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A Unified Field Approach on Fractional-Ordered Micropolar Thermoelasticity with Diffusion

Soumen Shaw[α], Basudeb Mukhopadhyay[Ω]

Abstract - **The present paper is completely devoted on derivation of some basic fundamental relations in generalized thermodiffusive micropolar elasticity with fractional-ordered derivatives. The generalized heat conduction and mass diffusion equations have been modified by using fractional calculus. A variational principle is obtained and hence the uniqueness theorem for those equations has been proved.**

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I. INTRODUCTION

It is well established that the thermoelasticity theory is a fusion of the theory of heat conduction and the theory of elasticity. In classical theory of thermoelasticity there was a diffusive phenomenon on the heat propagation and thermal signals propagate with infinite speed. This physically unacceptable drawback of infinite speed of heat propagation was inherent in that theory. Modifying the Fourier's law of heat conduction, Lord-Shulman [1] introduced one non-classical theory of heat propagation with one relaxation time which can avoid that paradox. Green-Lindsay [2], in the year 1972, Proposed another one with two relaxation times. These non-classical theories are referred as generalized theory of thermoelasticity. Dhaliwal and Sherief [3] extended that generalized theory for anisotropic media. Later on, during the year 1991-1993 Green and Naghdi [4, 5, 6] introduced a new theory of thermoelasticity and divide their theory into three parts, referred as types I, II and III. In an extensive review work on the development of generalized/ hyperbolic thermoelasticity till 1998 is available in the review article of Chandrasekharaiah [7].

Diffusion can be defined as the random movement of the particles from the higher concentrated regions to the lower concentrated regions because of the non-zero concentration gradient which can be expressed in terms of changes of the concentration at that position. In recent past it has been observed that there are so many researchers are interested to study on this aspects due to a great application in geophysics and in industry e.g. so many oil companies are interested in the thermodiffusion process for more efficient extraction of oil from the oil deposits.Diffusion is used to form base and emitter in bipolar transistors, form integrated resistors and used to introduce 'dopants' in controlled amounts into the semiconductor substrate. The thermodiffusion in elastic solids is due to the coupling among the temperature, elastic strain and mass diffusion in addition with the exchange between heat and mass in the nature.

In 1974, Nowacki [8-11] developed the theory of coupled thermoelastic diffusion. The generalized theory in thermoelastic diffusion was introduced by Sherief et al. [12] in 2004. Again, in the year 2005, Sherief et al. [13] studied a half space problem in the theory of generalized thermoelastic diffusion. The influence of diffusion on generalized thermoelastic problems of infinite body with a cylindrical cavity studied by Rong-hou et al. [14]. Singh [15, 16] in his couple of papers discussed the reflection of waves from the free surface in generalized thermoelastic diffusion. In recent times Kumar and Kansal [17, 18] studied about the Rayleigh and Lamb wave propagation on free surface in transversely isotropic thermoelastic diffusion. Sharma et al. [19-20] studied on thermodiffusive surface wave propagation in heat conducting materials and Kumar et al. [21] discussed on the plane strain deformation in generalized thermoelastic diffusion in 2007-2008.

The linear theory of micropolar thermoelasticity has been developed by extending the theory of micropolar elasticity including thermal effect by Eringen [22, 23] and Nowacki [24, 25]. Minagawa et al. [26] discussed the propagation of plane harmonic waves in a cubic micropolar medium. Kumar and Rani [27] studied time harmonic sources in a thermally conducting cubic crystal and Mechanical/ thermal sources in a micropolar thermoelastic medium with cubic symmetry by Kumar and Aliawalia [28]. In the year 2006, Kumar and Aliawalia [29] studied on deformation due to time harmonic sources in micropolar thermoelastic medium with two relaxation times.

Fractional calculus has been used successfully to modify many existing model of physical process. In the formulation of tautochrone problem, Abel applied fractional calculus to solve integral equation and that was first application of fractional derivatives. Using fractional derivatives, for the description of viscoelastic materials, Caputo [30], Caputo and Mainardi [31] found an agreement between the experimental results with theoretical one. Recently, Sherief et al. [32] introduced

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the fractional ordered thermoelasticity by using Caputo's [30] Phenomenon.

In the present investigation we are concerned about the interaction between thermoelastic diffusion with micropolar-elasticity by using fractional derivatives.

Nomenclature:

U : Internal energy per unit mass, K : Kinetic energy per unit mass, L : Power of external force, Q: Heat absorbed by the material body, W: Quantity of heat generated in unit time in unit volume, $\vec{q}$: Heat flux vector, $\vec{F}$: External body force, $\vec{l}$: Body Couple, σ_l : Microstretch rotatory inertia, ρ : Constant mass density of the medium, $\vec{u}$: Displacement vector, $\vec{\xi}$: Micro-displacement vector, $\vec{\phi}$: Microrotation vector, j : Microinertia, $\vec{n}$: Outward drawn normal vector, t_{ij} : Stress tensor, m_{ij} : Coupled stress tensor, ε_{ij} : Micropolar strain tensor, $\upsilon_i : \dot{\phi}_i$, time rate change of microrotation component, $\gamma_{ij} : \phi_{i,j}$, microrotation tensor, $\vec{\eta}$: Flow of diffusion mass vector, T : Absolute temperature, S : Entropy per unit mass, P : Chemical potential per unit mass, C : Concentration, λ, μ : Lame' constants, $\alpha, \beta, \gamma, \kappa$: Micropolar elastic constants, $\beta_1 = (3\lambda + 2\mu + \kappa)\alpha_t$, $\beta_2 = (3\lambda + 2\mu + \kappa)\alpha_c$, α_t : Coefficient of linear thermal expansion, α_c : Coefficient of linear diffusion expansion, C_E : Specific heat at constant strain, a : Measure of thermodiffusion effects, b : Measure of thermodiffusive effects, υ : Thermal relaxation time, τ : Diffusion relaxation time, D : Thermoelastic diffusion constant, K : Coefficient of thermal conductivity, ε_{ijk} : Permutation tensor, δ_{ij} : Kronecker delta.

II. FUNDAMENTAL EQUATIONS

The Riemann-Liouville fractional integral is introduced as a natural generalization of the convolution type integral [33, 34, 35, 36, 37],

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad (\alpha > 0) \qquad (2.1)$$

The Laplace transform for this fractional integral is defined by,

$$L\left[I^{\alpha} f(t)\right] = \frac{1}{s^{\alpha}} L\left[f(t)\right]. \qquad (2.2)$$

The Riemann-Liouville derivative of fractional of fractional order α is defined as the left-inverse of the fractional integral I^{α} as

$$D_{RL}^{\alpha} f(t) = D^{n} I^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau \quad , \quad n-1 < \alpha < n \qquad (2.3)$$

and for Laplace transform, the initial values of the fractional integral $I^{n-\alpha} f(t)$ and its derivatives of order $k = 1,2,3,\ldots,n-1$ are required, where

$$L\left[D_{RL}^{\alpha} f(t)\right] = s^{\alpha} L\left[f(t)\right] - \sum_{k=0}^{n-1} s^{n-1-k} D^{k} I^{n-\alpha} f(0), \quad n-1 < \alpha < n. \qquad (2.4)$$

An alternative definition of fractional derivative was proposed by Caputo [30] as,

$$D_{C}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^{n}}{dt^{n}} f(\tau)\, d\tau, \quad n-1 < \alpha < n \qquad (2.5)$$

and for the Laplace transform, this definition has an advantage, the initial values of $f(t)$ and its integer derivatives of order $k = 1,2,3,\ldots,n-1$ are required unlike the fractional ordered derivatives given by the equation (2.3), so that

$$L\left[D_{C}^{\alpha} f(t)\right] = s^{\alpha} L\left[f(t)\right] - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0), \quad n-1 < \alpha < n \qquad (2.6)$$

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Here we shall use the fractional derivatives of order $\alpha \in (0,1]$, according to Caputo [30].

Let V be any arbitrary volume element of a material body bounded by the closed surface A. Now the first law of thermoelasticity, the law of conservation of energy for the volume element V can be written in the following form:

$$\frac{d}{dt}\int_V \rho(\mathrm{U}+\mathrm{K}) = \mathrm{L} + \frac{d\mathrm{Q}}{dt} \tag{2.7}$$

Here,

$$\mathrm{L} = \int_V \rho\left(F_i\,\dot{u}_i + l_i\,\upsilon_i\right)dV + \int_A \left(t_{ji}\,\dot{u}_i + m_{ji}\upsilon_i\right)n_j\,dA \tag{2.8}$$

$$\dot{\mathrm{Q}} = -\int_A \mathrm{q}_i n_i\,dA + \int_V W dV \tag{2.9}$$

Now the equations of the motion are as follows:

$$t_{ji,j} + \rho F_i = \rho \ddot{u}_i \tag{2.10}$$

$$m_{kl,k} + \varepsilon_{lmn} t_{mn} + \rho l_l = \rho\sigma_l \tag{2.11}$$

where $\sigma_k = j_{kl}\dot{\upsilon}_l + \varepsilon_{klm}\, j_{mn}\upsilon_l\upsilon_n$

After using divergence theorem and the equations of motion (2.10), (2.11) we obtain from the equation (2.7),

$$\frac{d}{dt}\int_V \rho(\mathrm{U}+\mathrm{K}) = \int_V \left[\frac{d}{dt}(\rho\mathrm{K}) + t_{ji}\left(u_{i,j} + \varepsilon_{ijk}\upsilon_k\right) + m_{ji}\upsilon_{i,j} + \rho\varepsilon_{ijk}\, j_{kl}\,\upsilon_i\upsilon_j\upsilon_l - q_{j,j}\right]dV \tag{2.12}$$

Now for micropolar thermoelastic solids, the deformations, micro-rotations and the changes of temperature are very small in natural state of the body, so a linear approximation is possible. For linear approximation we introduce, as was done by Eringen [23], the norm function of the array

$$W = \left(\vec{u}_{,i}, \vec{\phi}, \vec{\phi}_{,i}, \theta, \theta_{,i}\right)$$

by $\varepsilon = \|W\| = (W\cdot W)^{1/2}$

and for some neighborhood of $\varepsilon = 0$, there exist positive constants k and n such that

$$\left\|\mathrm{O}\left(\varepsilon^n\right)\right\| \le k\,\varepsilon^n \quad , k > 0\,, n > 0.$$

Now, for linear (1st order) approximation equation (2.12) can be written in the following form:

$$\rho\dot{\mathrm{U}} = \mathrm{t}_{ji}\dot{\varepsilon}_{ji} + m_{ji}\dot{\gamma}_{ij} - q_{i,i} + W \tag{2.13}$$

Where,

$$\varepsilon_{ji} = u_{i,j} + \varepsilon_{ijk}\phi_k \tag{2.14}$$

Now we consider the entropy balance law,

$$\rho T\dot{S} = -q_{i,i} + P\eta_{i,i} + W \tag{2.15}$$

The equation of conservation of mass:

$$\eta_{i,i} = -\dot{C} \tag{2.16}$$

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where C is the concentration.
Now using the equations (2.15) and (2.16), from the equation (2.13) we obtain,

$$\rho\, d\mathrm{U} = \mathrm{t}_{ji} d\varepsilon_{ji} + m_{ji} d\gamma_{ij} + \rho T dS + P dC \tag{2.17}$$

For further proceedings, we introduce the helmholtz free energy function Ψ given by,

$$\Psi = \mathrm{U} - \mathrm{TS} \tag{2.18}$$

Now using the equation (2.17), the total differential of the function Ψ can be written as,

$$\rho\, d\Psi = t_{ji} d\varepsilon_{ji} + m_{ji} d\gamma_{ij} + P dC - \rho\, S dT \tag{2.19}$$

Here the function Ψ and all other functions under consideration can be expressed in terms of the independent variables $\varepsilon_{kl}, \gamma_{kl}, C, T$. Now by chain rule we get,

$$\rho\, d\Psi = \rho \frac{\partial \Psi}{\partial \varepsilon_{ji}} d\varepsilon_{ji} + \rho \frac{\partial \Psi}{\partial \gamma_{ij}} d\gamma_{ij} + \rho \frac{\partial \Psi}{\partial T} dT + \rho \frac{\partial \Psi}{\partial C} dC \tag{2.20}$$

Comparing the equations (2.19) and (2.20) we obtain,

$$t_{ji} = \rho \frac{\partial \Psi}{\partial \varepsilon_{ji}} \tag{2.21}$$

$$m_{ji} = \rho \frac{\partial \Psi}{\partial \gamma_{ij}} \tag{2.22}$$

$$P = \rho \frac{\partial \Psi}{\partial C} \tag{2.23}$$

$$S = -\frac{\partial \Psi}{\partial T} = -\frac{\partial \Psi}{\partial \theta} \tag{2.24}$$

Now expanding the function Ψ into the Taylor series in terms of independent variables about its natural state, we obtain,

$$\rho\Psi = \rho\Psi_0 + a_0\theta + A_{ji}\varepsilon_{ji} + b_0 C + B_{ij}\gamma_{ij} - \frac{\rho C_E}{2T_0}\theta^2 + \frac{1}{2}bC^2 + \frac{1}{2}A_{jilk}\varepsilon_{ji}\varepsilon_{lk} + a_{ji}\varepsilon_{ji}\theta + b_{ji}\varepsilon_{ji}C - a\theta C$$

$$+\frac{1}{2}B_{ijkl}\gamma_{ij}\gamma_{kl} + c_{ij}\gamma_{ij}\theta + d_{ij}\gamma_{ij}C + C_{jikl}\varepsilon_{ji}\gamma_{kl} + \ldots\ldots\ldots \tag{2.25}$$

where $\theta = T - T_0,\ T_0$ is the temperature of the medium in natural state such that $\left|\dfrac{\theta}{T_0}\right| << 1$.

In the natural state of the material body, we consider,

$$\Psi = 0,\ \ \theta = 0,\ \ t_{ij} = 0,\ \ C = 0,\ \ \varepsilon_{ij} = 0,\ \ \gamma_{ij} = 0.$$

and we obtain,

$$\Psi_0 = 0,\ a_0 = 0,\ A_{ji} = 0,\ b_0 = 0,\ \ B_{ij} = 0.$$

Now equation (2.25) can be recast in following form keeping only second order terms,

$$\rho\Psi = -\frac{\rho C_E}{2T_0}\theta^2 + \frac{1}{2}bC^2 + \frac{1}{2}A_{jilk}\varepsilon_{ji}\varepsilon_{lk} + a_{ji}\varepsilon_{ji}\theta + b_{ji}\varepsilon_{ji}C - a\theta C + \frac{1}{2}B_{ijkl}\gamma_{ij}\gamma_{kl} + c_{ij}\gamma_{ij}\theta + d_{ij}\gamma_{ij}C$$

$$+ C_{jikl}\varepsilon_{ji}\gamma_{kl}\,. \tag{2.26}$$

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Now from equations (2.21) – (2.24) and equation (2.26) we obtain,

$$t_{ij} = A_{ijkl}\varepsilon_{kl} + a_{ij}\theta + b_{ij}C + C_{ijlk}\gamma_{lk} \tag{2.27}$$

$$m_{ji} = B_{ijkl}\gamma_{kl} + c_{ij}\theta + d_{ij}C + C_{jikl}\varepsilon_{lk} \tag{2.28}$$

$$P = bC + b_{ji}\varepsilon_{ji} - a\theta + d_{ij}\gamma_{ij} \tag{2.29}$$

$$\rho S = \frac{\rho C_E}{T_0}\theta - a_{ji}\varepsilon_{ji} + aC - c_{ij}\gamma_{ij} \tag{2.30}$$

Now, in isotropic solids, from the equation (2.26) it has been seen that, since free energy is a scalar quantity, each terms of right hand side of equation (2.26) are scalars and hence,

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + (\mu+\kappa)\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk}\ , B_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{il}\delta_{jk} + \gamma\delta_{ik}\delta_{jl}\ , C_{ijkl} = 0\ ,$$
$$a_{ij} = -\beta_1\delta_{ij}\ , b_{ij} = -\beta_2\delta_{ij}\ , c_{ij} = 0\ .$$

Diffusion is a random walk of the particles of the material body from higher concentrated regions to lower concentrated regions. Here d_{ij} represents the coupling coefficients between γ_{ij} and the concentration C of the body, in an isotropic solid the gradient of microrotation and the concentration are independent. So we take $d_{ij} = 0$ in isotropic body. In addition, the material is called spin-isotropic if $j_{\alpha\beta} = j\delta_{\alpha\beta}$.

Hence, in isotropic solids constitutive equations are,

$$t_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + (\mu+\kappa)\varepsilon_{ij} + \mu\varepsilon_{ji} - \beta_1\theta\delta_{ij} - \beta_2 C\delta_{ij} \tag{2.31}$$

$$m_{ij} = \alpha\gamma_{kk}\delta_{ij} + \beta\gamma_{ij} + \gamma\gamma_{ji} \tag{2.32}$$

$$P = bC - \beta_2\varepsilon_{kk} - a\theta \tag{2.33}$$

$$\rho T_0 S = \rho C_E\theta + \beta_1 T_0\varepsilon_{kk} + aT_0 C \tag{2.34}$$

Now, the linear equations of balance law are given by,

$$t_{\beta\alpha,\beta} + \rho F_\alpha = \rho\ddot{u}_\alpha \tag{2.35}$$

$$m_{\beta\alpha,\beta} + \varepsilon_{\alpha mn}t_{mn} + \rho l_\alpha = \rho j\ddot{\phi}_\alpha \tag{2.36}$$

The linearized form of heat conduction is,

$$\rho T\dot{S} = -q_{i,i} + W \tag{2.37}$$

and without contradiction with the second law of thermodynamics, we assume a generalized form of Fourier's law of heat conduction equation as,

$$\left(1 + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)q_i = -K\theta_{,i} \tag{2.38}$$

where α is a constant such that $0 \le \alpha \le 1$.

Now using the equations (2.34) and (2.37), from the equation (2.38) we obtain,

$$K\theta_{,ii} = \rho C_E\left(1 + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial\theta}{\partial t} + \beta_1 T_0\left(1 + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial\varepsilon_{kk}}{\partial t}\varepsilon_{kk} + aT_0\left(1 + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial C}{\partial t} - \left(1 + \tau_0\frac{\partial^\alpha}{\partial t^\alpha}\right)W \tag{2.39}$$

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We may consider the equation (2.39) is an extended version of fractional ordered heat conduction equation in isotropic, micropolar, thermodiffusive elastic solids.

Now we consider the equation of mass flux vector,

$$\left(1+\tau_1\frac{\partial^\alpha}{\partial t^\alpha}\right)\eta_i = -DP_{,i} \tag{2.40}$$

Now using the equation (2.16) and (2.33), from the equation (2.40) we obtain,

$$DbC_{,ii} = D\beta_2\nabla^2\varepsilon_{kk} + Da\theta_{,ii} + \left(1+\tau_1\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial C}{\partial t} \tag{2.41}$$

This equation may be considered as the fractional ordered generalized mass diffusion equation in isotropic, micropolar elastic solids.

Special cases :

When $\alpha \to 1$, equations (2.39) and (2.41) reduces to

$$K\theta_{,ii} = \rho C_E\left(\dot{\theta}+\upsilon\ddot{\theta}\right) + \beta_1 T_0\left(\frac{\partial}{\partial t}+\upsilon\frac{\partial^2}{\partial t^2}\right)\varepsilon_{kk} + aT_0\left(\dot{C}+\upsilon\ddot{C}\right) - \left(1+\upsilon\frac{\partial}{\partial t}\right)W \tag{2.42}$$

$$DbC_{,ii} = D\beta_2\nabla^2\varepsilon_{kk} + Da\theta_{,ii} + \left(\frac{\partial}{\partial t}+\tau\frac{\partial^2}{\partial t^2}\right)C \tag{2.43}$$

These are the generalized heat conduction and mass diffusion equations in isotropic, micropolar elastic solids.

Again, when $\alpha \to 1,\ \kappa \to 0$ equations (2.39), (2.41) transform to

$$K\theta_{,ii} = \rho C_E\left(1+\upsilon\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial t} + \beta_1 T_0\left(1+\upsilon\frac{\partial}{\partial t}\right)\left(\vec{\nabla}\cdot\dot{\vec{u}}\right) + aT_0\left(1+\upsilon\frac{\partial}{\partial t}\right)\frac{\partial C}{\partial t} - \left(1+\upsilon\frac{\partial}{\partial t}\right)W \tag{2.44}$$

$$DbC_{,ii} = D\beta_2\nabla^2\left(\vec{\nabla}\cdot\vec{u}\right) + Da\theta_{,ii} + \left(1+\tau\frac{\partial}{\partial t}\right)\frac{\partial C}{\partial t} \tag{2.45}$$

Equations (2.44) and (2.45) represent the heat conduction and mass diffusion equations for isotropic elastic solids, as was done by Sherief et al. [12].

III. A VARIATIONAL PRINCIPLE

The variational theorem in classical thermoelasticity first derived by Biot [38] and explained their applications by means of several examples. The variational theorem on classical thermoelastic diffusion was done by Sherief et al. [12]. In micropolar thermoelasticity the variational principle and uniqueness theorem was done by Eringen [22, 23]. Recently, a variational principle of fractional order generalized thermoelasticity was done by Youssef and Al-Lehaibi [39]. Now we shall present a compact derivation of a variational principle on fractional-ordered micropolar thermoelastic diffusion.

We consider,

$$\boldsymbol{w} = \frac{1}{2}\int_V \left[A_{klmn}\varepsilon_{kl}\varepsilon_{mn} + B_{klmn}\gamma_{kl}\gamma_{mn}\right]dV\,, \tag{3.1}$$

where the integrand is homogeneous quadratic form of strain tensor and microrotation tensors.

Now we consider a virtual displacement i.e. for a neighboring state in which the displacement, strain tensor, microrotation tensors are changed by the quantities δu_i , $\delta\varepsilon_{ij}$, $\delta\gamma_{ij}$ respectively, we obtain,

$$\delta\,\boldsymbol{w} = \frac{1}{2}\int_V \left[A_{klmn}\left(\delta\varepsilon_{kl}\right)\varepsilon_{mn} + A_{klmn}\varepsilon_{kl}\left(\delta\varepsilon_{mn}\right) + B_{klmn}\left(\delta\gamma_{kl}\right)\gamma_{mn} + B_{klmn}\gamma_{kl}\left(\gamma_{mn}\right)\right]dV \tag{3.2}$$

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Now using the constitutive equations;

$$t_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + (\mu + \kappa)\varepsilon_{ij} + \mu \varepsilon_{ji} - \beta_1 \theta \delta_{ij} - \beta_2 C \delta_{ij} \tag{3.3}$$

$$m_{ij} = \alpha \gamma_{kk} \delta_{ij} + \beta \gamma_{ij} + \gamma \gamma_{ji} \tag{3.4}$$

Taking into account the equations of motion;

$$t_{\beta\alpha,\beta} + \rho F_\alpha = \rho \ddot{u}_\alpha \tag{3.5}$$

$$m_{\beta\alpha,\beta} + \varepsilon_{\alpha mn} t_{mn} + \rho l_\alpha = \rho j \ddot{\phi}_\alpha \tag{3.6}$$

Corresponding boundary conditions;

$$p_i = t_{ji} n_j \tag{3.7}$$

$$m_i = m_{ji} n_j \tag{3.8}$$

and using divergence theorem, we obtain,

$$\int_V \rho F_i \, \delta u_i \, dV + \int_A p_i \, \delta u_i \, dA + \int_V \rho l_i \, \delta\phi_i \, dV + \int_A m_i \, \delta\phi_i \, dA - \int_V \rho \ddot{u}_i \, \delta u_i \, dV - \rho j \int_V \ddot{\phi}_i \, \delta \phi_i \, dV$$

$$= \delta \mathrm{W} - \beta_1 \int_V \theta \, \delta \varepsilon_{kk} \, dV - \beta_2 \int_V C \, \delta\varepsilon_{kk} \, dV \tag{3.9}$$

This is the first variational equation and it would be complete for uncoupled thermoelasticity if the temperature θ and the concentration C in last two integrations of the right hand side of the equation (3.9) were known. Taking into account the coupling between the strain field, temperature and concentration it is observed that θ and C are unknown. Hence it is necessary to introduce other relations considering the phenomena of heat conduction as well as mass diffusion.

Now we introduce one vector $\vec{H}$ as was done by Biot [40], related with entropy by,

$$\rho S = -div\vec{H} = -H_{i,i} \tag{3.10}$$

Again we know the relations,

$$\rho T_0 S = \rho C_E \theta + \beta_1 T_0 \varepsilon_{kk} + a T_0 C \tag{3.11}$$

$$q_{i,i} = -\rho T_0 \dot{S} \tag{3.12}$$

$$\left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right) q_i = -K\theta_{,i} \tag{3.13}$$

Now using the equation (3.10), from the relations (3.11), (3.12) and (3.13) we obtain,

$$-H_{i,i} = \frac{\rho C_E}{T_0}\theta + \beta_1 \varepsilon_{kk} + aC \tag{3.14}$$

$$q_i = T_0 \dot{H}_i \tag{3.15}$$

$$T_0 \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) H_i = -K\theta_{,i} \tag{3.16}$$

Now multiplying both sides of equation (3.16) by δH_i and then integrating over the region V of the body we obtain,

$$\int_V \left[\theta_{,i} + \frac{T_0}{K}\left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right) H_i \right] \delta H_i \, dV = 0 \tag{3.17}$$

Now,

$$\int_V \theta_{,i} \, \delta H_i \, dV = \int_A \theta n_i \, \delta H_i \, dA + \frac{\rho C_E}{T_0} \int_V \theta \, \delta \theta \, dV + \beta_1 \int_V \theta \, \delta \varepsilon_{kk} \, dV + a \int_V \theta \, \delta C \, dV \tag{3.18}$$

Using the equation (3.18), from the equation (3.17) we obtain,

$$\int_A \theta n_i \, \delta H_i \, dA + \frac{\rho C_E}{T_0} \int_V \theta \, \delta \theta \, dV + \beta_1 \int_V \theta \, \delta \varepsilon_{kk} \, dV + a \int_V \theta \, \delta C \, dV + \frac{T_0}{K} \int_V \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right) H_i \delta H_i \, dV = 0 \tag{3.19}$$

Here we introduce heat potential function,

$$IP = \frac{\rho C_E}{2T_0} \int_V \theta^2 \, dV \tag{3.20}$$

And
$$\delta IP = \frac{\rho C_E}{T_0} \int_V \theta \, \delta \theta \, dV \tag{3.21}$$

Again, the heat dissipation function ID, where

$$\delta ID = \frac{T_0}{K} \int_V \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right) H_i \, \delta H_i \, dV \tag{3.22}$$

Therefore, from equation (3.19) we obtain,

$$\int_A \theta n_i \, \delta H_i \, dA + \delta(IP + ID) + \beta_1 \int_V \theta \, \delta \varepsilon_{kk} \, dV + a \int_V \theta \, \delta C \, dV = 0 \tag{3.23}$$

This is the second variational equation connected with heat conduction.

In order to obtain the variational equation connected with the process of thermodiffusion we introduce another vector as was done by Sherief et al [12] related with mass concentration by,

$$C = -div\vec{G} = -G_{i,i} \tag{3.24}$$

Now from the relations,

$$P = bC - \beta_2 \varepsilon_{kk} - a\theta \,, \tag{3.25}$$

$$\eta_{i,i} = -\dot{C} \tag{3.26}$$

$$\left(1 + \tau_1 \frac{\partial^\alpha}{\partial t^\alpha}\right) \eta_i = -DP_{,i} \tag{3.27}$$

and using the equation (3.24) we obtain,

$$\eta_i = \dot{G}_i \tag{3.28}$$

$$\left(\frac{\partial}{\partial t} + \tau_1 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\right) G_i = -DP_{,i} \tag{3.29}$$

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Now multiplying both sides of equation (3.29) by δG_i and then integrating over the region V we obtain,

$$\int_V \left[P_{,i} + \frac{1}{D}\left(\frac{\partial}{\partial t} + \tau_1 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) G_i \right] \delta G_i \, dV = 0 \tag{3.30}$$

Now,

$$\int_V P_{,i}\, \delta G_i \, dV = \int_A P\, n_i \, \delta G_i \, dA + b \int_V C\, \delta C \, dV - \beta_2 \int_V \varepsilon_{kk} \, \delta C \, dV - a \int_V \theta \, \delta \, C \, dV \tag{3.31}$$

Using the equation (3.31), from the equation (3.30) we obtain,

$$\int_A P\, n_i \, \delta G_i \, dA - \beta_2 \int_V \varepsilon_{kk} \, \delta C \, dV + b \int_V C\, \delta C \, dV - a \int_V \theta \, \delta \, C \, dV + \frac{1}{D} \int_V \left(\frac{\partial}{\partial t} + \tau_1 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) G_i \, \delta G_i \, dV = 0 \tag{3.32}$$

Now we define, the diffusion potential function,

$$IP' = \frac{b}{2} \int_V C^2 \, dV \tag{3.33}$$

And

$$\delta IP' = b \int_V C\, \delta C \, dV \tag{3.34}$$

Again, the diffusive dissipation function ID', where

$$\delta ID' = \frac{1}{D} \int_V \left(\frac{\partial}{\partial t} + \tau_1 \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \right) G_i \, \delta G_i \, dV \tag{3.35}$$

Therefore, from the equation (3.32) we obtain,

$$\int_A P\, n_i \, \delta G_i \, dA - \beta_2 \int_V \varepsilon_{kk} \, \delta C \, dV - a \int_V \theta \, \delta \, C \, dV + \delta\left(IP' + ID' \right) = 0 \tag{3.36}$$

Equation (3.36) represents the third variational equation.

Now from the equations (3.9), (3.23) and (3.36) we obtain,

$$\begin{aligned} \delta\Big[\; \mathcal{W} + IP + IP' + ID + ID' - \beta_2 \int_V C\, \varepsilon_{kk} \, dV \; \Big] &= -\int_A \theta \, n_i \, \delta H_i \, dA - \int_A P\, n_i \, \delta G_i \, dA + \int_V \rho F_i \, \delta \, u_i \, dV \\ &+ \int_A p_i \, \delta \, u_i \, dA + \int_V \rho \, l_i \, \delta \, \phi_i \, dV + \int_A m_i \, \delta \, \phi_i \, dA - \rho \int_V \ddot{u}_i \, \delta \, u_i dV - \rho j \int_V \ddot{\phi}_i \, \delta \, \phi_i \, dV \end{aligned} \tag{3.37}$$

Now, $\delta \, u_i = \dot{u}_i \, dt$, $\delta \, \theta = \dot{\theta} \, dt$, $\delta \, H_i = \dot{H}_i \, dt$, $\delta \, G_i = \dot{G}_i \, dt$ etc.

Therefore, equation (3.37) reduces to,

$$\begin{aligned} \frac{d}{dt}\Big[\; \mathcal{W} + \mathrm{K} + IP + IP' + ID + ID' + \frac{a}{b} \int_V P\, \theta \, dV \; \Big] &= \int_V \rho F_i \, \dot{u}_i \, dV + \int_V \rho \, l_i \, \dot{\phi}_i \, dV + \int_A p_i \, \dot{u}_i \, dA + \int_A m_i \, \dot{\phi}_i \, dA \\ &+ \frac{K}{T_0} \int_A \theta \frac{\partial \theta}{\partial n} dA + \int_A P \frac{\partial \eta_i}{\partial n} dA \end{aligned} \tag{3.38}$$

where the mass concentration C is written in terms of the chemical potential P.

IV. UNIQUENESS THEOREM

Uniqueness theorem states that there is only one solution of the equations (2.25), (2.36) (2.39) and (2.41) subject to the boundary conditions;

$$p_i = t_{ji}\, n_j\,,\ m_i = m_{ji}\, n_j\,,\ \theta = \theta_1(x,t)\,,\ P = P_1(x,t) \quad , \quad x \in A,\ t > 0$$

and initial conditions;

$$u_i(x,0) = u_{i0}(x),\ \dot{u}_i(x,0) = \dot{u}_{i0}(x),\ \phi_i(x,0) = \phi_{i0}(x),\ \dot{\phi}_i(x,0) = \dot{\phi}_{i0}(x),\ \theta(x,0) = \theta_0(x)$$

$$P(x,0) = P_0(x)\,,\ \dot{P}(x,0) = \dot{P}_0(x)\,,\ x \in V,\ t = 0\,,$$

where the body occupying the region V bounded by the surface A.

Proof : We consider, if possible, there exist two sets of solutions (u_i, ϕ_i, θ, P) and $(u'_i, \phi'_i, \theta', P')$.

We take, $\hat{u}_i = u_i - u'_i$, $\hat{\phi}_i = \phi_i - \phi'_i$, $\hat{\theta}_i = \theta_i - \theta'_i$, $\hat{P} = P - P'$

Hence the solution $(\hat{u}_i, \hat{\phi}_i, \hat{\theta}, \hat{P})$ must satisfy the equations of motion, heat conduction equation and equation of mass diffusion with no body forces, body couples and without heat sources term i.e.,

$$t_{\beta\alpha,\beta} = \rho \ddot{u}_\alpha \tag{4.1}$$

$$m_{\beta\alpha,\beta} + \varepsilon_{\alpha mn} t_{mn} = \rho j \ddot{\phi}_\alpha \tag{4.2}$$

$$K\theta_{,ii} = \rho C_E \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right) \frac{\partial \theta}{\partial t} + \beta_1 T_0 \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right) \frac{\partial \varepsilon_{kk}}{\partial t} \varepsilon_{kk} + a T_0 \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right) \frac{\partial C}{\partial t} \tag{4.3}$$

$$DbC_{,ii} = D\beta_2 \nabla^2 \varepsilon_{kk} + Da\theta_{,ii} + \left(1 + \tau_1 \frac{\partial^\alpha}{\partial t^\alpha}\right) \frac{\partial C}{\partial t} \tag{4.4}$$

with homogeneous boundary and initial conditions.

Thus we arrive with a system for which the displacement $\hat{u}_i$, microrotation $\hat{\phi}_i$, temperature $\hat{\theta}$ and the chemical potential $\hat{P}$ are vanishes inside the body initially and the surface traction $\hat{p}_i$, surface couple $\hat{m}_i$, temperature $\hat{\theta}$ and the chemical potential $\hat{P}$ are vanishes on surface A.

Hence, it is enough to prove that the measure of strain tensor, microrotation tensor, temperature and the chemical potential are vanishes inside the body.
Now from the equation (3.38) we obtain,

$$\frac{d}{dt}\left[\ \mathrm{W} + \mathrm{K} + IP + IP' + ID + ID' + \frac{a}{b}\int_V P\theta\, dV\ \right] = 0 \quad \text{or,}$$

$$\frac{d}{dt}\left[\ \mathrm{W} + \mathrm{K} + IP + IP'\ \right] = -\frac{a}{b}\int_V P\theta\, dV - \frac{T_0}{K}\int_V \dot{H}_i \left(1 + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha}\right) \dot{H}_i\, dV - \frac{1}{D}\int_V \dot{G}_i \left(1 + \tau_1 \frac{\partial^\alpha}{\partial t^\alpha}\right) \dot{G}_i\, dV \le 0 \tag{4.5}$$

The integral in the left hand side of equation (4.5) is zero initially. On the other hand the inequality proves that the left hand side of the equation is either zero or decreases taking negative values. Since the integrand is a sum of squares and vanishes at $t = 0$, therefore, first possibility holds.
Hence, $\mathcal{W} + \mathrm{K} + IP + IP' = 0$,

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and $\hat{u}_i = 0$, $\hat{\dot{\phi}}_i = 0$, $\hat{\theta} = 0$, $\hat{P} = 0$, $\hat{\varepsilon}_{ij} = 0$, $\hat{\gamma}_{ij} = 0$ for $t \geq 0$.

Since $\hat{t}_{ij}$ and $\hat{m}_{ij}$ are the linear functions of $\hat{\varepsilon}_{ij}$, $\hat{\gamma}_{ij}$, $\hat{\theta}$ and $\hat{P}$ which are zero for $t \geq 0$,

therefore, $\hat{t}_{ij} = 0 = \hat{m}_{ij}$, $t \geq 0$.

Hence, $\hat{u}_i = \hat{\phi}_i = \hat{\theta} = \hat{P} = 0$ for $t \geq 0$.

This completes the proof of uniqueness theorem.

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