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Principle of Quasi Work and its Import on Structural Analysis

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PRINCIPLEOFQUASIWORKANDITSIMPORTONSTRUCTURALANALYSIS

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Principle of Quasi Work and its Import on Structural Analysis

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Nomenclature

B = Total branch degrees of freedom

b_j = Total Degrees Of Freedom (DOF) associated with direction j.

$\{d\}_n$ = Nodal displacement in system 'n'.

F_i = Internal (branch) force.

$\{F\}_m$ = Set of internal forces in system 'm'.

$M_\#$ = Support moment reaction at support #.

m, n = Subscripts denoting topologically similar systems 'm' and 'n'.

N = Total degrees of freedom.

P_j = External (generalized) force.

$\{P\}_m$ = Self equilibrating set of external force acting on system 'm'.

$R_\#$ = Support reaction at support #.

S = Arbitrary parameter distributed over nodes.

U_{mn} = Quasi Strain Energy ($=\{F\}_m^T\{\delta\}_n$)

W_{mn} = Quasi Work ($=\{P\}_m^T\{d\}_n$)

ΔS_k = Difference of S between terminal nodes of branch k.

Γ, Λ = Linear operators.

α = A constant.

$\{\delta\}_n$ = Set of compatible deformations in system 'n'.

π_{mn} = Quasi total potential.

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I. Introduction

Discrete structural element models characterizing stiffness, inertia and damping properties of structural elements forming the lower end of the spectrum of finite elements in evolving a new methodology termed as Model Based Design, MBD, was given by Prasad [1]. This method utilizes structural models as an assemblage of appropriately interconnected discrete elements or modules (comprising of such elements). The striking similarity between such discrete structural models and electrical networks led to the adaptation of concept of Topologically Similar Systems (TSS) in the realm of structural analysis by Pandita [2], Panditta, Ambardhar, et al [3], Panditta and Wani [4], Panditta, Shimpi, et al [5], Panditta [6] and Panditta [7].

However, during a search for the analytical methods suitable for providing an analysis link between a pair of TSS, a glaring inadequacy of the existing energy methods and variational principles is noticed. Exploitation of topological similarity for analysis is beyond the scope of existing energy methods (Argyris and Ashley [8] and Shames [9]), variational principles (Reissener [10,11]) and finite element methods (Cook, Malkus et al [12] and Akin [13]); since these principles/ methods can be applied only to one structural configuration at a time.

In this paper, general form of Principle of Quasi Work (PQW) and its derivatives based on Tellegen's theorem for electrical network analysis (Penfield, Spencer, et al [14]) governing a pair of TSS are derived and illustrated.

II. Basic Theorem

Equation for nodal equilibrium in direction 'j' of a discrete model of any structural system can be written as:

$$\sum_{i=1}^{b_j} F_i + P_j = 0 \tag{1}$$

Where, F_i is internal (branch) force, P_j is external (generalized) force and b_j represents total Degrees Of Freedom (DOF) associated with direction j.

Multiplication of left hand side of the above equation with any non-trivial nodal parameter will not

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alter the right hand side of this equation (even if the resulting product may not have any physical significance).

Hence, multiplying Eqn. (1) by any parameter S (distributed over the nodes) and taking the sum over all 'N' DOF of the system, one obtains:

$$\sum_{j=1}^{N}\left(\sum_{i=1}^{b_j} F_i S_j + P_j S_j\right) = 0 \qquad (2)$$

First term of the left hand side of Eqn. (2) takes the sum of the product over each branch twice (once on each node to which these branches are connected). This is equivalent to taking the sum of the product of the difference of S between terminal nodes of each branch (represented as ΔS) and the force due to this branch at one of the nodes. Hence, this double sum can be replaced by a single sum taken over all branches. Thus Eqn. (2) takes the form

$$\sum_{k=1}^{B} F_k(\Delta S_k) - \sum_{j=1}^{N} P_j S_j = 0 \qquad (3)$$

Where, $B = \sum_{j=1}^{N} {b_j}/{2}$.

Here, 'B' represents, in general, the total branch DOF of the structural system, ΔS_k is a branch parameter defined as the difference between the parameter values associated with the pair of the generalized directions corresponding to the k^{th} branch DOF and 'N' is the total number of DOF of the system.

The negative sign in Eqn. (3) is a consequence of the definition of F_k and ΔS_k together with the associated sign relevant to the self equilibrating force system in k^{th} branch or more generally the k^{th} branch DOF (in the sense that there can be more than one self equilibrating system of forces concurrently in the branch corresponding to tension, torsion, etc.).

Taking advantage of arbitrary nature of parameter S_j it will be prudent to define these S_j's as nodal parameters of another conveniently chosen TSS which could be distinctly different from the given structural system. Equation (3) can then be deduced to provide the mathematical statement of the basic theorem as:

$$\sum_{k=1}^{B} (F_k)_m\, (\Delta S_k)_n = \sum_{j=1}^{N} (P_j)_m\, (S_j)_n \qquad (4)$$

Where, the subscripts 'm' and 'n' refer to two distinct structural systems with topological similarity as their connecting link. This can also be stated as:

Sum of the product of internal branch forces of a system with the corresponding branch nodal parameter differences of another topologically similar system is equal to the sum of the product of external (self equilibrating) nodal forces of the system with corresponding nodal parameters of the topologically similar system.

Here, it may be relevant to mention that if *nodal parameter 'S_j'* is nodal displacement then the *product* has units of work/ energy and if it represents rate of nodal deformation then the *product* has units of power and so on.

Equation (4) can be written in matrix notation as:

$$\{F\}_m^T\{\Delta S\}_n - \{P\}_m^T\{S\}_n = 0 \qquad (5)$$

or

$$\Phi_{mn} = \phi_{mn} - \psi_{mn} = 0 \qquad (6)$$

Where, ϕ_{mn} represents the first term and ψ_{mn} second term in the left hand side of the Eqn.(5).

Even though, Eqn.(4), Eqn.(5), and Eqn.(6) are derived for systems where one deals with discrete set of finite nodes and branches, this theorem is equally applicable to continuous structures. Since continuum can be treated as consisting of infinite DOF and the above equations can be used for continuum by replacing vectors by functions and the vector products by integrals (over the appropriate domain) which represent the two terms in Eqn.(5). Each of the distribution functions can also be approximated by appropriate number of interpolation functions (i.e. generalised coordinates) and the resulting integrals of Eqn.(5) can also be represented by matrix products (e.g. as in FEM formulations). In fact, this equation is very wide in its scope and it can be applied to various fields of science. It only assumes the state of (static or dynamic) equilibrium for its applicability.

III. General form of basic Theorem

Let Γ and Λ be two linear operators which when operated upon forces F and generalized parameters S of Eqn. (5), result in the most general form of the theorem:

$$\{\Gamma F\}^T\{\Delta(\Lambda S)\} = \{\Gamma P\}^T\{\Lambda S\} \qquad (7)$$

Above equation holds good for any type of element, loading/ excitation and boundary/ initial conditions. These operators when operated upon F and S should not change basic characteristics of F and S. These operators can be given a broader meaning which allows these operators to represent TSS also. For example let Γ, Λ represent TSS_m and TSS_n respectively, then Eqn. (7) reduces to Eqn. (5)

a) Weak Form

Interchanging the role of operators in Eqn.(7) we get:

$$\{\Lambda F\}^T\{\Delta(\Gamma S)\} = \{\Lambda P\}^T\{\Gamma S\} \qquad (8)$$

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A linear combination of Eqns. (7) and (8) yields:

$$\{\Gamma F\}^T\{\Delta(\Lambda S)\}+\alpha\{\Lambda F\}^T\{\Delta(\Gamma S)\} = \{\Gamma P\}^T\{\Lambda S\}+\alpha\{\Lambda P\}^T\{\Gamma S\} \quad (9)$$

Where, α is any arbitrary constant. This equation is designated as the 'weak form' of the theorem. It will be useful when the theorem has to be applied twice.

b) Variational Form

Taking suitable variations over Eqn.(6) gives:

$$\delta[\Phi_{mn}]=0 \quad (10)$$

Since, force(s) F and parameter(s) S belong to two different systems, it is possible to vary a single parameter set of one of the systems at a time without affecting all other parameter sets. This formulation can have two variants owing to choice of TSS sequence (m and n). A brief illustration of the concept of TSS adapted to Structural mechanics follows.

IV. Topologically Similar Systems

To evolve the definition of topologically similar system, one has to go to Eqn. (3). In this equation second term is the summation over nodes hence, number of nodes in TSS should be same. First term of the equation is summed over braches hence number of branches should be same. As it also involves the parameter ΔS which in turn involves two nodes to which a branch is connected hence connectivity of branches should also be same. If one assures same interconnectivity of nodes it will in turn ensure that number of branches is same. Hence, for two systems to be topologically similar one has to ensure that total number of nodes is same and connectivity of branches is also same. Topology can now be defined as *unique layout of nodes with specified interconnectivity of nodes*. Systems with same topology are TSS.

Moreover, in the derivation of this equation the manner in which a branch force is developed is immaterial. Hence, systems with same topology (TSS) may differ in other details (e.g. material properties, boundary conditions, etc.). Illustrations of pairs of TSS for discrete structural models and continuum structures are given in Ref. [3]. For a given problem there are infinite number of TSS, wherein any branch/ element parameter can even assume limiting values of zero/ infinity (making such branch/ element on a load path vanish/ rigid)[3]. Conditions that continuum system should satisfy for being TSS have to be derived in each case. For beams and rods/ shafts conditions have been derived in Panditta, Ambardhar, et al. [3] and Panditta and Maruf [4], respectively.

Obtaining equations that can link such systems would be a boon for structural analysis. If one succeeds in this crucial step, all advantages of the structural analysis theorems available for the solution of a single system can now be extended to an unlimited group of structures which have topological similarity as their link (and the only constraint in their choice).

Now, the energy principle (PQW) applicable to a pair of *topologically similar structural systems* will be deduced from the basic theorem.

V. Principal of Quasi Work

If generalized parameters $(S_j)_n$ in the Basic theorem are replaced by generalized displacements $(d_j)_n$ of TSS_n, the following Principle of Quasi Work results:

*In a pair of TSS, quasi work done by (self equilibrating set of) external forces of any one of the systems while going through the **corresponding** (compatible) displacements of the other system, is equal to quasi energy due to internal forces of former system while going through **corresponding** deformations of the latter system.*

In the mathematical form it can be stated as:

$$W_{mn}=U_{mn} \quad (11)$$

In case of continuum systems, quasi energy is computed by utilising stresses of one system and strains of other system.

a) Proof

By replacing $\{S\}_n$ by displacements $\{d\}_n$ and branch parameters $\{\Delta S\}_n$ by branch deformations $\{\delta\}_n$, equation (5) becomes:

$$\{P\}_m^T\{d\}_n=\{F\}_m^T\{\delta\}_n \quad (12)$$

And, hence, Eqn.(11) is proved.

Panditta, Shimpi, et al. [5] validated PQW and derived useful theorems based on PQW for discrete structural models. Panditta, Ambardhar, et al. [3] after validating PQW for beams applied it to redundant beams with advantage. Panditta and Maruf [4] applied PQW to one dimensional structural elements for getting deflection without resorting to internal force/ moment distributions. Panditta [6] used PQW for calculating nodal deflection of trusses with great advantage and Panditta [7] applied PQW to columns for obtaining Euler critical load without resorting to the solution of differential equations. In the next section direct application of PQW to indeterminate structures is given.

VI. Application of PQW to Indeterminate Structure

Figure 1a shows a uniformly loaded indeterminate beam built in at both the ends with length L and flexural rigidity E_1I_1. From symmetry and equilibrium considerations $R_A = R_B = w_1L/2$ and $M_A = M_B$. Hence, only unknown to be determined is either M_A or M_B. This given beam is designated as TES_1. In order to apply PQW, a pair of TES is needed. In this example, a simply supported beam with overhang on both the

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sides as given in Fig.1b is selected as TES_2. TES are topologically equivalent systems in which $E_1\ I_1 = E_2\ I_2 = EI$ (for beams). Quasi energy U_{21} and quasi work W_{21} for this pair is given by:

$$\begin{aligned} U_{21} &= M_2\ L\,(12\,M_A - w_1\,L^2)/24\,EI \\ W_{21} &= \{R_C\}_2\{v(L/4)\}_1 + \{R_D\}_2\{v(3L/4)\}_1 \\ &\quad + \{M_2\}_2\{v'(L)\}_1 = 0 \end{aligned} \tag{13}$$

$W_{21} = 0$ as reactions $R_C = -R_D$ in TSS_2 and in TSS_1 deflections v (L/4) = v (3L/4) due to symmetry and v' (L) = 0. Applying PQW (i.e. $U_{12} = W_{12}$), one gets MA = MB = $wL^2/12$. It can be seen from this simple example that PQW connects two distinct structural systems and provides solution for one system using the solution of other system. This is not possible through conventional theorems unless the later beam is a statically determinate part of the given problem, which is not the case in the present example.

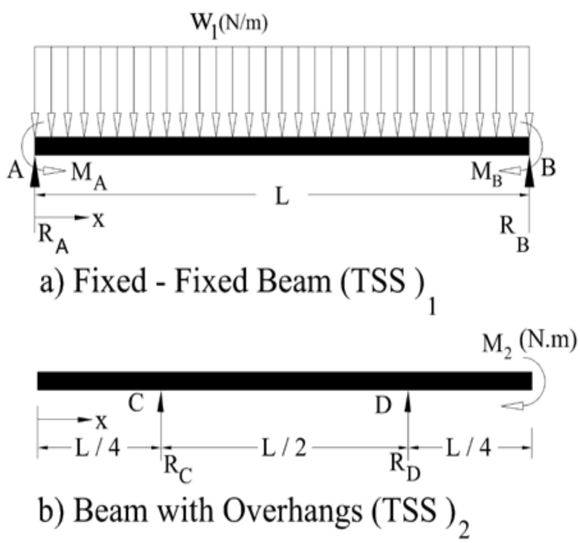

Figure 1 : Beam with both ends built - in

VII. Theorems Based on PQW

Counterpart of some of the well known energy theorems which will be applicable to TSS will now be derived from PQW. Pandita [6] has obtained deflection theorem, load theorem (counterparts of Castigliano's theorems) and unit load theorem for TSS and has also derived relative displacement theorem and its two corollaries which make calculation of nodal deflections of trusses very easy. Relative displacement theorem[6] does not have its counterpart in structural analysis. Equivalent forms of some other conventional theorems are given below:

a) Variational Principles for TSS

Variational principles applicable to topologically similar systems can be derived by considering variational form of Eqn. (11):

$$\delta(U_{mn} - W_{mn}) = \delta\pi_{mn} = 0 \tag{14}$$

$$\text{i.e}\quad \sum_{j=1}^{N} \frac{\partial \pi_{mn}}{\partial d_{jn}}\ \delta d_{jn} + \sum_{k=1}^{B} \frac{\partial \pi_{mn}}{\partial P_{jm}}\ \delta P_{jm} = 0 \tag{15}$$

Where, π_{mn} is the quasi total potential.

Making an appropriate choice of variations, either in forces or in displacements, Eqn.(14) gives rise to:

$$\frac{\partial}{\partial d_{jn}}[\pi_{mn}] = 0 \tag{16}$$

$$\text{or}\qquad \frac{\partial}{\partial P_{jm}}[\pi_{mn}] = 0 \tag{17}$$

It may be relevant to state that Eqns.(16) and (17) in respect of Topologically identical systems (when m=n) correspond to the familiar variational principles with the significant difference that π_{mn} should be replaced by the total complementary potential energy.

Unlike in the concept of total potential where one considers only applied loads and ignores reactions due to constraints which do no work, in the present context both applied forces and constraint reactions have to be considered, since the displacement field of the TSS can contribute to work terms due to reactions. Here, the total quasi potential becomes zero which is not the case of the total potential where the condition of its stationary value generates the necessary equations.

In Eqns.(16) and (17), the term 'virtual' is conspicuous by its absence as here one deals with real displacements and real forces. Since, both forces $\{P\}_m$ and displacements $\{d\}_n$ are independent of each other (as these belong to different systems), it is possible to obtain variations with either of these. For the same reason, even the use of the term 'complementary energy' does not find a place in these formulations/ theorems.

b) Reciprocal Flexibility Theorem for TSS

Consider a pair of topologically similar systems TSS_m and TSS_n with corresponding pair of directions i and j specified within each of these systems.

For a pair of global directions (i and j) defined in each of the given pair of TSS_m and TSS_n; ratio of $\bar{d}_{jm}$ (displacement in direction j due to a unit load in direction i of TSS_m) and $\bar{d}_{in}$ is directly proportional to ratio of their respective generalized reciprocal flexibilities $(f_{ji})_m$ and *$(f_{ij})_n$ corresponding to the pair of directions.* Mathematically, this can be stated as:

$$\frac{\bar{d}_{jm}}{\bar{d}_{jn}} = \frac{(f_{ji})_m}{(f_{ij})_n} \tag{18}$$

i. *Proof*

From the definition of flexibility coefficients f_{ij} with respect to a pair of global generalized directions *i* and *j* for a pair of TSS, the displacements in directions j and i in system m and n respectively are given by:

$$d_{jm} = (f_{ji})_m \; P_{im} \qquad (19)$$

$$d_{in} = (f_{ij})_n \; P_{jn} \qquad (20)$$

By dividing these equations and substituting $P_{im}=P_{jn}=1$, one obtains Eqn.(18) which reduces to Reciprocal Theorem (i.e. $f_{ij}=f_{ji}$), when the pair of systems is identical.

VIII. Application of PQW to Nonlinear Structure

Application of PQW and its derivative theorems are given Ref. [3-7] for discrete models[5] and linear structures. In this section, an example is included to illustrate application of deflection theorem[6] to a typical nonlinear structure.

a) *Illustration: A Typical Nonlinear Structure*

Equation for mid span deflection of a simply supported beam with a nonlinear elastic prop at the centre and subjected to a general case of transverse loading (vide Fig.2a) will now be obtained. Here, nonlinear characteristic of the prop are chosen to be the same as those used in Argyris and Kelsey [8] while illustrating the principle of virtual displacement. A typical symmetric load distribution as in Fig. 2b is chosen to get expression for mid span deflection as given in Ref. 8, as a special case there of.

For this purpose, a TSS_2 is chosen as in Fig 1c, here, it is termed as TES_2 by taking $E_2I_2 = E_1I_1 = EI$. Displacement of TES_2 can be written as:

$$\begin{aligned} v_2(x) &= P_2(x^3 - 2 < x - L >^3 - 3L^2 x)/12EI \\ &= P_2\, \bar{v}_2(x) \quad \text{and} \\ v_2(L) &= -P_2 L^3/6EI = -P_2 \beta / k \\ \text{where} \quad & \beta = k\, L^3/6EI \end{aligned} \qquad (21)$$

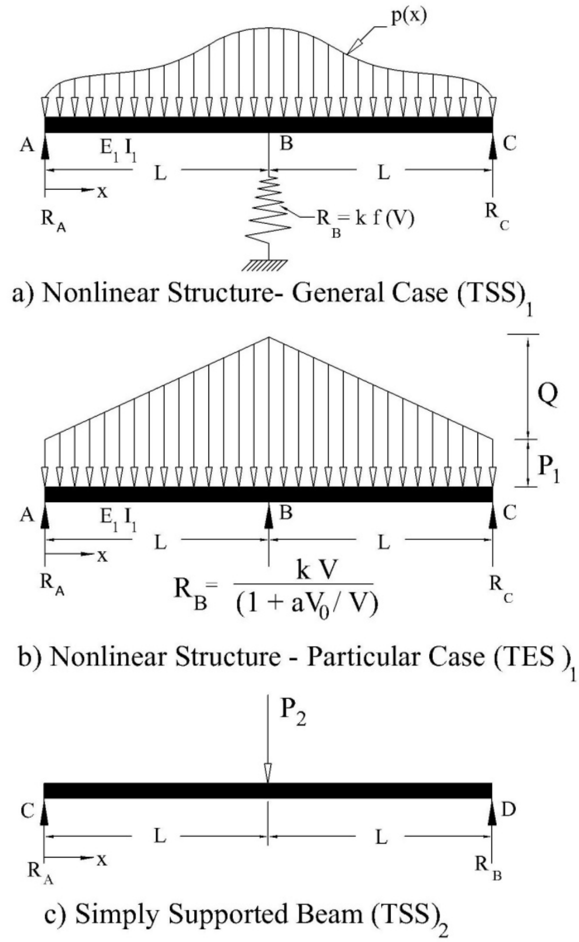

Figure 2 : Indeterminate Beam on a nonlinear support

The quasi work W_{12} is given by:

$$\begin{aligned} W_{12} &= \int_0^{2L} \{p(x)\}_1 \{P_2 \bar{v}_2(x)\}_2 \, dx \\ &+ \{R_B\}_2 \{P_2\, \bar{v}_2(L)\}_2 \\ &= \{P_2\}_2 \, [V_1 + R_B\, \bar{v}_2(L)] \end{aligned} \qquad (22)$$

$$\text{where, } V_1 = \int_0^{2L} p(x)\, \bar{v}_2(x)\, dx \quad \text{and}$$

$$\bar{v}_2(L) = v_2(L)/P_2 = -\beta / k$$

The deflection V at x = L of TES_1 is:

$$V = \partial W_{12} / \partial \{P_2\}_2 = V_1 - \beta\, f(V) \qquad (23)$$

Substituting $f(V) = V\,[1 + a/(1 - V/V_0)]$ (vide Fig. 1b, where V_0 is the limiting deformation of the prop) yields the following quadratic equation:

$$\begin{aligned} &(1+\beta)(V/V_0)^2 - [1 + \beta\,(1+a) \\ &+ V_1/V_0](V/V_0) + V_1/V_0 = 0 \end{aligned} \qquad (24)$$

The value of V_1 can be calculated for any given loading p(x). For the typical symmetrical linear load distribution shown in Fig. 1b, we get:

$$V_1 = \frac{5P_1L^4}{24EI} + \frac{2QL^5}{15EI} \tag{25}$$

It may be noted that Q = 0 results in an expression for V which is identical to the one given in Ref. [8], page 10.

IX. Import on Structural Analysis

Introduction of this Principle of Quasi Work (PQW), in the realm of structural mechanics heralds a new phase of structural analysis and has the following unique features:

a) Conventional energy theorems and variational principles are special cases of PQW and its derivatives respectively. Hence, this principle has far reaching utility.

b) It incorporates the advantages of both the force and displacement analysis procedures and unifies these distinct procedures.

c) It dispenses with the concept of '*virtual displacement*', '*virtual force*' and '*complimentary energy*'.

d) It offers simpler procedures for redundant structural analysis.

X. Conclusions

- A new theorem in its various forms has been derived. Though, this theorem has potential for application in several fields, this paper addresses its applications to the field of structural analysis through Principle of Quasi Work.
- PQW has the distinction of being able to form a link between any two distinct topologically similar structural systems, thereby, offering a wide choice for solving complex structural problems. Such a link, for the first time, has paved way for solution of many a problem with the help of the solution of a suitably chosen topologically similar problem. This has made it possible to have new procedures for analysis of statically indeterminate structures.
- Conventional energy/ variational principles fall out as a special case of PQW and its derivatives.
- PQW dispenses with the concepts of virtual displacements, virtual forces and complementary energy.
- Utility of PQW through its various derivatives is demonstrated by its application to nonlinear structures.
- Versatility of PQW stands already established by various authors through applications to discrete and linear continuum structures.

XI. Acknowledgments

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