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Homotopy Analysis Method to Solve the Multi-Order Fractional Differential Equations

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Homotopy Analysis Method to Solve the Multi-Order Fractional Differential Equations

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I. Introduction

In recent years, fractional differential equations have attracted many researchers [7-9] due to their very important applications in Physics, Science and Engineering such as damping law, rheology, diffusion process, description of fractional random walk and so on. Most fractional differential equations do not have exact solutions, so approximation and numerical techniques must be used, such as Laplace transform method [10], Adomian decomposition method [6,11,12], Variational iteration method [13,14], Homotopy perturbation method [15,16], Hamotopy analysis method [1,17,18] and so on. The homotopy analysis method (HAM) was first proposed by Liao [1] in his Ph.D. Thesis. This method (HAM) given in Liao [17] also provides a systematic and an effective procedure for explicit and numerical solutions of a wide and general class of differential equations system representing real physical and engineering problems.

In this paper, the homotopy analysis method (HAM) Liao [1] is applied to solve multi-order fractional differential equations studied by Diethelm and Ford [2]. We also present an algorithm to convert the multi-order fractional differential equation into a system of fractional differential equations without putting any of the restrictions. This algorithm is valid in the most general case and yields fewer number of equations in a system compared to those in Diethelm-Ford algorithm. In last the solutions of the system of FDE have been obtained by applying the Homotopy analysis method.

II. Some Basic Definitions

Definition 2.1:

A real function $F(x), x > 0$ is said to be in space $c_\mu, \mu \in R$ is there exists a real number p ($>\mu$) such that $F(x) = x^p F1(x)$ where $F1(x)$ $c[0,\infty]$ and it is said to be in the space C_μ^m iff $F^{(m)} \in c_\mu, m \in N.$

Definition 2.2:

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $F \in c_\mu, \mu \geq$, is defined as

$$J^\alpha F(x) = \frac{1}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} F(t)\, dt \ ; \ \alpha > 0, x > 0 \tag{1}$$

$$J^0 F(x) = F(x)$$

Properties of the operator J^α can be found in [10,19] we mentioned only the following

$$\text{(i)} \quad J^\alpha J^\beta = J^{\alpha+\beta} \qquad \text{(ii)} \ J^\alpha J^\beta = J^\beta J^\alpha \tag{2}$$

$$\text{(iii)} \quad J^\alpha x^\gamma = \frac{\Gamma\gamma+1}{\Gamma\alpha+\gamma+1} x^{\alpha+\gamma}$$

For $F \in c_\mu, \ \mu \geq -1, \ \alpha,\beta \geq 0$ and $\gamma > -1.$

Definition 2.3 :

The Fractional derivative of F(x) in the Caputo sense is defined as

$$D^\alpha F(x) = J^{n-\alpha} D^n F(x) = \frac{1}{\Gamma n-\alpha} \int_0^x (x-t)^{n-\alpha-1} F^{(n)}(t)\, dt \tag{3}$$

For

$$x > 0, \ n-1 < \alpha < n, \ n \in N, \ x > 0, \ F \in c_{-1}^{\mu} \tag{4}$$

Caputo's fractional derivative has a useful property [19]

$$J^\alpha [D^\alpha F(x)] = F(x) - \sum_{k=0}^{n-1} F^{(k)}(0^+) \frac{x^k}{k!} \qquad (n-1 < \alpha < n) \tag{5}$$

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For the Caputo's derivative

$$D^{\alpha}(c)=0$$

$$D^{\alpha}x^{\beta}=\begin{cases}0 & ;\ \beta<\alpha-1\\ \dfrac{\Gamma\beta+1}{\Gamma\beta-\alpha-1}x^{\beta-\alpha} & ;\ \beta>\alpha-1\end{cases} \qquad (6)$$

$$D^{\beta}J^{\alpha}F(x)=\begin{cases}J^{\alpha-\beta}F(x) & ;\ \alpha>\beta\\ F(x) & ;\ \alpha=\beta\\ D^{\beta-\alpha}F(x) & ;\ \alpha<\beta\end{cases} \qquad (7)$$

Caputo's fractional derivative is linear operator, similar to integer order derivative

$$D^{\alpha}[a\,f(x)+b\,g(x)]=a\,D^{\alpha}f(x)+b\,D^{\alpha}g(x) \qquad (8)$$

where a and b are constants. Also this operator satisfies the so-called Leibnitz rule.

$$D^{\alpha}[g(x)\,f(x)]=\sum_{k=0}^{\infty}\binom{\alpha}{k}g^{(k)}(x)\,D^{\alpha-k}f(x) \qquad (9)$$

For n to be the smallest integer that exceeds α, the Caputo space fractional derivative operator of order $\alpha>0$ is defined as

$$D_t^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{dt^{\alpha}}=\begin{cases}\dfrac{1}{\Gamma n-\alpha}\displaystyle\int_0^t(t-\tau)^{n-\alpha-1}\frac{\partial^{n}u(x,\tau)}{\partial\tau^{n}}d\tau & n-1<\alpha<n\\ \dfrac{\partial^{n}u(x,t)}{\partial t^{n}} & \alpha=n\in N,\end{cases} \qquad (10)$$

For the purpose of this article, the Caputo's definition of fractional differentiation will be used.

Definition 2.4 : The Mittag-Leffler function $E_{\alpha}(z)$ with $\alpha>0$ is defined the following series representation valid in the whole complex plane [3] .

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma\alpha n+1},\quad \alpha>0,\ z\in C.$$

Lemma 2.5. Diethelm and Ford [4]. Let $Y(t)\in c^{k}(o,t)$ for some $T>0$ and $K\in IN$ and let $q\notin IN$ be such that $0<q<k$ then $D_{\alpha}^{q}y(0)=0.$

III. Algorithm to Convert the Multi-Order FDE into a System of FDE

Let the given fractional differential equation is

$$D_*^{n_k}y(x)=g\,(x,y(x),D_*^{n_1}y(x),D_*^{n_2}y(x),\ldots,D_*^{n_{k-1}}y(x)) \qquad (11)$$

Subject to the initial conditions

$$y^{j}(0)=y_0^{(j)}\ ;\quad j=0,1,2,\ldots,\lceil n_k\rceil-1 \qquad (12)$$

where $0<n_1<\ldots<n_{k-1}<n_k,\ n_i-n_{i-1}\le 1$ for all $i=1,2,\ldots,k$ and $0<n_i\le 1$, assume that $n_i\in Q$.

In Daftardor-Gejji and Jafari [5], Jafari, Das and Tajadodi [6] it was proved that the FDE (11) can be represented as a system of FDE, without any additional restrictions mentioned in equation (2). Here is above mentioned approach. Let us define

$$y_1(x)=y(x)$$

then

$$D_*^{n_1}y_1=y_2 \qquad (13)$$

Here two cases arise

Case (i) : If $m-1\le n_1\le n_2\le m$ then define

$$D_*^{n_2-n_1}y_2(x)=y_3=D_*^{n_2}y(x) \qquad (14)$$

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Case (ii) : Consider $m-1 \le n_1 < m \le n_2$. If $n_1 = m-1$ then define

$$D_*^{n_2-n_1} y_2(x) = y_{3\bullet} \tag{15}$$

$$D_*^{n_2-n_1} y_2(x) \; D_*^{n_2-m+1} y_1^{(m-1)} = D_*^{n_2} y_1(x)$$

If $m-1 < n_1 < m \le n_2$ then define

$$D_*^{m-n_1} y_2(x) = y_{3\bullet} = y^{(m)}. \quad \text{Further define:} \tag{16}$$

$$= D_*^{n_2-m} y_3(x) = y_{4\bullet}$$

and continuing similarly one can convert the initial value problem (11) into a system of FDE.

The following example will illustrate the method. Consider

$$D_*^{3.3} y(x) = F(x, y(x), D_*^{0.1} y(x), D_*^{1} y(x), D_*^{1.2} y(x), D_*^{1.5} y(x), D_*^{1.7} y(x), D_*^{2} y(x)$$
$$D_*^{2.2} y(x), D_*^{2.6} y(x), D_*^{3} y(x)) \tag{17}$$

where

$$y^{(j)}(0) = y_0^{(j)} \; : j = 0,1,2,3 \tag{18}$$

This initial value problem can be viewed as the following system of FDE.

Let $y_1(x) = y(x)$

$$D_*^{0.1} y_1(x) = y_2(x) \quad ; \; y_1(0) = y_0^{(0)}$$

$$D_*^{0.9} y_2(x) = y_3(x) \quad ; \; y_2(0) = 0$$

$$D_*^{0.2} y_3(x) = y_4(x) \quad ; \; y_3(0) = y_0^{(1)}$$

$$D_*^{0.3} y_4(x) = y_5(x) \quad ; \; y_4(0) = 0$$

$$D_*^{0.2} y_5(x) = y_6(x) \quad ; \; y_5(0) = 0$$

$$D_*^{0.3} y_6(x) = y_7(x) \quad ; \; y_6(0) = 0$$

$$D_*^{0.2} y_7(x) = y_8(x) \quad ; \; y_7(0) = y_0^{(2)}$$

$$D_*^{0.4} y_8(x) = y_9(x) \quad ; \; y_8(0) = 0$$

$$D_*^{0.4} y_9(x) = y_{10}(x) \quad ; \; y_9(0) = 0 \tag{19}$$

$$D_*^{0.3} y_{10}(x) = F(x, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9); \; y_{10}(0) = y_0^{(3)}.$$

This algorithm is valid in the most general case, because we do not impose any of the restriction on α and $n\ell$ as mentioned in equation (12).

IV. Basic Idea of HAM and a System of FDE

We can present the multi-order equation (11) as system of fractional differential equations:

$$D^{\alpha_\ell} y_\ell(x) = y_{\ell+1} \quad ; \ell = 1,2,\ldots,n-1$$

$$D^{\alpha_n} y_\ell(x) = F(x, y_1, y_2, \ldots, y_n)$$

$$y_\ell^{(k)}(0) = c_k^\ell, \quad 0 \le k \le m_\ell, \quad m_\ell < \alpha_\ell \le m_\ell + 1, \quad 1 \le \ell \le n. \tag{20}$$

According to the HAM, we construct the so-called zeroth order deformation equations

$$(1-q)D^{\alpha_\ell}[\phi_\ell(x:q)-y_{\ell 0}(x)]=q\,h_\ell H_\ell(x)[D^{\alpha_\ell}\phi_\ell(x;q)-\phi_{\ell+1}(x;q)]$$

$$\ell = 1,2,\ldots,n-1$$

$$(1-q)D^{\alpha_n}[\phi_n(x;q)-y_{n0}(x)]=q\,h_n H_n(x)[D^{\alpha_n}\phi_n(x;q)-F(x,\phi_1,\phi_2,...,\phi_n] \tag{21}$$

where $h_\ell \neq 0$ denotes an auxiliary parameter, $H_\ell(x)$ is an auxiliary function, $q\in[0,1]$ is an embedding parameter, $y_{\ell 0}(x)$ is initial guess of $y_\ell(x)$ and $\phi_\ell(x;q)$ unknown function of independent variables x and q.

Obviously, when $q = 0$ and $q = 1$ it holds

$$\phi_\ell(x;0) = y_{\ell 0}(x)$$

$$\phi_\ell(x;1) = y_\ell(x) \qquad \ell = 1,2,...,n. \tag{22}$$

Thus as q increases from 0 to 1 the solution $\phi_\ell(x;q)$ varies from the initial guess $y_{\ell 0}(x)$ to the solution $y_\ell(x)$. Expanding in Taylor's series with respect to q, we have

$$\phi_\ell(x;q) = y_{\ell 0}(x) + \sum_{m=1}^{\infty} y_{\ell m}(x)\,q^m \tag{23}$$

where

$$y_{\ell m} = \frac{1}{m!}\frac{\partial^m \phi_\ell(x;q)}{\partial q^m}\bigg|_{q=0} \qquad \ell = 1,2,...,n \tag{24}$$

If the auxiliary linear operator, initial guess, the auxiliary parameters $\hbar$ and the auxiliary function are so properly chosen the series (23) converges at $q = 1$, then

$$y_\ell(x) = y_{\ell 0}(x) + \sum_{m=1}^{\infty} y_{\ell m}(x) : \qquad \ell=1,2,,...,n \tag{25}$$

Define the vector

$$\vec{y}_\ell(x) = \{y_{\ell 0}(x), y_{\ell 1}(x),..., y_{\ell n}(x)\} \tag{26}$$

Differentiating equation (21) m times with respect to q and then putting setting $q = 0$ and finally dividing them by m! we obtain the m^th order deformation equation

$$D^{\alpha_\ell}[y_{\ell m}(x)-\chi_m y_{\ell m-1}(x)] = \hbar_\ell H_\ell(x)\,R_{\ell m}(\vec{y}_{1m-1},...,\vec{y}_{nm-1},x)\ ;\ \ell = 1,2,...,n-1$$

$$D^{\alpha_n}[y_{nm}(x)-\chi_m y_{nm-1}(x)] = \hbar_n H_n(x)\,R_{nm}(\vec{y}_{1m-1},...,\vec{y}_{nm-1},x) \tag{27}$$

where

$$R_{\ell m}(\vec{y}_{1m-1},...,\vec{y}_{nm-1},x) = \frac{1}{(m-1)!}\frac{\partial^{m-1}[D^{\alpha_\ell}\phi_\ell(x;q)-\phi_{\ell+1}(x;q)}{\partial q^{m-1}}\bigg|_{q=0}$$

$$R_{nm}(\vec{y}_{1m-1},\ldots,\vec{y}_{nm-1},x)=\frac{1}{(m-1)!}\frac{\partial^{m-1}[D^{\alpha_n}\phi_n(x;q)-F(x,\phi_1,\phi_2,\ldots,\phi_n)]}{\partial q^{m-1}}\bigg|_{q=0} \quad (28)$$

and

$$\chi_m=\begin{cases}0\ , & m\le 1\\ 1\ , & m>1\end{cases} \quad (29)$$

Applying $J^{\alpha l}$ the inverse operator D^{α_ℓ} of on both sides of equation (27), we have

$$y_{\ell m}(x)=\chi_m y_{\ell m-1}(x)-\chi_m\sum_{j=0}^{m-1}y^{(j)}_{\ell m-1}(0^+)\frac{x^j}{j!}+\hbar_\ell H_\ell(x)\,J^{\alpha_\ell}R_{\ell m}(\vec{y}_{\ell m-1},\ldots,\vec{y}_{nm-1},x)$$

$$y_{nm}(x)=\chi_m y_{nm-1}(x)-\chi_m\sum_{j=0}^{m-1}y^{(j)}_{nm-1}(0^+)\frac{x^j}{j!}+\hbar_n H_n(x)\,J^{\alpha_n}R_{nm}(\vec{y}_{1m-1},\ldots,\vec{y}_{nm-1},x) \quad (30)$$

The m-th order deformation equations are linear and thus can be easily solved. We have

$$y_\ell(x)=\sum_{m=0}^{\infty}y_{\ell m}(x) \qquad \ell=1,2,\ldots,n \quad (31)$$

when $M\to\infty$, we get an accurate approximation of original equation (11).

V. Test Examples

Example 1.

$$D^4y(x)+D^{3.5}\,y(x)+y^3(x)=x^9 \quad (32)$$

with the initial conditions

$$y(0)=y'(0)=y''(0)=0\ ,\ y'''(0)=6 \quad (33)$$

In view of the discussion in the last section the equation (32) can be viewed as the following system of FDE

$$y_1(x)=y(x)$$

then

$$D^{3.5}\,y_1(x)=y_2(x);\ \ y_1(0)=y_1'(0)$$

$$=y_1''(0)=0\ ,\ y_1'''(0)=6$$

and

$$D^{0.5}\,y_2(x)\ \ =-y_2-y_1^3+x^9,\ \ \ y_2(0)=0$$

Using equation ( ), we get the following scheme:

$$y_{10}=x^3\ ,\ y_{20}=0$$

$$y_{1m}=(\chi_m+h_1)[y_{1m\text{-}1}(x)-y_{1m-1}(0)]+h_1\,J^{3.5}[-y_{2m-1}]$$

$$y_{11}=h\,J^{3.5}[-y_{20}]=0$$

$$y_{2m}=(\chi_m+h_2)[y_{2m-1}(x)-y_{2m-1}(0)]+h_2J^{0.5}\left[y_{2m-1}+\sum_{i=0}^{m-1}y_{1i}\sum_{j=0}^{m-1-i}y_{1j}y_{1m-1-i-j}-(1-\chi_m)x^9\right]$$

$$y_{21}=h_2\,J^{0.5}[y_{20}+y_{10}^3-x^9]$$

$$y_{21}=0$$

and hence

$$y_{1m}=0,y_{2m}=0\ ;\ m\ge 1$$

In view of above terms, we find $y_1(x)=x^3$, $y_2(x)=0$ so $y(x)\ x^3$ is the required solution of the given equation.

Example 2: Consider the following initial value problem

$$D_*^2y(x)-D_*^{3/2}\,y(x)+y(x)=1+x \quad (34)$$

with the initial conditions

$$y(0)=y'(0)=1 \quad (35)$$

Equation (34) is equivalent to the following system of equations

$$y_1(x) = y(x)$$

$$D^{1.5}\, y_1(x) = y_2(x)\,; \quad y_1(0) = y_1'(0) = 1$$

$$D^{0.5}\, y_2(x) = y_2(x) - y_1(x) + (1+x)$$

as the initial guess we assume $y_{10} = 1+x$, $y_{20} = 0$.

By HAM the m-th order deformation equations are given by

$$y_{1m}(x) = \chi_m y_{1m-1} + h_1\, J^{1.5}[-y_{2m-1}]$$

$$y_{2m}(x) = \chi_m\, y_{2m-1} + h_2[-y_{2m-1} + y_{1m-1} - (1-\chi_m)(1+x)]$$

$$= h_2\, jJ^{0.5}[-y_{20} + y_{10} - (1-\chi_1)(1+x)] = 0$$

and hence $y_{1m} = 0, y_{2m} = 0\,; m \geq 1$.

In view of above, we get the exact solution $y(x) = 1+x$.

VI. Conclusion

This paper deals with the approximate solution of a class of multi-order fractional differential equations by Homotopy analysis method. Thus it has been demonstrated that Homotopy analysis method proves useful in solving linear as well as non-linear multi-order fractional differential equation by reducing them into a system of fractional differential equations.

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