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A Probabilistic Approach to Portfolio Optimization under Epistemic Uncertainty

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APROBABILISTICAPPROACHTOPORTFOLIOOPTIMIZATIONUNDEREPISTEMICUNCERTAINTY

A Probabilistic Approach to Portfolio Optimization under Epistemic Uncertainty

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Abstract- Portfolio is an appropriate collection or list of investments held by an organization or an individual where Portfolio optimization means to calculate the optimal weights for the collection or list of investments that gives the highest return with least risk. In previous works on this topic did not consider correlation factor and epistemic uncertainty on returns. In our model both the factors are considered because correlation between the assets exists in real life and epistemic uncertainty may exist in different rate of return of different assets. In this work two models are developed to optimize a portfolio. Two models are Optimization model without EU and Optimization model with EU. By optimization we can decide the assets on which we can invest, how much to invest and can take divestment decisions.

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I. Introduction

Portfolio is an appropriate collection or list of investments held by an organization or an individual. Portfolio optimization means to calculate the optimal weights for the collection or list of investments that gives the highest return with least risk. In this work we have used a probabilistic approach to optimize the portfolio. Existing models on this topic by other authors [e.g., 1] did not consider correlation factor and epistemic uncertainty on returns. In our model both the factors are considered. Epistemic uncertainty is the situation where we do not do things correctly. Epistemic uncertainty results in data because of not measuring a quantity with sufficient accuracy, or because of building a model by neglecting certain effects, or because of hidden data. Moreover in "portfolio optimization (2 Assets)" by Markowitz [1] proportion of investment was calculated with considering a riskless asset and two risky assets. In another paper [2] optimization of investments were calculated by the linear combination of the assets. This means in that model the co-variances of assets were not considered. Covariance measures the specified value by which two random variables change together. In our model co-variances between assets are considered. Kawas and Thiele [3] developed Log robust model for both the independent and correlated assets to optimize the portfolio. The continuous probability distribution of a random variable whose logarithm is normally distributed is called Log normal distribution [4]. They had considered the uncertainty on the rate of return. But Log robust model incorporates the randomness in the rate of returns of the assets. In our model no trends in the rate of return is required. Buckley et al [5] formulated the optimization of the investment where the rates of return of the assets have Gaussian mixture distribution. The continuous probability distribution that has a bell shaped probability density function is known as Gaussian distribution [6]. In our model we have considered that the rate of return or the range of rate of return will be known.

II. Epistemic Uncertainty (EU), Correlation and Covariance

a) Epistemic Uncertainty (EU)

Epistemic uncertainty is the situation where we do not do things correctly. Epistemic uncertainty results in data because of not measuring a quantity with sufficient accuracy, or because of building a model by neglecting certain effects, or because of hidden data [7]. In our model epistemic uncertainty exists in the rate of returns of assets. We do not know what will be the future rate of return of an asset. Moreover, we have used the rate of return on particular time interval where the rate of return is changing continuously. Therefore, by considering the epistemic uncertainty we will know the range of rate of return for an asset rather than the particular rate of return.

b) Correlation

One of the most commonly used statistics is correlation. It measures and describes the degree of relationship between two variables. To compute the correlation of two variables „Pearson's product-moment" method can be used. "Pearson's product-moment" correlation is computed by dividing the covariance with standard deviation [8].

$$\text{So, } corr(x,y) = \frac{\text{cov}(x,y)}{\sigma_x * \sigma_y} = [\, n*\Sigma xy - (\Sigma x)(\Sigma y) / \sqrt{([n*\Sigma x^2 - (\Sigma x)^2][n*\Sigma y^2 - (\Sigma y)^2])}] \qquad (1)$$

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Here,

$corr(x,y)$ = Correlation between x and y

$cov(x,y)$ = covariance of x and y

σ_x = standard deviation of x

σ_y = standard deviation of y

n = Number of values or elements

x = First Score

y = Second Score

Σxy = Sum of the product of first and Second Scores

Σx = Sum of First Scores

Σy = Sum of Second Scores

Σx^2 = Sum of square First Scores

Σy^2 = Sum of square Second Scores

Correlation is +1 in the case of a positive or increasing linear relationship, −1 in the case of a decreasing or negative linear relationship and some value between −1 and 1 in all other cases, indicating the degree of linear dependence between the variables. As it approaches zero, it implies that there is no relationship. The closer the coefficient is to either −1 or 1, the stronger the correlation between the variables.

c) Covariance

Covariance measures the specified value by which two random variables change together. If larger values of one variable correspond to the larger values of the other variable and the same holds for the smaller values, the covariance is positive. In the opposite case, when the larger values of one variable correspond to the smaller values of the other, the covariance is negative. Therefore, the sign of the covariance shows the trend in the linear relationship between the variables. Covariance can be calculated as follows [9]:

$$Cov(x,y) = E[(x-E[x])(y-E[y])] \tag{2}$$

$$Cov\,(x,y) = E[xy]-E[x]*E[y] \tag{3}$$

Here,

$Cov(x,y)$ = Covariance of x and y

$E[x]$ = Mean of x

$E[y]$ = Mean of y

$E[xy]$ = Mean of xy

III. Model Structure

Optimization refers to the selection of a best element from some set of available alternatives. Portfolio is an appropriate collection or list of investments held by an organization or an individual. Portfolio optimization means to calculate the optimal weights for the collection or list of investments that gives the highest return with least risk. We have developed two models for portfolio optimization:

1. Optimization Model without EU
2. Optimization Model with EU

a) Optimization Model without EU

Let us assume that a set of *n* assets and a sum of money are given, which to be invested to maximize the total return on investment. We have to find the investment proportions *xi* (*i* = 1, 2...*n*) to be invested in assets *i* = 1, 2 ...*n* so that the total return on investment over a period is maximized. It is not possible to determine with certainty because the actual returns of the assets are not known. However, if we assume that the expected returns of the assets and their variances and co-variances are known, we can calculate the investment proportions that results in high expected return with low variance. As the model does not consider epistemic uncertainty, we know the specified value of the returns. In this model we will maximize the total return while simultaneously minimizing the risk.

Let,

n = Number of available assets

r_i = Expected return of assets

x_i = Fraction of the capital invested in asset i

σ_{ij} = The covariance of the return for assets i and j

Therefore, the optimization formulation becomes,

$$Max\, f_1(x) = w*\sum x_i * r_i - v*\sum\sum \sigma_{ij} * x_i * x_j \tag{4}$$

Subject to,

$$f_2(x) = \sum\sum \sigma_{ij} * x_i * x_j \le \varepsilon \tag{5}$$

$$\sum x_i = 1 \tag{6}$$

$$x_i \ge 0 \tag{7}$$

where i and j = 1, 2 ……n

Here,

w and v are the weighting coefficients that represent the relative importance of the total return and the risk respectively, where $w+v = 1$

If $Smin$ and $Smax$ are the minimum and maximum attainable values of $f_2(x)$

then $\varepsilon \in [S_{min}, S_{max}]$

By using Equ. (3) we can calculate the co-variances of the assets.

b) Optimization Model with EU

In this model we have used multi objective functions. There are a number of method by which this optimization can be done like Deterministic design optimization, Robustness-based design optimization, Robustness-based design optimization under data uncertainty, Robustness-based design with sparse point data, Determination of optimal sample size for sparse point data, Robustness-based design with interval data [10]. We have used Robustness-based design optimization under data uncertainty. Let us assume a set of n assets and a sum of money are given, which are to be invested to maximize the total return on investment. As the model considers the epistemic uncertainty, we know the specified value of the returns of some assets and a range of the returns for the rest number of assets. Let a set of p assets whose specified return values are known and a set of q assets whose range of returns are known [where $p+q = n$]. We have to find the investment proportions xi to be invested in assets $i = 1, 2 ...n$ so that the total return on investment over a period is maximized. Therefore, we consider additional constraints in this model. The solution is obtained by simultaneously solving two objective functions. We have to find the proportion of investments in one function, the returns and co-variance of asset set q in another function. As for the asset set q, the ranges of returns are known; we can get the range of co-variances for asset set q.

Let,

n = Number of available assets

r_i = Expected return of assets

x_i = Fraction of the capital invested in asset i

σ_{ij} = The covariance of the returns of assets i and j

lb = lower bound of returns for asset set q

ub = upper bound of returns for asset set q

Therefore, the objective function 1 becomes,

$$\underset{x_i}{\arg Max} f_1(x) = w * \sum x_i * r_i - v * \sum\sum \sigma_{ij} * x_i * x_j \tag{8}$$

Subject to,

$$f_2(x) = \sum\sum \sigma_{ij} * x_i * x_j \le \varepsilon \tag{9}$$

$$\sum x_i = 1 \tag{10}$$

$$x_i \ge 0 \tag{11}$$

where i and $j = 1, 2 \ldots\ldots n$

And the objective function 2 becomes,

$$\underset{r_k, \sigma_{ij}}{\arg Max} f_3(x) = w * \sum x_i * r_i - v * \sum\sum \sigma_{ij} * x_i * x_j \tag{12}$$

Subject to,

$$f_2(x) = \sum\sum \sigma_{ij} * x_i * x_j \le \varepsilon \tag{13}$$

$$lb_k \le r_k \le ub \tag{14}$$

$$min\ \sigma_{ij} \le \sigma_{ij} \le max\ \sigma_{ij} \tag{15}$$

where k is the asset identity of asset set q.

For the asset set q described by epistemic uncertainty, the additional constraints in Eqs. (14) and (15) are incorporated in to the model.

Here,

w and v are the weighting coefficients that represent the relative importance of the total return and the risk respectively, where $w+v = 1$

If $Smin$ and $Smax$ are the minimum and maximum attainable values of $f_2(x)$

then $\varepsilon \in [S_{min}, S_{max}]$

We can calculate the co variances of the assets by Equ (3).

IV. Numerical Data Analysis

a) Model of Optimization without EU

Under this model we do not consider the epistemic uncertainty. That means we know the specified value of the returns for assets.

Example Problem 1

Let, we have two assets (mutual funds). We will use the EPS (Earning Per Share) values as the rate of return.

The earnings per share (EPS) values from year 1 to 10 (10 values) of two mutual funds named 2NDICB and 3RDICB are as follows:

Table 1 : EPS value of 2NDICB and 3RDICB

YEAR	2NDICB	3RDICB
1	89.32	45.01
2	70.75	54.04
3	41.06	53.41
4	48.86	42.74
5	78.77	41.21
6	80.75	52.31
7	66.06	66.10
8	76.15	69.11
9	107.14	92.86
10	135.55	121.40

In the following discussion, we solve the above problem using our proposed model that does not consider the epistemic uncertainty.

<u>Average rate of return:</u>

Here, $r_1 = 79.441$ and

$r_2 = 63.819$ (mean values)

<u>Covariance:</u>

As we know, $\sigma_{ij} = E(xy) - E(x)*E(y)$

Here, $E(x) = 79.441$; $E(y) = 63.819$ and $E(xy) = 5562.914$

So, $\sigma_{12} = \sigma_{21} = 493.07$; $\sigma_{11} = 2906.9$; $\sigma_{22} = 583.79$.

<u>Optimized portfolio:</u>

By using (4), (5), (6) and (7) the equations become:

$$Max\ f_1(x) = w*(79.441*x_1+63.819*x_2) - v*(2906.9*x_1^2+986.14x_1x_2+583.79*x_2^2)$$

Subject to,

$$f_2(x) = 2906.9*x_1^2+986.14x_1x_2+583.79*x_2^2 \leq \varepsilon;$$

Letting x_1 and $x_2 = .5$; ε becomes 1119.21.

$x_1+x_2 = 1; \quad x_1$ and $x_2 \geq 0$.

By using MATLAB code we get,

Table 2 : Value of x1 and x2 by MATLAB code

w	v	x_1	x_2	Optimized return
1	0	.5	.5	-71.63
.1	.9	.0366	.9634	516.0148
.2	.8	.0370	.9630	451.5250
.3	.7	.0376	.9624	387.0342
.4	.6	.0383	.9617	322.5419
.5	.5	.0393	.9607	258.0474
.6	.4	.0409	.9591	193.5487
.7	.3	.0435	.9565	129.0420
.8	.2	.0487	.9513	64.5149
.9	.1	.0643	.9359	-.0933
0	1	.0362	.9638	580.5039

b) Model of Optimization with EU

Under this model we will consider the epistemic uncertainty. So we know the specified value of the returns of some assets and a range of the returns for the rest number of assets.

Example Problem 2

Let, we have three assets (asset 1, asset 2 and asset 3). The rate of return of asset 1 and 2 are known. But we know the limit of the rate of return of asset 3.

Table 3 : The rates of returns of asset 1 and asset 2

Rate of return of asset 1	Rate of return of asset 2
89.32	45.01
70.75	54.05
41.06	53.41
48.86	42.74
78.77	41.21

Table 4 : The limits of asset 3

Lower limit	Upper limit
62	71
60	72
61	75
59	73
58	70

In the following discussion, we solve the above problem using our proposed model that considers the epistemic uncertainty.

Average rate of return:

Average rate of return of asset 1 (r_1) = 65.75

Average rate of return of asset 2 (r_2) = 47.28

But we will have a limit for the rate of return of asset 3.

Lower limit of the average rate of return of asset 3 (lb) = (62+60+61+59+58)/5 = 60

Upper limit of the average rate of return of asset 3 (ub) = (71+72+75+73+70)/5 = 72.2

Covariance

If we want to optimize the assets portfolio then we have to determine six co variances.

Table 5 : Co-variances for Example Problem 2

1	σ_{11}
2	$\sigma_{12} = \sigma_{21}$
3	$\sigma_{13} = \sigma_{31}$
4	σ_{22}
5	$\sigma_{23} = \sigma_{32}$
6	σ_{33}

We can easily calculate σ_{11}, σ_{22} and $\sigma_{12} = \sigma_{21}$ because we know the mean rate of return of asset 1 and 2. But we can know the lower (minimum value) and the upper (maximum value) limit of σ_{33}, $\sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$ because we know the limit for asset 3.

$$\sigma_{ij} = E(xy) - E(x)*E(y).$$

So, $\sigma_{11} = 329.25;$

$\sigma_{22} = 29.58;$

$\sigma_{12} = \sigma_{21} = -34.31$

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Now we will calculate the limit for σ_{33}, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$.

σ_{33} can easily calculated by using the MATLAB code.

By solving, the limit for σ_{33} = [4*e-10, 245.2];

To solve $\sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$ we have to know the maximum and minimum value of *E(13)* and *E(23)*.

Solving by MATLAB we get *E(13)* = *E(31)* = [3947.8, 4719.2];

Similarly, *E(23)* = *E(32)* = [2840.7, 3419.6];

So we get, $\sigma_{13} = \sigma_{31}$ = [-799.39, 774.2] and $\sigma_{23} = \sigma_{32}$ = [-572.92, 582.8].

Optimized portfolio:

We will construct two objective functions to optimize the returns.

By using Eqs. (8), (9), (10) and (11) we get,

$$\max_{x_1, x_2, x_3} f_1(x) = w*(r_1*x_1+r_2*x_2+r_3*x_3) - v*(x_1^2*\sigma_{11}+x_2^2*\sigma_{22}+x_3^2*\sigma_{33}+2*x_1*x_2*\sigma_{12}+2*x_2*x_3*\sigma_{23}+2*x_3*x_1*\sigma_{31}) \quad (16)$$

Subject to,

$$w*(r_1*x_1+r_2*x_2+r_3*x_3) - v*(x_1^2*\sigma_{11}+x_2^2*\sigma_{22}+x_3^2*\sigma_{33}+2*x_1*x_2*\sigma_{12}+2*x_2*x_3*\sigma_{23}+2*x_3*x_1*\sigma_{31}) \le 994.26;$$

$x_1+x_2+x_3 = 1;$

$x_1, x_2, x_3 \ge 0.$

By using w =.5 and v =.5 with assumed value of x_1, x_2, x_3; we get ε = 994.26.

By using Eqs. (12), (13), (14) and (15) we get,

$$\min_{r_3, \sigma_{33}, \sigma_{23}, \sigma_{31}} f_1(x) = w*(r_1*x_1+r_2*x_2+r_3*x_3) - v*(x_1^2*\sigma_{11}+x_2^2*\sigma_{22}+x_3^2*\sigma_{33}+2*x_1*x_2*\sigma_{12}+2*x_2*x_3*\sigma_{23}+2*x_3*x_1*\sigma_{31}) \quad (17)$$

Subjected to,

$60 \le r_3 \le 72.2;$

$-572.92 \le \sigma_{23} \le 582.8;$

$-799.39 \le \sigma_{31} \le 774.2;$

$4*e\text{-}10 \le \sigma_{33} \le 245.2;$

$$w*(r_1*x_1+r_2*x_2+r_3*x_3) - v*(x_1^2*\sigma_{11}+x_2^2*\sigma_{22}+x_3^2*\sigma_{33}+2*x_1*x_2*\sigma_{12}+2*x_2*x_3*\sigma_{23}+2*x_3*x_1*\sigma_{31}) \le 994.26;$$

By using MATLAB code we can calculate the value of x_1, x_2, x_3 from Equ. (16) to use them in Equ. (17) and simultaneously calculating the value of $r_3, \sigma_{33}, \sigma_{23}, \sigma_{31}$ to use them in Equ. (16). Thus we can calculate the optimum return.

V. Findings

To calculate the optimal weights for the collection or list of investments that gives the highest return with least risk is called portfolio optimization. In this work we have used a probabilistic approach to optimize the portfolio. We have developed two models to optimize the asset portfolio. The two models optimize the portfolio in different approach. One considers the epistemic uncertainty and the other does not consider the epistemic uncertainty. By using these models we can optimize the portfolio of different investments, different shares. If we look through the results then we can see how the results define how much to invest in which asset.

Firstly, the two models optimize the portfolio by considering not only the rate of returns but also the risks. We have considered the risk portion in the constraint area from where we can minimize it and simultaneously we have considered it in the objective function also.

Secondly, we have not only calculated the range of co variances but also optimized the values of the co variances. The risks become less while the co variances become less.

Thirdly, we have shown the examples of the models by using two or three assets. However, the models can optimize as many assets as are there. But in that case we have to calculate the co variances with proper situation. That means we have to calculate the co variances by considering whether there exists epistemic uncertainty or not.

Fourthly, in the second model "Optimization Model with EU" we have considered the epistemic uncertainty. We have optimized the proportions of investments with simultaneously optimizing the co variances and the rate of returns for the assets under uncertain returns. We could optimize the co variances and rate of returns for the assets under uncertain returns by considering a separate objective functions and not considering the proportions of assets. In that case the calculation will be simpler than the present calculation but we will not have a properly optimized solution.

Finally, in optimizing the portfolio we have considered two notations w and v, where they are weighting coefficients that represent the relative importance of the objective functions. That means w and v represent the relative importance of the total return and the risk respectively. If we have a portfolio where the risk part has less importance then we can use a lower v value than the w value. And if we have a portfolio where the risk part has a greater importance then we can use a v value with comparison to the w value.

So the models optimize a portfolio regardless the number of assets by both considering and not considering the epistemic uncertainty, giving relative importance factor to the total return part and the risk part.

VI. Conclusion

Optimization refers to the selection of a best element from some set of available alternatives. Portfolio is an appropriate collection or list of investments held by an organization or an individual. Portfolio optimization means to calculate the optimal weights for the collection or list of investments that gives the highest return with least risk. In this work we have developed two models using probabilistic approach to optimize the portfolio. As we have stated earlier the findings of my thesis that the models optimize a portfolio regardless of the number of assets by both considering and not considering the epistemic uncertainty, giving importance factor to the total return part and the risk part. By using these models we can optimize the portfolio of investments, shares etc. We can know the proportions of investment on particular assets in efficient way that minimizes the risk on profitability. However, the models use discrete time interval values of the rate of returns. Moreover, the code used for the model "Optimization Model with EU" is not more flexible. Further research can be done by developing the model for the continuous values of the rate of return and developing the code for the model "Optimization Model with EU" by incorporating more flexibility.

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