Common Fixed Point Theorems for Self-Maps on Metric Spaces with Weak Distance

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1. INTRODUCTION

Let \((X; d)\) be a metric space. If \(f\) is a self-map on \(X\), and \(x_0 \in X\), we denote by \(fx\) the \(f\)-image of \(x_0\).

As a weaker form of the metric, Kada et al. [5] introduced the notion of weak distance (or simply \(w\)-distance) on \(X\) as follows:

Definition 1.1. Let \((X; d)\) be a metric space and \(p : X \times X \to [0, \infty)\) satisfy the following conditions:

\((w_1)\) \(p(x, y) \leq p(x, z) + p(z, y)\) for all \(x, y, z \in X\).

\((w_2)\) For any \(x \in X\), \(p(x, \cdot) : X \to \mathbb{R}_+\) is lower semi continuous in the second variable, that is \(p(x, y) \leq \liminf_{n \to \infty} p(x, y_n)\) whenever \(y_n \to y\) as \(n \to \infty\) for some \(x \in X\), and \(w_3\) Given \(\epsilon > 0\), there is a \(\delta > 0\) such that \(p(z, x) \leq \delta\) and \(p(z, y) \leq \delta\) imply that \(d(x, y) < \epsilon\).

Then \(p\) is known as a \(w\)-distance on \(X\).

Obviously, every metric \(d\) on \(X\) satisfies the conditions \((w_1)-(w_3)\), that is \(d\) is a \(w\)-distance on \(X\).

Example 1.1. Let \(X = \left\{ \frac{1}{m} : m = 1, 2, 3, \ldots \right\} \cup \{0\}\) with metric \(d(x, y) = x + y\) if \(x \neq y\) and \(d(x, y) = 0\) if \(x = y\) for all \(x, y \in X\). Note that \((X, d)\) is a complete metric space. Define \(p(x, y) = y\). Then \(p\) a \(w\)-distance on \(X\).

For other examples one can refer to [5].

Recently Ume and Sucheoel [4] have proved two common fixed point theorems, given below, for self-maps on a complete metric space with a \(w\)-distance on \(X\), which generalize and improve the results of Fisher [1], Dien [3] and Liu et al. [6].

Theorem 1.1 ([4], Theorem 3.1). Let \(X\) be a complete metric space \((X, d)\) with \(w\)-distance \(p\) on it. Suppose that \(f, g : X \to X\) and \(\phi : X \to [0, \infty)\) satisfy the conditions:

\[(1.1)\] \(g(X) \subseteq f(X)\)

\[(1.2)\] \(p(t, gx) \leq rp(t, fx) + \phi(fx) - \phi(gx)\) for all \(x, y \in X, 0 \leq r < 1\),

\[(1.3)\] \(
\lim_{n \to \infty} p(t, f^n(x)) = 0 = \lim_{n \to \infty} p(t, g^n(x)),
\)

we have

\[(1.4)\] \(
\lim_{n \to \infty} \max \{p(t, f^n(x)), p(t, g^n(x)), p(fg^n(x), fg^n(x))\} = 0,
\)

and

\[(1.5)\] \(\{p(u,fx) + p(u, gx) + p(fgx, fgx) : x \in X\} > 0\).

Then \(f\) and \(g\) will have a unique common fixed point.

Theorem 1.2 ([4], Theorem 3.6). Let \(X\) be a complete metric space \((X, d)\) with \(w\)-distance \(p\) and the mappings \(f, g : X \to X\) satisfy the conditions (a) and (d). Suppose that \(\phi, \psi : X \to [0, \infty)\) are such that

\begin{itemize}
  \item (a) \(g(X) \subseteq f(X)\)
  \item (b) \(p(t, gx) \leq rp(t, fx) + \phi(fx) - \phi(gx)\) for all \(x, y \in X, 0 \leq r < 1\),
  \item (c) \(\lim_{n \to \infty} p(t, f^n(x)) = 0 = \lim_{n \to \infty} p(t, g^n(x))\),
  \item (d) \(\lim_{n \to \infty} \max \{p(t, f^n(x)), p(t, g^n(x)), p(fg^n(x), fg^n(x))\} = 0\),
  \item (e) \(\{p(u,fx) + p(u, gx) + p(fgx, fgx) : x \in X\} > 0\).
\end{itemize}
for every sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in \( X \) with \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = t \),
we have \( \lim_{n \to \infty} \max \{ p(t, f x_n), p(t, g x_n), p(g x_n, g f x_n) \} = 0 \),
and
\[
p(gx, gy) \leq a_1 p(fx, fy) + a_2 p(fx, gx) + a_3 p(fy, gy) + a_4 p(fx, gy) + a_5 \sqrt{p(gx, fy) d(fy, gx)} + [\phi(fx) - \phi(gx)] + [\psi(fy) - \psi(gy)]
\]
for all \( x, y \in X \) where \( a_i \in [0, 1] \), \( i = 1, 2, 3, 4, 5 \) are such that
\[ a_1 + a_4 + a_5 < 1 \quad \text{and} \quad a_1 + a_2 + a_3 + 2a_4 < 1 \] (1.6)

Then \( f \) and \( g \) will have a unique common fixed point.

The purpose of this paper is to establish two fixed point theorems, which generalize those of Brian Fisher [1], Dien [3] and Liu et al. [6].

11. Preliminaries

First we state the following lemma, proved in [5]:

**Lemma 2.1.** Let \( X \) be a metric space with \( w \)-distance \( p \) on it. Then
\( p(x, y) = 0 \) and \( p(x, z) = 0 \) imply that \( y = z \).

Also \( \langle x_n \rangle_{n=1}^{\infty} \subset X \) is a Cauchy sequence in \( X \), provided
\( p(x_n, x_m) \leq \alpha n \) for all \( m > n \geq 1 \)
and
\( p(x_n) \leq \alpha n \) for all \( n \geq 1 \) each \( x \in X \).

We now introduce an orbit notion that is followed in the rest of the paper.

**Definition 2.1.** Let \( f \) and \( g \) be self-maps on \( X \). Given \( x_0 \in X \), if there exist points \( x_1, x_2, x_3, \ldots \) in \( X \) such that
\( y_n = g x_{n-1} = f x_n \) for \( n \geq 1 \), (2.1)
the sequence \( \langle y_n \rangle_{n=1}^{\infty} \) is called a \( g \)-orbit relative to \( f \) at \( x_0 \) or simply a \( (g, f) \)-orbit at \( x_0 \).
We call \( \langle x_n \rangle_{n=1}^{\infty} \) a base sequence associated with the \( g \)-orbit (2.1). Note that when \( f \) is the identity map \( i \) on \( X \), (2.1) and the base sequence coincide with the \( g \)-orbit \( g x_0, g x_1, g x_2, \ldots \) at \( x_0 \).
This notion was adopted in [8]. The notion of \( (g, f) \)-orbit is not unique. For instance, Nesci [7] defined a \( (g, f) \)-orbit at \( x_0 \) by the iterations:
\[ x_{2n-1} = g x_{2n-2}, \quad x_{2n} = f x_{2n-1} \quad \text{for} \quad n \geq 1 \] (2.2)
which was employed by Fisher [1] though no name was mentioned.

**Remark 2.1.** If the self-maps \( f \) and \( g \) on \( X \) satisfy the inclusion (1.1), then by a routine induction, it can be easily shown that \( (g, f) \)-orbit at each \( x_0 \) exists with the choice (2.1). Given \( x_0 \in X \), there can be more than one base sequence \( \langle x_n \rangle_{n=1}^{\infty} \) as the following examples reveal:

**Example 2.1.** Let \( X = \mathbb{R} \) with usual metric \( d(x, y) = |x - y| \) for all \( x, y \in X \). Define \( f, g : X \to X \) by
\[ f(x) = x^2 \quad \text{and} \quad g(x) = \frac{x^2}{4} \quad \text{for} \quad x \in X \].
Then (1.1) is obvious and hence by Remark 2.1, orbits can be specified at each \( x_0 \). Given \( x_0 \in X \), choose \( x_n = \pm \frac{x_0}{2^n} \) for \( n \geq 1 \).

Since each \( x_n \) has two choices, several base sequences \( \langle x_n \rangle_{n=1}^{\infty} \) can be specified to get the respective \( (g, f) \)-orbit.

We now prove

**Lemma 2.2.** Suppose that \( (X, d) \) is a metric space with \( w \)-distance \( p \) on \( X \). Let \( f, g : X \to X \) and \( \phi : X \to [0, \infty) \) satisfy the inclusion (1.1) and the condition (b) of Theorem 1.1. If \( X \) is complete metric space and \( x_0 \in X \), then
\[ \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z \quad \text{for some} \quad z \in X \]. (2.3)

**Proof.** Given \( x_0 \in X \), Suppose that \( \langle x_n \rangle_{n=1}^{\infty} \) is a base sequence at \( x_0 \) and \( f, g \) are such that (2.1) holds good. Now, by condition (b) with \( x = x_{n-1} \), we have
\[
p(t, f x_n) = p(t, g x_{n-1}) \leq r \cdot p(t, f x_{n-1}) + \phi(f x_{n-1}) - \phi(g x_{n-1})
\]
so that for any \( k \geq 2 \)
\[
\sum_{n=1}^{k} p(t, f x_n) \leq r \cdot \sum_{n=1}^{k} p(t, f x_{n-1}) + \sum_{n=1}^{k} [\phi(f x_{n-1}) - \phi(f x_n)]
\]
which gives

$$\sum_{n=2}^{k} p(t, fx_n) \leq \frac{r}{1-r} p(t, fx_1) + \frac{1}{1-r} [\phi(fx_0) - \phi(fx_k)]$$

$$< \frac{r}{1-r} p(t, fx_0) + \frac{1}{1-r} \phi(fx_0)$$

Showing that $\sum_{n=2}^{\infty} p(t, fx_n)$ converges so that $n$th term tends to 0 as $n \to \infty$, that is $\lim_{n \to \infty} p(t, fx_n) = 0$.

Now, by (i) of Lemma 2.1, it follows that $\langle fx_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in the $(g, f)$-orbit $X$. Since $X$ is complete, there is a $z \in X$ such that $fx_n \to z$ as $n \to \infty$.

Similar argument shows $\langle gx_n \rangle_{n=1}^{\infty}$ converges to $z'$ in $X$. Proceeding the limit as $n \to \infty$ in (2.1) and using these limits, it follows that $z = z'$, proving the lemma.

Remark 2.2. The converse of Lemma 2.2 is not true. In fact, the example given below shows that we can find a metric space $(X, d)$ with a $w$-distance $p$ on it satisfying condition (a) and (b) of Theorem 1.1 such that for any $x_0 \in X$ and for any base sequence $\langle x_n \rangle_{n=1}^{\infty}$ at $x_n$, both $\langle fx_n \rangle_{n=1}^{\infty}$ and $\langle gx_n \rangle_{n=1}^{\infty}$ converge to the same point in $X$, but $X$ is not complete.

Example 2.2. Let $X = [0, 1]$ with $d(x, y) = |x - y|$ for all $x, y \in X$. Clearly $(X, d)$ is an incomplete metric space. Define $f, g : X \to X$ by $fx = \frac{2x + 1}{4}$ and $gx = \frac{1}{2}$ for $x \in X$. Then $g(X) = \left\{\frac{1}{2}\right\}$ and $f(X) = \left\{\frac{1}{4}, \frac{3}{4}\right\}$ so that $g(X) \subset f(X)$. Let

$$p(x, y) = \frac{1}{4} \max \left\{ |2x - 1|, |2x - 4y + 1|, 2|x - y| \right\}$$

for $x, y \in X$,

which will be a $w$-distance on $X$. Also for any $t \in X$: $p(t, gx) = \frac{1}{4} |2t - 1|$ and

$$p(t, fx) = \frac{1}{8} \max \left\{ |2(2t - 1)|, |4(t - x)|, |(4t - 2x - 1)| \right\}$$

from which it follows $p(t, gx) \leq \frac{1}{2} p(t, fx) + \phi(fx) - \phi(gx)$ for any $x \in X$ where $\phi(x) = 1$ for all $x \in X$.

Note that for any $x_0 \in X$ there is only one base sequence $\langle x_n \rangle_{n=1}^{\infty}$ given by $x_n = \frac{1}{2}$ for all $n \geq 1$ so that both $\langle fx_n \rangle_{n=1}^{\infty}$ and $\langle gx_n \rangle_{n=1}^{\infty}$ are constant sequences with each term equal to $\frac{1}{2}$; and hence they converge to $\frac{1}{2} \in X$.

Lemma 2.3. Suppose $(X, d)$ is a metric space with $w$-distance $p$ on it. Let $f, g : X \to X$ and $\phi, \psi : X \to [0, \infty)$ be such that (a) of Theorem 1.1 and (f) of Theorem 1.2 hold. If $(X, d)$ is a complete metric space, then for any $x_0 \in X$ and for any base sequence $\langle x_n \rangle_{n=1}^{\infty}$ at $x_0$, both the sequences $\langle fx_n \rangle_{n=1}^{\infty}$ and $\langle gx_n \rangle_{n=1}^{\infty}$ converge to the same point in $X$.

Proof. Suppose that $\langle x_n \rangle_{n=1}^{\infty}$ is a base sequence at some $x_0 \in X$ with the choice (2.1). Write

$$\gamma_n = p(fx_n, fx_{n+1}) = p(gx_{n-1}, gx_n) \text{ for } n \geq 1$$

Then by (f) of Theorem 1.2, we have
\[\gamma_n = p(gx_{n-1}, gx_n)\]
\[\leq a_1p(fx_{n-1}, fx_n) + a_2p(fx_{n-1}, gx_{n-1}) + a_3p(fx_n, gx_n)\]
\[+ a_4p(fx_{n-1}, gx_n) + a_5\sqrt{p(gx_{n-1}, fx_n)d(fx_n, gx_{n-1})}\]
\[+ [\phi(fx_{n-1}) - \phi(gx_{n-1})] + [ (fx_n) - (gx_n)]\]
\[\leq \alpha \gamma_{n-1} + a_2\gamma_{n-1} + a_3\gamma_n + a_4(\gamma_{n-1} + \gamma_n)\]
\[+ [\phi(fx_{n-1}) - \phi(fx_n)] + [ (fx_n) - (fx_{n+1})]\]
\[= (a_1 + a_2 + a_4)\gamma_{n-1} + (a_3 + a_4)\gamma_n\]
\[+ [\phi(fx_{n-1}) - \phi(fx_n)] + [ (fx_n) - (fx_{n+1})]\]
from which we get
\[\gamma_n \leq \alpha \gamma_{n-1} + \beta \{ [\phi(fx_{n-1}) - \phi(fx_n)] + [ (fx_n) - (fx_{n+1})] \}, \; n \geq 2,\]

where \(\alpha = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4}\) and \(\beta = \frac{1}{1 - a_3 - a_4}\)

Therefore for any integer \(k \geq 2\),
\[\sum_{n=2}^{k} \gamma_n \leq \alpha \sum_{n=2}^{k} \gamma_{n-1} + \beta \left[ \phi(fx_1) - \phi(fx_k) \right] + \left[ (fx_2) - (fx_3) \right] \]

which gives
\[\sum_{n=2}^{k} \gamma_n \leq \frac{\gamma_1 \alpha}{1 - \alpha} + \frac{\beta [\phi(fx_1) + (fx_2)]}{1 - \alpha}\]

showing that \(\sum_{n=2}^{\infty} \gamma_n\) converges and hence \(n \to 0\) as \(n \to \infty\). Also from the above lines, for \(m > n \geq 2\), we see that \(p(fx_n, fx_m) \leq \alpha_n\) where \(\alpha_n = \gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1}\). Since \(\alpha \to 0\) as \(n \to \infty\), it follows from (h) of Lemma 2.1 that \(\langle fx_n \rangle_{n=1}^{\infty}\) is a Cauchy sequence in \(X\) and hence converges to some \(z \in X\). Similarly we can prove that \(\langle gx_n \rangle_{n=1}^{\infty}\) converges to some \(z' \in X\). But \(gx_{n-1} = fx_n\) for all \(n \geq 1\) it follows that \(z = z'\), completing the proof of lemma.

**Remark 2.3.** The converse of Lemma 2.3 is not true. In fact, it is not difficult to exhibit a metric space \((x, d)\) with \(w\)-distance \(p\) on it for which (a) of Theorem 1.1 and (f) of Theorem 1.2 in which for any \(x_0 \in X\) and for any base sequence \(\langle x_n \rangle_{n=1}^{\infty}\) at \(x_0\) both sequences \(\langle fx_n \rangle_{n=1}^{\infty}\) and \(\langle gx_n \rangle_{n=1}^{\infty}\) converge to the same point, yet \((x, d)\) is not complete.

\[\lim_{n \to \infty} \max(p(z, fu_n), p(z, gu_n), p(fu_n, gu_n)) = 0.\]  

(3.1)

Then \(z\) is a unique common fixed point of \(f\) and \(g\).

**Proof.** Writing \(x = x_n\) in (k) we get
\[p(z, fx_{n+1}) = p(z, gx_n) \leq rp(z, fx_n) + \phi(fx_n) - \phi(gx_n).\]
Then as in Lemma 2.2, we can prove that \( \sum_{n=1}^{\infty} p(z, f x_n) \) converges hence
\[
\lim_{n \to \infty} p(z, f x_n) = \lim_{n \to \infty} p(z, g x_n) = 0.
\]
Using this in (3.1), it follows that
\[
\lim_{n \to \infty} p(f g x_n, g f x_n) = 0.
\]

Now if \( z \) is not a common fixed point of \( f \) and \( g \), then either \( f z \neq z \) or \( g z \neq z \); and therefore, by the condition (1.4) of Theorem 1.1
\[
0 < \inf \{p(z, f x) + p(z, g x) + p(f g x, g f x) : x \in X\}
\]
\[
\leq \inf \{p(z, f x_n) + p(z, g x_n) + p(f g x_n, g f x_n) : n \geq 1\}
\]
\[
= 0,
\]
a contradiction. Hence \( f z \neq z \) and \( g z \neq z \).

The uniqueness of the common fixed point \( z \) follows as in the proof of Theorem 3.1 of [4].

**Remark 3.1.** In view of Remark 2.3, Theorem 3.1 generalizes Theorem 1.2. Also since \( d \) is a \( w \)-distance, the results proved by Dien [3] and Liuet.al [6] will be particular cases of Theorem 3.1.

**Theorem 3.2.** Let \( (X, d) \) be a metric space with \( w \)-distance \( p \) on it. Suppose that \( f, g : X \to X \) and \( \phi : X \to [0, \infty) \) satisfy the inclusion (1.1) and the condition (1.4) of Theorem 1.1 and the condition (f) of Theorem 1.2. If \( (j) \) and \( (l) \) hold good, then \( z \) is the unique common fixed point of \( f \) and \( g \).

**Proof.** The proof is similar to the first main result and is omitted here.

**Remark 3.2.** In view of Remark 2.6, Theorem 3.2 generalizes Theorem 1.2. Also since \( d \) is a \( w \)-distance on \( X \), the fixed point theorem of Fisher [1] is a particular case of Theorem 3.2 with \( p = d \).

**References Références Referencias**