



Delta- Hedging Models: Comments

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I. INTRODUCTION

Delta is the ratio comparing the change in the price of the underlying asset to the corresponding change in the price of a derivative¹. For example, if a stock option has a delta value of 0.65, this means that if the underlying stock increases in price by \$1, the option will rise by \$0.65, all else equal. Delta hedging is a derivative trading strategy. It is an options strategy that aims to reduce, or hedge, the risk associated with price movements in the underlying asset, by offsetting long and short positions. The delta of an option helps you determine the quantity of the underlying asset to buy or sell. This is known as delta hedging. Delta hedging involves trading another security to create a delta-neutral position, or a position that has a zero delta. For example, a long call position may be delta hedged by shorting the underlying stock. This strategy is based on the change in premium, or price of option, caused by a change in the price of the underlying security. The theoretical change in premium for each basis point or \$1 change in price of the underlying is the delta, and the relationship between the two movements is the hedge ratio. The price of a put option with a delta of -0.50 is expected to rise by 50 cents if the underlying asset falls by \$1. The opposite is true as well. The delta of a call option ranges between zero and one, while the delta of a put option ranges between negative one and zero. So, delta hedging is a strategy used to mitigate the risk associated with the

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¹ Those industries which are connected to the finance and commodity markets and are trading derivatives are the most likely to use delta hedging techniques. This may include institutional investors, banks, hedge funds and natural resource companies, among others. Delta hedging is a technique that attempts to manage risk for an option position by hedging the exposure with shares or contracts in the underlying asset.

price move in the underlying asset of an option by entering an offsetting position. Although this hedge reduces a portfolio's exposure to the underlying asset, it has its limitations. One limitation of delta hedging is that a position still has risk exposure even if the position is delta-neutral. Delta hedging needs to be constantly readjusted with movements of the underlying asset.

II. DELTA-HEDGE VOLATILITY

The delta hedge portfolio has the value

$$\pi = -\mu + w' p \quad (1)$$

where $w(p, t)$ is the option price. The instantaneous return on the portfolio is

$$\frac{d\pi}{\pi dt} = \frac{-dw + w' dp}{(-w + p w') dt} \quad (2)$$

We can formulate the delta hedge in terms of the returns variable x . Transforming to returns $x = \ln p / p_o$, the delta hedge has the value

$$\pi = -\mu + \mu' \quad (3)$$

where $\mu(x, t) = w(p, t)$ is the price of the option. If we use the stochastic differential equation.

The general theory of volatility of fat-tailed returns distribution with $H=1/2$ was formulated as follows. Beginning with a stochastic differential equation

$$dx = (\mu - D(x, t) / 2) dt + \sqrt{D(x, t)} dB(t) \quad (4)$$

where $B(t)$ is a Weinerprocess, $(dB) = 0$ $(dB^2) = dt$ and $x = \ln(p(t) / p_0)$ where $p_0 = p(t_0)$. In what follows let $R(x = \mu - D(x, t) / 2)$. The Solution is given by the stochastic integral equation

$$\Delta x = \int_x^{t+\Delta t} R(x(s), t) dx + (D(x, t))^{1/2} \circ \Delta B \quad (5)$$

Transforming to returns $x = \ln p / p$, the delta hedge portfolio has the value

$$\pi = -\mu + \mu' \quad (6)$$

where $\mu(x, t) / p = w(p, t)$ is the price of the option. If we use equation (5), then the portfolio's instantaneous return is (by Ito's calculus) given by

$$\frac{d\pi}{\pi dt} = \frac{-(\mu - \mu' D / 2) - \mu'' D / 2}{(-\mu + \mu')} \quad (7)$$

And is deterministic; because the stochastic differential terms $O(dx)$ have cancelled. Setting $r = d\pi / \pi dt$ we obtain the equation of motion for the average or expected option price $\mu(x, t)$ as

$$r\mu = \mu + (r - D / 2)\mu' + D / 2\mu'' \quad (8)$$

With simple transformation

$$\mu = e^{\int_T^t r(s)ds} \quad (9)$$

Equation (8) becomes

$$0 = v + (r - D / 2)v' + D / 2v'' \quad (10)$$

Note that if the equation does not exist due to the non vanishing of higher moments in which case the option pricing partial differential equation (6) must be used, then the option pricing partial differential equation also does not exist.² Therefore, in order to bring the 'expected price' option pricing principle into agreement with delta hedge, we see that it would be necessary to choose $\mu = r_d = r$. in order to make prediction risk neutral. How, then to choose r ?

Let r denote any rate of the expected portfolio return r may be constant or may depend on t . Calculation of the mean square fluctuation of the quantity $(d\pi / \pi dt - r)$, shows that the hedge is risk free to $O(dt)$ whether or not $D(x, t)$ is constant or variable and whether or not the portfolio return r is chosen to be the risk-free rate of interest³. Would an application of arbitrage argument would lead to the choice $r = r_0$?^{4, 5}

² The equation (9) is the same as Kolmogorov equation, see Gnedenko (1967), with the choice of $\mu = r$, both equations have exactly the same as the Green function so that no information is provided by solving the option pricing that is not already contained in the Kolmogorov equation. solution

³ Practical examples of so-called risk-free rates of interest r_0 are provided by the rate of interest for the money market, bank deposits, CDs or U.S. Treasury Bills. So, we are left with the important question: what is the right choice of r in option pricing.

⁴ Arbitrage plays a critical role in the analysis of securities markets, because its effect is to bring prices to fundamental values and to keep markets efficient. For this reason, it is important to understand how well the textbook description of arbitrage approximates reality. Many argue that the textbook description does not describe realistic arbitrage trades and, moreover, the discrepancies become particularly important when arbitrageurs manage other people's money. See Grossman and Miller (1988).

⁵ Even a simplest trade becomes a case of what is known as risk arbitrage. In risk arbitrage, no arbitrageur does not make money with probability one and may need substantial amounts of capital to both execute his trades and cover his losses. Most real world arbitrage trades in bond and equity markets are in a sense examples of risk arbitrage. When arbitrage requires capital, arbitrageurs can become most constrained when they have the best opportunities, i. e., when the mispricing they have bet against gets even worse. Moreover, the

Finance theorists treat the formal no-arbitrage argument as holy but mathematicians know that every proposition about the market must be tested and retested. We must therefore pay close attention to the traders' practices because traders are the closet analog of experimenters that we can find in finance⁶. No-arbitrage argument assumes that the portfolio is kept globally risk free via dynamic challenging. The delta hedge portfolio is instantaneously risk-free but has finite risk over finite time intervals Δt unless continuous-time updating is accomplished to within observational error.

III. NO- ARVITRAGE ARGUMENTS

The no-arbitrage argument assume that the portfolio is kept globally risk free through dynamic rebalancing. The delta-hedge portfolio is instantaneously risk free but finite risk over finite time intervals, Δt . unless continuous time updating is accomplished to within observational error. However one cannot update too often because of trading fees and all that and this introduces error that in turn produce risk. This risk is recognized by traders who do not use the risk-free interest rate. The reason for this is also theoretically clear: why bother to construct a hedge that must be dynamically balanced, frequently updated, merely to get the same rate of return r_0 that a money market account or a CD would provide?⁷ In our present era since the beginning of the collapse of the bubble and under the current non-conservative regime in Washington, it would be pretty risky to assume positive stock returns over time intervals on the order of a few years. Let us pursue the matter a little further:

$$\hat{\delta} = r - D(\delta, t) / 2 \quad (11)$$

$$D(\bar{\delta}, t) = \begin{cases} b^2 & x > \delta \\ b'^2 & x < \delta \end{cases} \quad (12)$$

We must take $r(t)$ and also $\mu(t)$ to be discontinuous in δ as well. The value of t is then fixed by the condition related to the cost of carry r_d but with the choice $\mu = r$ the solution for a call with $\ln(K/P) \leq \delta$. this then will have the form:

fear of this scenario would make them more cautious when they put on their initial trades and hence less effective in bringing about market efficiency. See De Long *et al* (1990), Dow and Gorton (1994).

⁶ Shleifer and Vishny (1990) in their arbitrage model focus on the market for a specific asset, in which they assumed there are three types of participants, noise traders, arbitrageurs, and investments in arbitrage funds who do not trade on their own. Arbitrageurs specialize in trading only in this market, whereas investors allocate funds between arbitrageurs operating in both this and many other markets.

⁷This choice also agrees with historic stock data, which shows that from 1900 to 2000 a stock index or bonds would have provided a better investment than a bank savings account.

$$C(K, P, \Delta t) = e^{-r, \Delta t} \int_{\ln(k/p)}^{\delta} (p e^x - K) f(x, \Delta t) dx + e^{-r, \Delta t} \int_{\ln(k/p)}^{\infty} (p e^x - K) f(x, \Delta t) dx \quad (13)$$

where $\Delta t = T - t$. Now we have two discounting factors for two separate regions divided by the singular point $x = \delta$. Note finally that because the singular point $P = p_0 e^b$ of the price distribution evolve deterministically, we could depart from the usual no-arbitrage argument to assert that we should identify $\delta = r_0 \Delta t$ where r_0 the risk-free rate is. The weakness in the argument is that it requires $\mu > 0$ and $\delta > 0$, meaning that expected asset returns are always positive which is not necessarily case. The trouble with this line of argument is that the millions of traders are typically not the ones who have the knowledge and information to engage in arbitrage. More commonly, arbitrage is conducted by relatively few professional, highly specialized investors who combine their knowledge with the resources of outside investors to take large positions of these markets.⁸⁹

A market made up only of noise is a market in agreement with the efficient market hypothesis). Arbitrage is impossible systematically in a market consisting of pure noise. This is the complete opposite of the neo-classical notion of perfect information (zero entropy)¹⁰. Rather, financial markets show that 'the neo-classical emperor wears no clothing at all'¹¹

⁸ The fundamental feature of such arbitrage is that brains and resources are separated by an agency relationship. The money comes from wealthy individuals, banks, endowments and other investors with only a limited knowledge of individual markets and is invested by arbitrageurs with highly specialized knowledge of these markets.

⁹ Scaling exponents and extreme events will not help. If the exponential density in terms of the variable $y = p / p(0)$

$$f(y, t) = f(\ln y, t) / y$$

Exhibits fat- tail scaling with time dependent tail price exponents $\gamma - 1$

and $\nu + 1$. These tail exponents become smaller as Δt increases. From our standpoint the scaling itself is neither useful nor important in application like option pricing, nor is helpful in understanding the underlying dynamics.

¹⁰ If $f(x, t)$ is the empirical return density then the entropy is

$$S(t) = \int_{-\infty}^{\infty} f(x, t) \ln f(x, t) dx$$

but, again, equilibrium is impossible

because this entropy is always increasing. The entropy can never reach a maximum because f , which is exponential in returns x spreads without limit. The same could be said of the Gaussian approximation to the returns distribution. Das (2015), Giles (2016), Lavenda (2010).

¹¹ In particular, as Lavenda (2010), Das (2015) even the central limit theorem cannot be used to derive a Gaussian without the assumption of a microscopic invariance in the form of step sizes and probabilities for the underlying discrete random walk. If one make other microscopic assumptions about step sizes and corresponding probabilities, then one gets an exponential distribution, a Levy distribution, or some other distributions. There is no universality

No-arbitrage condition does not guarantee market equilibrium, which is defined by vanishing total excess demand for an asset. Consider two spatially separated markets with two different price distributions for the same asset. If enough traders go long in one market and short in the other, then the market price distributions can be brought into agreement. However, if there is positive excess demand for the asset then the average of the asset will continue increasing with time, so that there is no equilibrium,¹²

IV. CONCLUDING REMARKS

This paper describes the workings of markets in which arbitrageurs' performance to ascertain their ability to invest profitably is limited. The avoidance of volatility by arbitrageurs suggest an approach to understanding persistent excess returns in security prices. Specifically, we expect problems to reflect not some exposure of securities to difficult-to-handle macroeconomic and statistical risks. The more realistic view of arbitrage can shed light on a variety of observations in securities markets that are difficult to understand in more conventional mathematical models. Unlike in the efficient market model, the risk need not be correlated with any macroeconomic factors and can be purely idiosyncratic fundamental or noise trader risk.

REFERENCES RÉFÉRENCES REFERENCIAS

1. Das Amaresh (2015) Does Adam Smith's Invisible Hand Work for Financial Markets *Global Journal of Management and Business Research* Vol XV Version1 pp 31-36.
2. Das Amaresh (2015) Does the Black- Scholes Models Follow a Gaussian or Exponential Distribution? A Note, *APRN Journal of Science and Technology* Vol 15, No 10, pp 506 – 508.
3. Delong J Bradford, Andrei Shleifer, Lawrence Summers and Robert Waldmann (1990) Noise Trader Risk in Financial Markets, *Journal of Political Economy*.
4. Dow James and Gary Gorton (1994) Noise Trading Delegated Portfolio Management and Economic Welfare, *NBER Working Paper* 4858.

independent of the microscopic assumptions: different local laws of time –evolution of probability distributions.

¹² The excess demand $\xi(p, t)$ is defined by $dp / dt = \xi(p, t)$ is defined by drift plus pure noise... So markets that are not in equilibrium can satisfy the arbitrage condition. The equilibrium would then be the absence of arbitrage possibilities, that is, there is only one price of an asset. Smith and Foley (2002) have proposed as shown in Das () a thermodynamic interpretation of one price based on utility maximization. In their discussion a quantity labeled as entropy is formally defined in terms of utility, but the quantity so defined cannot represent disorder/uncertainty because there is no liquidity, no analog of the heat bath, in neo-classical equilibrium.

5. Giles R (2016) *Mathematical Foundation of Thermodynamics, International Science Monographs on Pure and Applied Mathematics.*
6. Gnedenko B (1969) *The Theory of Probability* B DSeckler, Chelsea, New York.
7. Lavenda B(2010) *A New Perspectives on Thermodynamics*, New York, Springer.
8. Shleifer A and Robert Vishny (1992) Equilibrium Short Horizons of Investors and Firms, *American Economic Review, Papers and Proceedings*, 80, pp 148-153.

