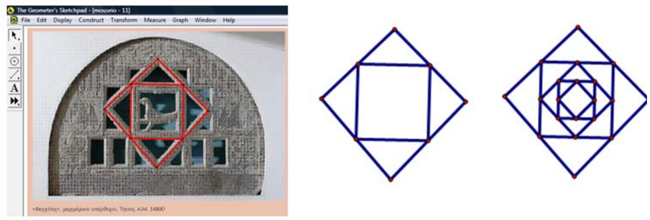

Figure 32 : Snapshot of the image's input in the Geometer's Sketchpad dynamic geometry system

The question was the following: "What kind of quadrilateral is shaped by joining the midpoints of the external quadrilateral?" For any quadrilateral, we can prove that the internal quadrilateral constructed by the midpoints of the sides of the external quadrilateral is a parallelogram. The students learn to prove this through a procedure of the application of the midpoint-connector theorem. In the image above, the interior figure is a square, as is the exterior figure.

If the exterior quadrilateral becomes a rectangle, then the interior--constructed by joining the midpoints of the initial--will become a rhombus, the next interior constructed will become a rectangle, etc.

The students can visualize a secondary property of the rectangle (for example that the axes of symmetry of the rectangle can be interpreted as diagonals of the rhombus, in other words can be interpreted differently and acquire a second role. Then the symbol of rectangle is transformed to the signal of rhombus. It is what many researchers have discussed (e.g., van Hiele, 1986; Patsiomitou & Emvalotis, 2010a, b). The Toulmin's model diagram below is a representation of the way students expressed their thoughts. They told that "If the figure is a rectangle, then its diagonals are congruent, so these segments --that join the midpoints of the opposite sides-- are parallel and half the length of the diagonals".

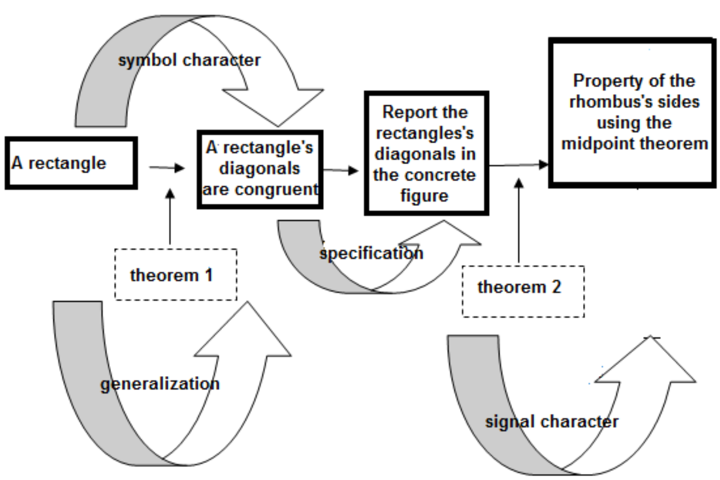

Figure 33 : A pseudo-Toulmin model for the representation of the way students expressed their thoughts

f) *Situation Sixth (Componential analysis). Modeling fractal objects using static and dynamic means*

i. *Phase One: Introduction to the world of fractals*

The 6th situation led students to think about self similarity, which is not included in high school curriculum. The objective of the situation seventh was to awaken students in mathematics that are not included in their class curriculum. Moreover, because of the scheduled curriculum is difficult for them to explore. Martinez (2003) writes that "Mandelbrot coined the word "fractal" (from the Latin word "fractus", meaning fractured, broken) to label objects, shapes or behaviors that have similar properties (self-similarity) at all levels of magnification or across all times, and which dimension, being greater than one but smaller than two, cannot be expressed as an integer" (reported in http://www.fractovia.org/art/people/mandelbrot.html). The plan was to incorporate and illustrate fractal geometry --or facilitate the understanding of topics from geometry-- in already existing curriculum (e.g., fractions, proportion and ratio, calculations of area and volume, logarithms and exponentials, sequences and series, convergence of geometric series, geometry of plane transformations etc.) Furthermore, the enrichment with fractals into existing curriculum helps students to develop their imagination and apply mathematics outside the classroom, in real-world activities in cases that other students couldn't see the relevance. For example, among students' kites a highlighted one existed, constructed with Baravelle spirals (e.g., Chopin, 1994; Patsiomitou, 2005) fractals, of a student 12 years-old who participated in Fractal group.

Figure 34 : Students' constructions of kites

Mathematical concepts related to the construction and investigations of a fractal are divided into geometric and algebraic segments, which cover almost all concepts included in the high school curriculum. For the fractal constructions the Geometer's Sketchpad dynamic geometry software has been used which is the best dynamic geometry program for facilitating fractal constructions because of the in-depth iteration process that helps students gain strong intuition for the meanings (Patsiomitou, 2005, 2007).

The students watched videos exploring the fascinating world of fractals. The videos were posted, in the Moodle environment. The language of the videos was English, which did not cause dissatisfaction or difficulty for the students. Moreover, it is well-known that the language of mathematics is common internationally, and in the videos, common notations for mathematical concepts were presented. The students could also cooperate to collaboratively answer questions and complete a text for the golden rectangle, gathering information from websites or creating their own constructions. Mandelbrot or Julia fractals fascinated the students because of the beauty of the objects they observed. Some of the students processed natural fractals (e.g., broccoli, cauliflower) to understand that a fractal structure does not change. The shape and the size of the object do not affect the structure and the self-similarity of the objects.

ii. Phase Two: Modeling fractal objects

The design of the activities and the experimental process that is reported here is an excerpt of my Master's thesis (Patsiomitou, 2005). This process has been repeated in the students' fractal group in the previous school year. For example: the construction of a "Pythagorean Theorem" custom tool, as well as the application of a "Pythagorean Theorem" custom tool recursively, led them to create Pythagorean fractal trees. Via the proposed activities we are able to investigate whether the construction of the fractals implementation or via the custom tools or of the process iteration can assist in investigating open–ended problems whose objective is the standardisation of intuitive ideas and the development of abstract processes. Moreover, we are able to investigate whether the students can be imported into the basic notions of infinitesimals and their use in calculus.

I conceived of LVAR representations when creating linked pages in Sketchpad files to construct fractals for my Masters thesis. Here is explained the rationale I followed in the design process. The most important parts of the design and research process are going to be mentioned here, enriched, to explain the importance of linking visual active representations, instrumental decoding, and RVR-–as LVARs have been illustrated later (e.g., Patsiomitou, 2012a, b). The modeling and construction of an in-depth fractal structure is difficult or impossible with familiar geometry instruments (ruler and compass). Although the students' construction started in the paper-pencil environment, they felt it necessary to continue their construction in dynamic geometry software. The construction of the Sierpinski triangle fractal was one of the favorite subjects for the students. Moreover, the discussion expanded on the concept of a golden rectangle and golden spiral, and other spirals, such as the Fibonacci sequence, concepts that enriched the mathematical world of the students. Below I describe the way in which the students constructed a Sierpinski triangle in the DGS environment through two different ways: that of a custom tool (script) and thereafter application of the tool in (n) steps or the application of functional process of iteration (Steketee, 2002, Jackiw & Sinclair, 2004) to the initial construction (or even the composition of the two modes).

Phase Two -Part 1: The construction of the Sierpinski triangle via custom tools

For the construction of the Sierpinski triangle, the students started with an isosceles triangle (or an equilateral) and the midpoint of its sides (Patsiomitou, 2005, 2007). Then they guided to build a custom tool in order to continue the process. The students had to grasp the process in order to construct a Sierpinski triangle in-depth.

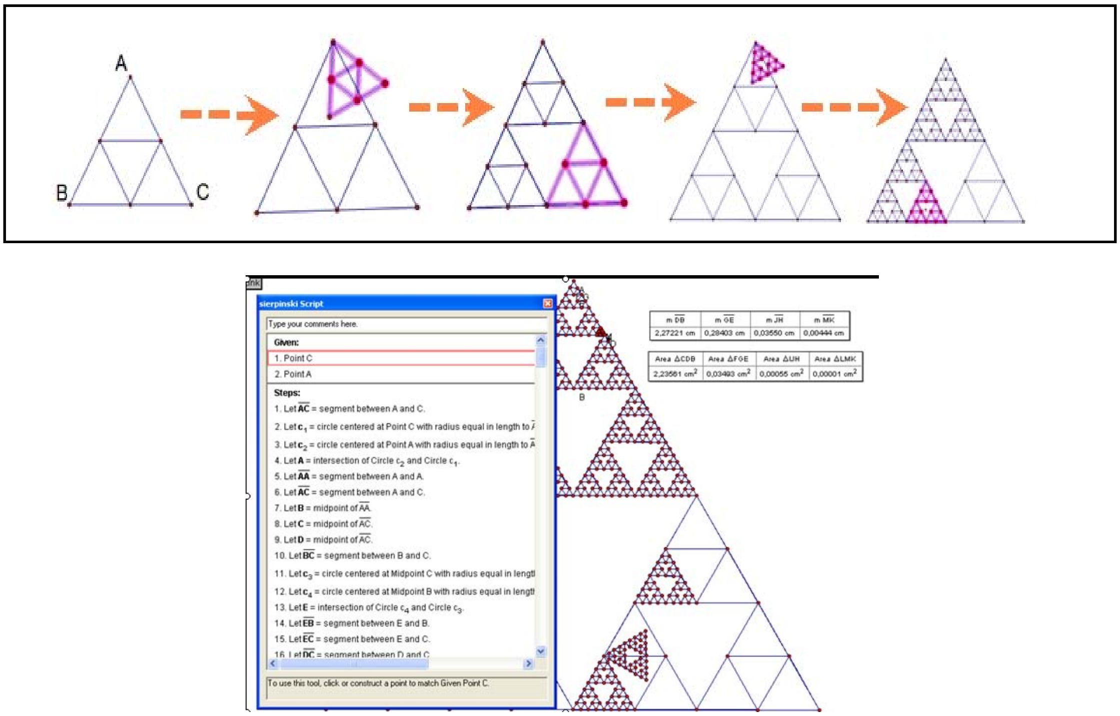

Figure 35 : Sierpinski's construction (Patsiomitou, 2005) via the utilization of sequential custom tools

The process is represented in the pseudo – Toulmin diagram below

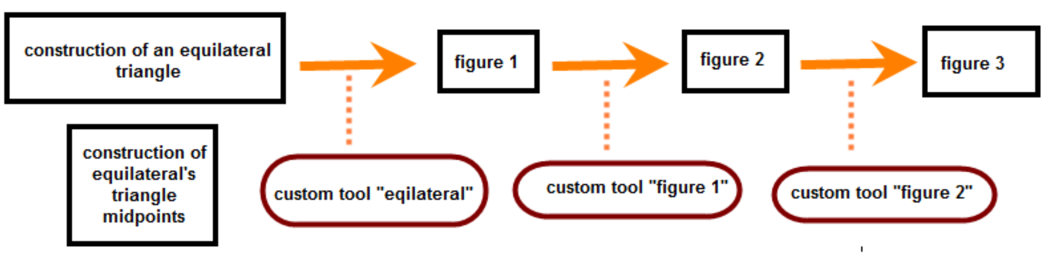

Figure 36 : A pseudo-Toulmin model for the representation of the process

From an instrumental genesis perspective, the students can construct an instrumented action scheme by using the custom tool, and then a higher order instrumented action scheme. Therefore, the custom tool 'equilateral' acts as a building unit in the genesis of the higher-order scheme, exactly as Drijvers & Trouche (2008) argue:

"The difference between elementary usage schemes and higher–order instrumented action schemes is not always obvious. Sometimes, it is merely a matter of the level of the user and the level of observation: what at first may seem an instrumented action scheme for a particular user, may later act as a building block in the genesis of a higher-order scheme. [...] a utilization scheme involves an interplay between acting and thinking, and that it integrates machine techniques and mental concepts [...] the conceptual part of utilization schemes, includes both mathematical objects and insight into the 'mathematics of the machine"(p. 372).

The sequential creation of custom tools led the students to grasp meanings; however, most of the students had difficulties in understanding the structure of the triangle as the process evolved.

Phase Two -Part 2: The iteration process

In order to approach the task we constructed an equilateral triangle and from the midpoints of its sides the next equilateral and so on. The problem that we discussed concerned the calculation of the sum of the areas of the successive equilateral triangles in the interior of the shape. The whole iteration process can be demonstrated using Geometer's Sketchpad software to make it understandable for children aged 13-16. If we build a custom tool ("Area's sum," for example) that finds the sum of the successive triangles, divide the sum with the area of the initial triangle, and repeat continually, we will get a result (e.g. 1, 25). The structural repetition of the triangles in-depth, as well as of the calculations, will not change the results. The next figures (Figure 37) demonstrate the linking of the visual active representations of the calculations, which generalizes the process. The final result is equal to 1, 3333…, meaning that the limit of the sequence of the infinite sum of the areas approaches the 1, 3333…number as is strictly proven. The resulting sequence is formed by the sum of the areas of triangles (>8 triangles) made in each iteration. This means that we finally have a sequence of terms equal to 1.33333. In this way, the students understand that the size of the triangles does not affect the ratio of the sum of the area, which is approximately (~ 1.33) and remains stable, even if we continue the process.

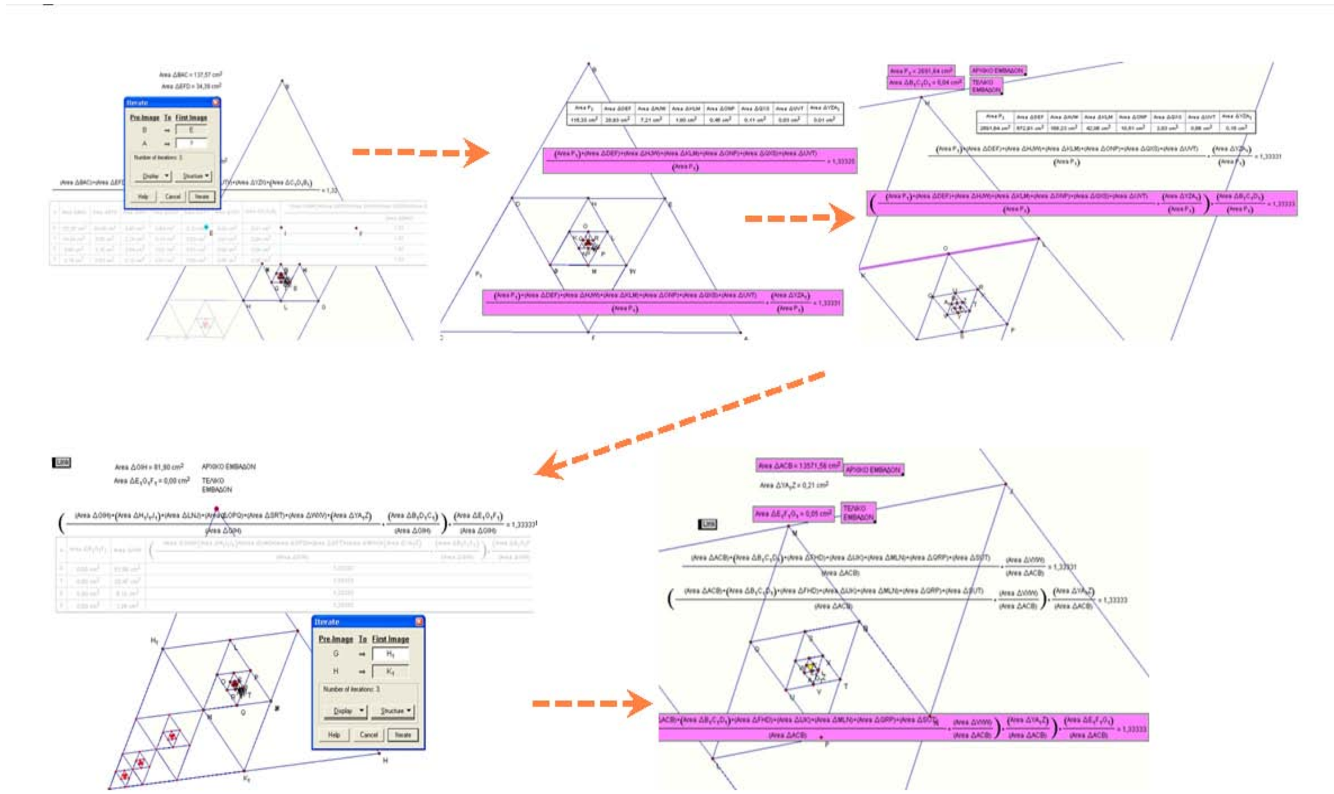

Figure 37 : Linking Visual Active Representations of Sierpinki's iteration process

How easy is it for a teacher or student working in the paper–pencil environment to create these representations with the software's accuracy or to synthesize all these together with precision and speed? From a mathematical perspective, we could mention the following:

If the area of the initial triangle is equal to E (the first term), every one of the triangles being built by joining its sides' midpoints has an area equal to $\frac{E}{4}$.

This series is geometric, with the constant ratio = 1/ 4. The question is about the calculation of the addition of triangles' areas in depth.

Meaning, the sum of E+E/4+E/16 + … whose each successive term can be obtained by multiplying the previous term by r = 1 / 4 (|r| < 1).This infinite sum can be calculated by applying the formula $\frac{\alpha}{1-r}$ with a constant ratio between successive terms, or equally

$$\sum_{n=0}^{\infty}\frac{E}{4^{n}}=\frac{E}{1-r}$$

The sum is equal to 4/ 3 = 1, 33333….

$$\frac{1}{1-r}=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$$

iii. Phase Three: An experimental process with the Fractal's Group

The experimental process that is reported here is an excerpt of the paper "Fractals as a context of comprehension of the sequences and the limit in a Dynamic Geometry environment" (Patsiomitou, 2007) included in the Electronic Proceedings of the 8th International Conference on Technology in Mathematics Teaching (ICTMT8) which took place in Hradec Králové (excerpt of my Master's thesis). The most important parts of the research process are going to be mentioned here, enriched, to explain the importance of linking visual active representations, instrumental decoding, and RVR—as LVARs have been illustrated. This process has been repeated in the students' fractal group in the previous school year.

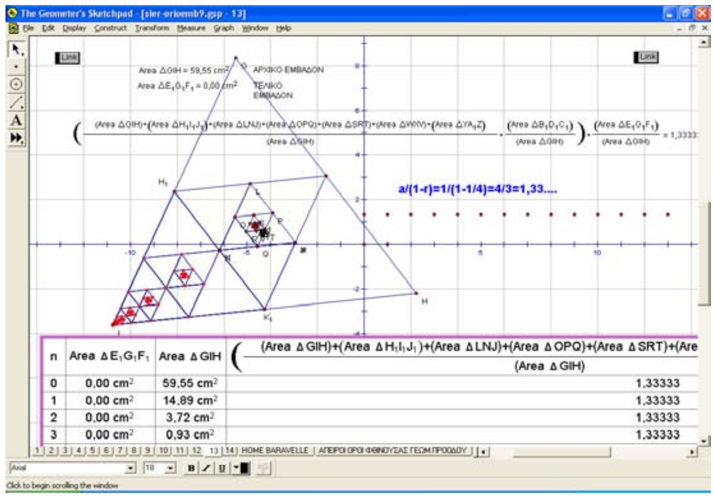

Figure 38 : The repetition of the number 1, 3333.

The team was constituted from 6 students 15-16 years old. The students at the school had not been taught about the sum of infinite terms of a geometric progression, because it was not included in their curriculum. The students initially observed that the areas in the interior of the shape were decreased by the ratio $r = 1/4$. This led them to the definition of the geometric progression for the areas of the shape and to the calculation of the sum of the infinite sequence $E + E/4 + E/16 + \ldots$ where E is the area of the initial triangle, E/4 the area of the next internal triangle and so on. The inquiry process investigated if the students could perceive the meaning of the limit of the sequence of the infinite sum of the areas (approximate result almost equal to 1, 33333…).

The students through guided questions calculated the sums of the areas of the 2 first equilateral triangles and then divided it by the area of the initial triangle. Thereafter they calculated the sums of the areas of the three first triangles and divided this again by the area of the initial triangle. The process continued with the construction of the suitable custom tool that repeated this inductive process. When the process reached the 9 first steps the sum of the 9 internal repetitions of the areas of the equilateral triangles within the shape and the calculation of the ratio was stabilised at 1,33333 even when the initial triangle's shape was increased by dragging. Therefore, the generalisation of the process resulted from the process of iteration. With the assistance of the dilate tool and zooming into the depth of the construction thus dilating the structure the afforded impression was that of an infinitely continuous structure which had in actual fact remained unaltered and constant. The students confirmed the repetition of the number 1,33333 on the table for (n) first steps of iterative constructional steps. In the beginning they applied the process and a shape resulted at the centre of the initial shape. Dilate tool assisted them to see into the centre of the shape and extend their mental representations.

[…] In the latter activity we were led towards the construction of a branch of the Pythagorean tree using the modes that were mentioned before. The students had not comprehended the graphic representation of the sequence when it had been discussed with static means in their class during their course of mathematics. Their initial reaction was to connect the isolated points that resulted from the plotting of the areas of the successive squares, in order to produce a continuous curve. This reaction of the students was a result of misconception of the definition of the domain of any sequence which is the natural numbers, but more so the result of the correlation of the graphic representation of the functions as it has been introduced by static means. The female student that with certainty repeated the definition of the sequence, but had comprehended this fine point about the graphic representation of any sequence, in no way displaying her misconception by inquiring about the fate of the intermediate points of the continuum.

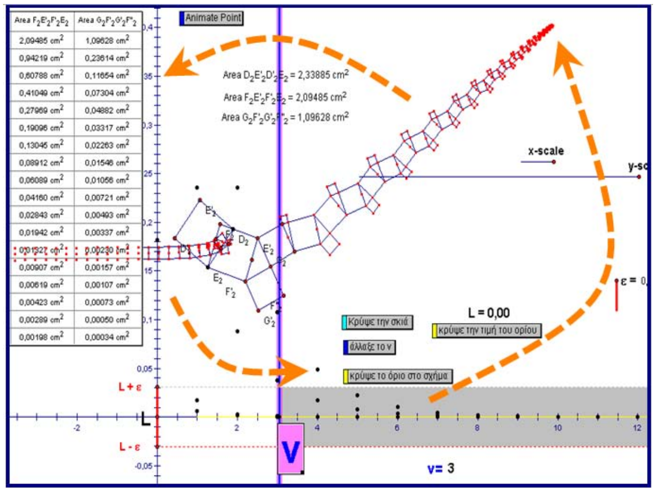

Figure 39 : Multiple linking visual active representations

The connection of the concept image with the concept definition of the meaning (Vinner, 1983) and finally the graphic representation was created through the environment of the software. At this point in the shape we have used as base for the development of the activity the file seqlimit.gsp. (Retrieved from http://www.teacherlink.org/content/math/activities/sketchpadv4.html). As it is described by the authors: "[The file has been designed] to help students graphically visualize the concepts behind the formal definition of the limit of a sequence. Given a value for epsilon, students can manipulate N to find a value for N beyond which all further terms of the sequence lie within the distance epsilon from the limit".

In this sketch I had created an adaptation of the shape of fractal Pythagorean tree (Figure 39). The process of animation can produce the changes in the tabulated measurements (calculations) that allow the user to examine the dynamic process. These changes come as result of the fluctuations in the size of an artefact-fractal which have the possibility of increasing (decreasing) and altering orientation. The students consequently had an environment of multiple linking visual active representations in which the shape of the fractal had been linked with the table of the measurements via the functional process of iteration, which continuously could be linked with the graphic representation of the sequence. […] (Patsiomitou, 2005, 2007).

As a result of the construction and application of the custom tool as much as the process of iteration the direct perception of the user is attained in regard to the steps in the development of the construction pertaining to (Patsiomitou, 2005, 2007):

- The repetitions in the measurements or calculations of the areas of initial shapes
- The developmental way of the construction of the shape and
- Its orientation towards the sequential steps of the construction on the screen's diagram or in successive pages of the same file.

Through the application of the custom tool the possibility is given to the user to acquire an inductive way of thinking for the finite steps of the construction but the generalisation with regard to the constructional result can be achieved from the process of iteration which inductively renders the construction theoretically to infinity (Patsiomitou, 2005, 2007, 2008d). This function of the software also constitutes a certain crucial and essential particularity, while the construction with a compass and a ruler as formal tools of static geometry has a beginning and an end. Moreover, "a teacher is able to distinguish different levels of acquisition and mathematical engagement with a fractal topic [as] scripts [/ iteration command] represent an abstraction of his/her own work or process, and thus using them as "abstract" tools require one level more advanced or sophisticated a conceptualization than using "literal" tools like the compass or straightedge. " (Personal e-mail communication with Nicholas Jackiw on September 29, 2005).

In the software, via the process of iteration we have the potential of the constructions thus becoming more complex being in theory rendered inductively to infinity. The result of the process of iteration is the construction of the tables that repeat the process of initial measurements and calculations in dynamic connection with the shape, thus increasing (or decreasing) the level of the process of iteration while the software adds (or removes) the next level of measurements (or even calculations), whereas in the first column of the table the sequence of the natural numbers is presented. In that way through this operation, the environment of the software promotes the exploration of the sequences and of the series. The iteration process by functioning thus has integrated or embodied the meaning of sequence while there is a direct connection between the user's perception and the abstract mathematical meaning (Patsiomitou, 2005, 2007). Therefore, I think that The Geometer's Sketchpad v4 [-v5] is the best tool to introduce fractals in classroom not only for aesthetic purpose rather than for the pursuit of their very interesting mathematics.

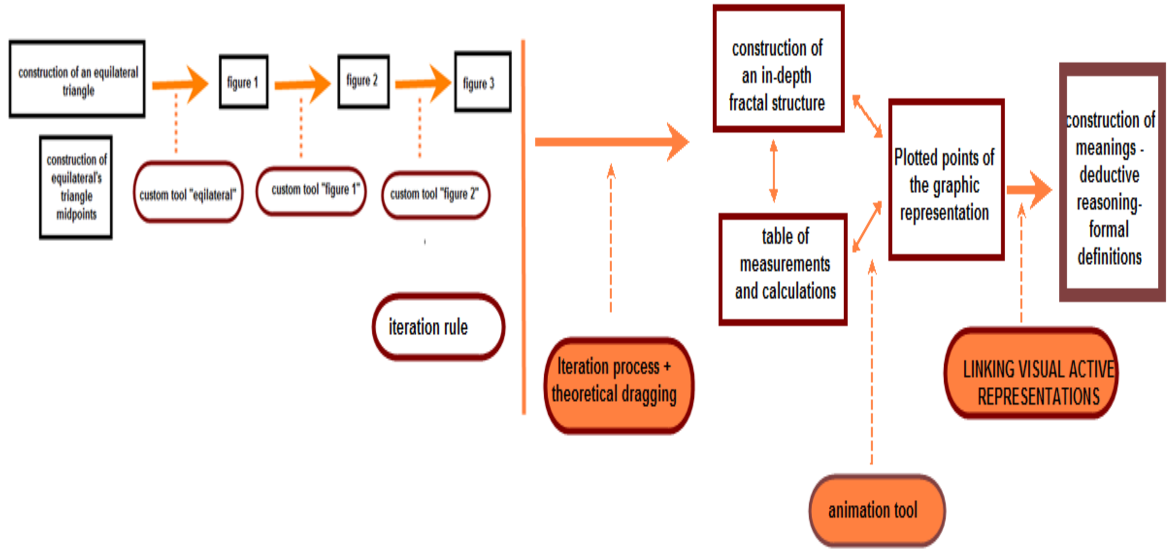

Figure 40 : A Pseudo Toulmin model explaining the birth of the meaning 'Linking Visual Active Representations'.

The structures of fractals, [by applying the meaning of dynamic LVAR representations], aims that students (a) review most of theorems, (b) identify the potential weaknesses and cognitive obstacles that students face in their effort to understand the process, (c) develop the links between the virtual representations and the formulations with which students justify their construction, as a result of understanding the figures' transformations and symmetry, and (d) develop most of the competencies described in the beginning of the article and higher–order level skills (e.g., generalize patterns using recursion, use algebraic formulae and symbolic expressions to explain mathematical relationships, etc.) than those that they are able to develop through traditional mathematics. This is very important for their movement through vH levels.

V. Discussion

a) Developing a theory on dynamic transformations

The emphasis on construction using the Transform menu in GSP was shaped to facilitate the understanding of symmetries and strengthen the development of structures in the students' minds. The thought that the shapes have symmetries can lead students to dynamically reinvent new ways of constructing them through the dynamic geometry software.

The focus on transformations is in accordance to Coxford & Usiskin (1975), who report inter alia that, the use of different types of transformations in the curriculum simplifies the mathematical development (for example, the definitions of congruence and similarity cover all figures). Therefore, the proofs of many theorems are simpler and more accessible to all students. Furthermore, the authors argue that transformations facilitate the understanding of mathematical concepts for students from different mathematical competence and prepare the ground for future processing concepts of algebra and analysis. Transformations used by the students in the DGS environment can be distinguished through the following:

- Theoretical and experimental dragging, as mentioned at the beginning of the article.
- Transformation elicited from the reflection, dilation, rotation, or translation of the object. Dragging on rotated (dilated, reflected, or translated) objects maintain the congruency and structural relationship between the elements of the construction.
- Transformations elicited from the utilization of the action buttons tools (for example, the hide/show action button, the link button, the movement button, or animation).
- Transformations elicited from the annotation of the dynamic diagram (for example, use of colors, formulations, and the trace tool). Moreover, the combination of transformations (e.g., the trace tool and dragging tool, the calculations and the dragging of the geometrical object's points).
- Transformations elicited from the application of the custom tools. The application of custom tools reorganizes the external representation. The application of a custom tool (or the repetition of the application of a custom tool) is accomplished in a sequence of steps directly perceived by the user. Consequently, custom tools operate as a referent point for organizing, pursuing, and retrieving information.
- Transformations elicited from the synthesis of the dynamic diagram.

- Combinations of transformations due to the synthesis of the software's interaction techniques (Sedig & Sumner, 2006).
- Complex transformations of the LVAR dynamic representations.

Therefore, *dynamic geomatric transformations* are defined as the modifications of the diagram on screen that result in the modification in one or more included geometric objects. This could be an elicitation from the addition, cancelation of the diagram's elements that cause the rearrangement of the diagram, its anasynthesis, or even the modification of any object's size or orientation. Moreover, it could seen as we apply one or more interaction techniques, or their combination, on the diagram's objects. Transformations on prototype elements (e.g., points, line segments) led the students to (1) visualize the objects that were constructed in the first phase of the process and (2) perceive a few properties of the figure's symmetry initially at the visual level. It was observed that during the process the students connected, in their minds, representations that helped them to respond to the next level, according to the theory of van Hiele.

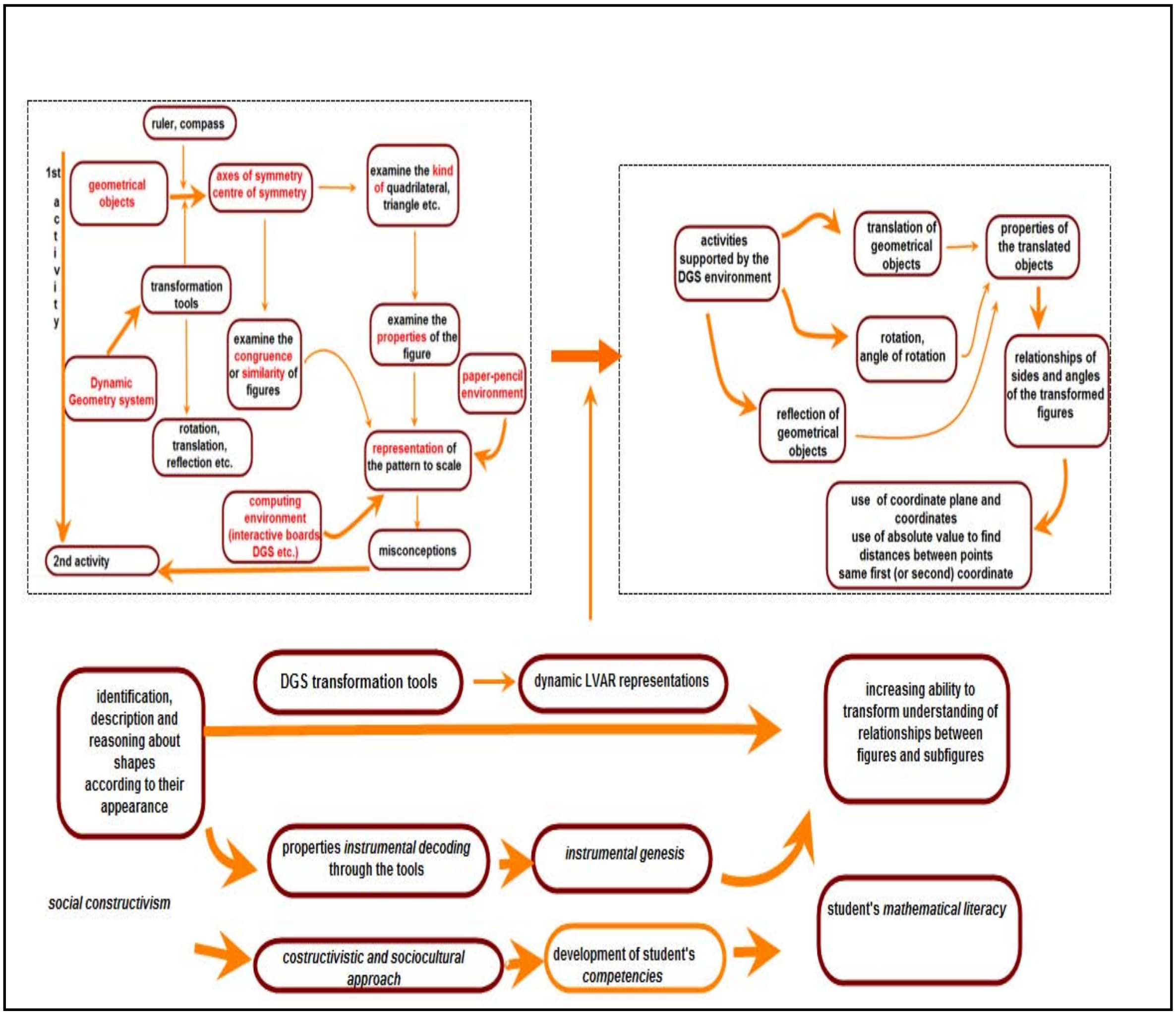

Figure 41 : A pseudo-Toulmin model explaining the transformations in the DGS environment

The dynamic manipulation of objects in software led the students to construct the properties of the shapes. The use of software transformation tools influenced the direction of their thinking, since their use allowed the properties of shapes to be analyzed and then synthesized into shapes. As a result, the construction and transformation of semi-preconstructed LVAR led the students to a theoretical way of thinking, and cognitive transformations through related cognitive connections.

"If we accept that mathematical growth coincides with constructing new mathematical reality, we may conceive mathematics education as supporting students in constructing new mathematical reality. This fits with Freudenthal's (1973) notion of "mathematics as a human activity". In his view students should be given the opportunity to reinvent mathematics. The challenge then is: How to make students invent what you want them to invent?" (Gravemeijer, 2004, p.3)

Many students do not have the ability to dynamically visualize and mentally manipulate geometric objects, which is important for solving problems in geometry. In that case they are not able to reflect on or to anticipate a possible solution to the problem. Therefore, geometric transformations in the software help the students to form an intermediate stage between the concrete and the abstract. They help them to instrumental decode the mathematical symbols and to connect their use with the pre-existing knowledge. Then the interaction with the software incorporates the steps and the mental or cognitive actions that facilitate the understanding of the solution.

The use of the transformations in the DGS environment strongly influenced the formation of the 'dynamic' teaching cycle process which is described in the next section.

b) A 'dynamic' teaching cycle process through LVAR

The data presented here focused on teaching situations, including instructional units, classroom activities, and simulated or modeled problems in the DGS environment. In this section, I shall analyze my role as teacher, researcher, and instructor of the activities as it emerged from the teaching situations, as well as the students' role in the formation of a mathematical teaching cycle. The design or selection of teaching activities and problems that stimulate and excite mathematical reinvention (Freudenthal, 1973) on the part of students is a "challenge for the teacher, [who must] try to see the world through the eyes of the student." (Gravemeijer, 2004, p.8)

If the teaching and learning of concepts through the use of real problems in a DGS environment is compared with the traditional approach, we conclude that, "the modelling perspective [using a DGS environment] offers major advantages. The process of modelling constitutes the bridge between mathematics as a set of tools for describing aspects of the real world, on the one hand, and mathematics as the analysis of abstract structures, on the other" (Corte, Verschaffel & Greer, 2000, p.71). Moreover, the intrinsic design of dynamic representational systems has essential impacts on the mental representations of the student, that is, the ways in which students construct their personal representations of meaning during the activity, whether these representations are directed at an individual student or in the student's collaborative environment with others. Accordingly, the conclusions can be used to analyze the potential of these tools for mathematics teaching and learning, to design new tools, and to better understand the ways in which these tools can be (instrumentally) decoded by teachers and students to be transformed into theoretical knowledge built through mediation. As teachers (or teacher- researchers) design teaching concepts and ways of interacting with their students, they increasingly feel the need to understand the minds of the students, looking for methods to lead their students to understand the concepts. Therefore, the determining factor is the teacher who decides on the objectives/aims of the teaching method and chooses the means for effective implementation of the objectives or of the educational process. The positive attitudes/behaviors of the teachers of mathematics with regard to mathematics, their positive position with regard to technology, and their interest in the students' understanding of the concepts, are the most important factors for the development of innovative applications in schools.

As a teacher-researcher, I know that the students encounter difficulties in order to understand the concepts in geometry. The connection between the represented and the representation can create conflicts to students because they are not able to control the information that comes from the outside world (Mesquita, 1998). The question is how we can overcome the cognitive obstacles they face and what are these teaching situations which can provide the scaffolding to the next van Hiele level.

The instructional units aimed to challenge students and "elicit, support and extend children's mathematical thinking, facilitating mathematical discussions, using the representations of concepts and encourage use of alternative solution methods" (Fuson et al., 2000, p. 277). Many times I tried to shift mentally from an observer's point of view to an actor's point of view (Cobb, Yackel & Wood, 1992 in Gravemeijer, 2004), and consider now the design of the activity of this regard. My approach was as follows:

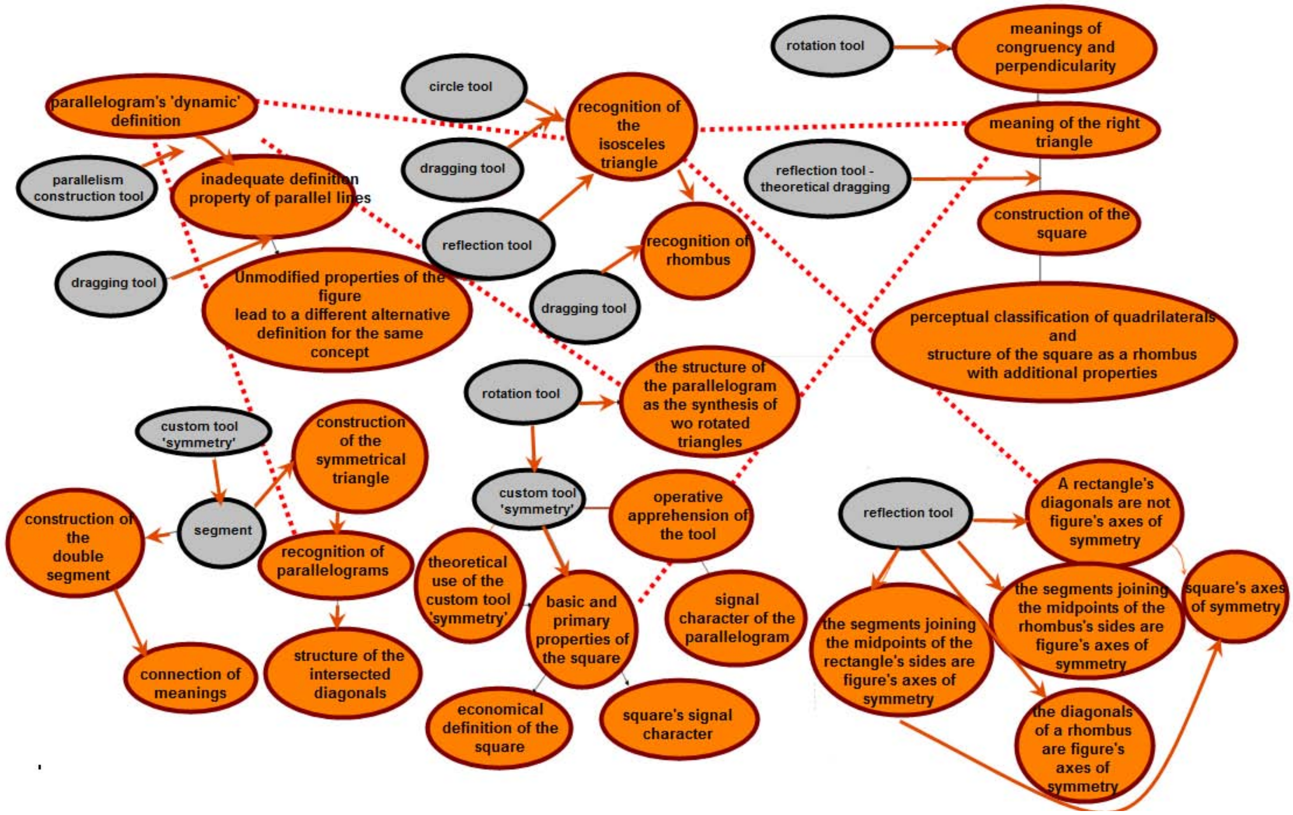

Figure 42 : A pseudo-Toulmin's model representing the instructional design through LVAR

Situation 1 : The highlighted idea in mathematics, which is 'symmetry,' is interdisciplinary, connected with art and culture. The aim is to 'see' mathematics in any context.

Situation 2 : The challenge was to connect the transformations in static and dynamic means conceptually and procedurally. Instrumental decoding of students' mathematical ideas played a major role for the overcoming of cognitive and instrumental obstacles.

Situation 3: This situation aims to accomplish the figure's symbol character. The grid in the DGS environment provides a challenge for the experimentation. The important points in this situation were the students' methods of dealing with the questions: "Under what conditions does the rhombus become a square?" or "What are the similarities and differences between a kite and a square?"

Situation 4: The motivation for this situation was that my students understand the parallelograms from their symmetry properties and, if they have a set of properties, to understand the kind of quadrilateral. This phase is very crucial for the students to acquire the ability to replace a figure with a set of properties that represent it and from these properties to construct the figure. In other words, the figure will acquire the signal character.

Situation 5: The recognition of differences and similarities between figures' symmetry properties demarcates the scope of this situation. The teaching and didactic process must focus at the understanding of the structuring process and not the learning of ready-made structures.

Situation 6: The development of structures in students' minds has been achieved with the synthesis of a more complex construction. The situations aim to develop the abstraction. Pythagorean Theorem's reconfigurations have been used as a tool for the development of students' instrumental decoding of a complex figure's anasynthesis. The 6th situation led students to think about self similarity, which is not included in high school curriculum.

Situation 7: Self -similarity, Pythagorean Theorem and the midpoint theorem are the mathematical backgrounds of this situation. Here is explained the rationale in the design process and the importance of linking visual active representations and instrumental decoding.

The use of a computing environment such as dynamic geometry helps students to build 'a model of the meaning' (Thompson, 1987, p.85) and overcome the difficulties of translation between representations through the automatic translation or "dyna-linking" (Ainsworth, 1999, p. 133), since [they] "encode causal, functional, structural, and semantic properties and relationships of a represented world – either abstract or concrete" (Sedig & Sumner, 2006, p.2). The design and redesign of activities for the teaching and learning processes, with real problems or simulations of real-world problems through LVAR in the dynamic geometry software, and the results obtained from the research data (Patsiomitou, 2012 a, b), suggest that a student develops his/her abstractive competency when his/her cognitive structures are linked through representations that the student develops during the learning process.

"Apart from the aspect of anticipating the mental activities of the students, a key element of the notion of a hypothetical learning trajectory is that the hypothetical character of the learning trajectory is taken seriously. The teacher has to investigate whether the thinking of the students actually evolves as conjectured, and he or she has to revise or adjust the learning trajectory on the basis of his or her findings. In relation to this, Simon (1995) speaks of a mathematical teaching cycle. In a similar manner, Freudenthal (1973) speaks of thought experiments that are followed by instructional experiments in a cyclic process of trial and adjustment. If we accept this image of the role of the teacher in instruction that aims at helping students to invent some (to them) new mathematics, we may ask ourselves, what type of support should be offered to teachers. Apparently, we will have to aim at developing means of support that teachers can use in construing and revising hypothetical learning trajectories" (Gravemejer, 2004, p.9).

The whole action is an innovative production of a new approach to the educational process based on theoretical underpinning. This innovation is introduced for the first time in the school of established practice, and thus, proposes the redevelopment / redesign of the everyday teaching practice by using LVAR, with proper interventions in school curriculum. Specifically, linked representations that the student is able to construct (Patsiomitou, 2012a, b):

- When the student builds a representation (e.g., a rectangle) in order to create a robust construction externalizing his/her mental approach, using software interaction techniques by externalizing his/her mental approach or by transforming an external or internal representation to another representation in the same representational system or another one.
- When s/he gets feedback from the theoretical dragging to mentally link figures' properties so that, because of the addition of properties, subsequent representations stem from earlier ones.
- When s/he transforms representations so that the subsequent representations stem from previous ones due to the addition of properties.
- When s/he links mentally the developmental procedural aspects in a process of a dynamic reinvention
- When s/he reverses the procedure in order to create the same figure in a phase of the DHLP or between phases of the same DHLP.
- Adding to the initial [procedural] structure so that the first component parts of a construction lead to a structure and to eventually becoming more and more complex,
- Linking the conceptual steps of the construction (p. 76).

Moreover, the procedures, due to their design, "prompted" the cooperation of students, contributed to the development of positive behavior, and strengthened the weak students to understand the concepts and procedures while interacting with their classmates. The process resulted in the cooperation of students with me which often "forgot" my role and "took" on the role of a student playing the 'game' to ask questions that some of my students did not have the courage to ask.

With regard to the problems, many teachers prefer algebra to geometry. The reasons are as follows: (a) the awareness of the risk of the student's failure or (b) the teacher's lack of confidence for their knowledge of the subject of geometry. How would the LVAR process (i.e. the utilization of LVAR concept for the construction of activities) change this weakness when students are able to process on official electronic platforms from the Ministry of Education? How will this affect the confidence of teachers who handle this platform for their students, giving feedback on their knowledge?

These questions should be discussed, as well as discussing who will educate the designers of these activities so that the material is consistent with the idea. On the other hand, it is obvious that there is possible misuse of the LVAR concept for the construction of activities by the way that every teacher thinks, which could lead to opposing results. It is therefore necessary to train the agents who will spread the LVAR idea, with consistent processes of meaning. Still, the implementation of the idea can be generalized and repeated in any group of students, at different times and in any thematic framework (e.g., the objects of physics or chemistry).

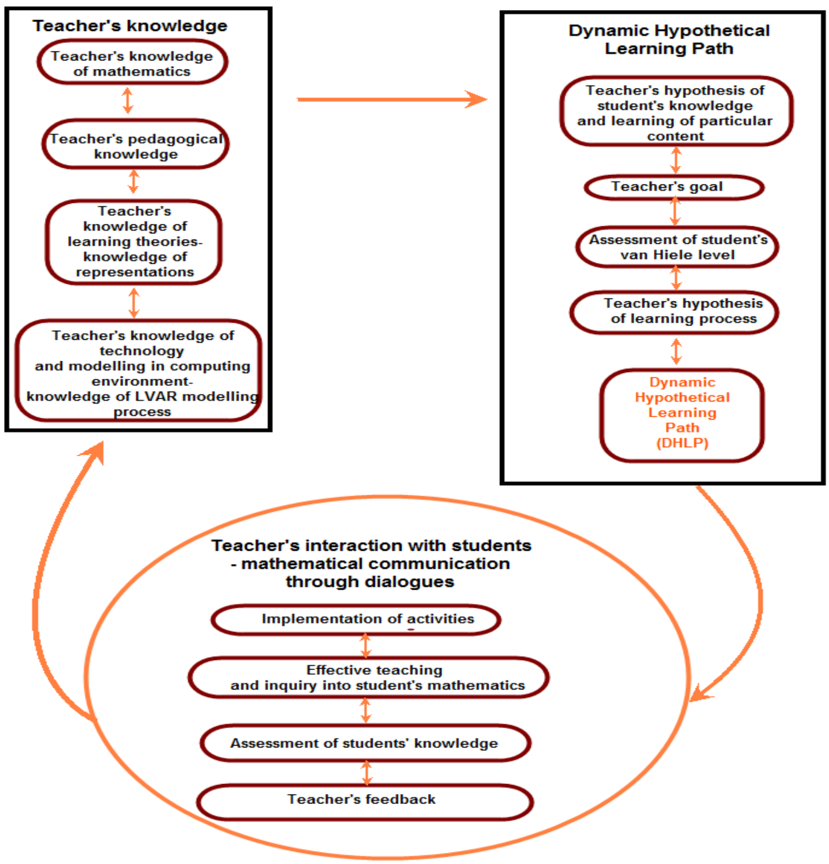

Figure 43 : An adaptation of Simon's (1995) Mathematics Teaching Cycle for the current study

The analysis of the teaching situations and the development of the Mathematics Teaching Cycle have led to an iterative diagram that is an adaptation of Simon's (1995, p.136) work, taking into account also the work of McGraw (2002, p.10). The Mathematics Teaching Cycle portrays the relationship between the following areas of knowledge (Simon, 1995): "the teacher's knowledge of mathematics mathematics and his hypotheses about the students' understandings, several areas of teacher knowledge come into play, including the teacher's theories about mathematics teaching and learning; knowledge of learning with respect to the particular mathematical content; and knowledge of mathematical representations, materials, and activities" (p. 133).

What has been examined is the use of technology in the teaching cycle which plays an important role in the development of discussions, as well as students' vH level. The diagram aims to include the incorporation of technology practices in class. The teacher's interaction with students and the mathematical communication through dialogues is accomplished in sequential situations: the implementation of activities, effective teaching and inquiry into students' mathematics, the assignment of students' knowledge, all of which leads to the teacher's feedback. These processes go on continually and can suggest adaptations in various domains of a teacher's knowledge, including in the following areas: mathematics, pedagogy, representations, technology, and modeling through LVAR representations. The whole process leads to a modification of the hypothetical learning path that includes a continuous interaction between the teacher's knowledge of particular content, the teacher's goal, and assessment of the students' vH levels.

VI. Conclusions

The modeling of a problem in the dynamic environment can 'carry' any [mathematical] object to the classroom in two ways: through the use of digital images or through the use of their simulations. On the other hand, a technological tool is important as the design of artifacts can be generalized and replicated in any group of students, at different times and in any thematic framework (e.g., science, geography). Therefore, referring to LVAR is concluded in the following (Patsiomitou, 2012a, p 498):

- How could this affect the students' understanding of the utilization of LVAR in the teaching and learning of other disciplines (e.g., physics or ancient Greek and history)? [or] Would students understand the obscure points of other disciplines, because of the interaction with the [appropriate] dynamic LVAR representations?

- Can the students develop their linking of the conceptual and procedural representations of these objects?

On the other hand, new cognitive tools are not included [or included in a very slow way] for the teaching of concepts. It is particularly important for the 'movement' of a process by applying innovative practices to change the negative views that a large portion of teachers have regarding technology. This seems to focus on a lack of knowledge because of the phobias surrounding technological tools in the mathematics classroom, leading to an adherence to traditional teaching methods.

In general, the whole issue has to do with the way we perceive the world, the natural objects (unconscious), how we compare them mentally (consciously) with theoretical constructs of geometry in order to represent them and how we instrumental decode them using technology. Finally, it is important to continue teaching and research concepts in this vital field, through activities that involve children in the learning process, so using linked visual representations they will learn how to develop, interpret, and make sense of geometric concepts. This argument recognizes and underlines the force of Kant's argument (1929, "Critique of Pure Reason") that:

There can be no doubt that all our knowledge begins with experience. For how should our faculty of knowledge be awakened into action did not objects affecting our senses partly of themselves *produce representations,* partly arouse the activity of our understanding to *compare these representations*, and, by *combining or separating them*, work up the raw material of the sensible impressions into that knowledge of objects which is entitled experience? [Because] "Understanding is the faculty of knowledge and […] *knowledge consists in the determinate relation of given representations to an object"*.

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