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An Adaptive Filter to Pick up a Wiener Filter from the Error using MSE with and Without Noise

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Introduction- In this paper we explain the suboptimum channel equalization approach. This approach employ linear transversal filter that we will explain. This filter structure has computational complexity that is linear with the order.

This filter is shown in figure 1. Its input is the sequence v_k , its output is the estimate of the output sequence l_k . The estimate might be expressed as

$$\hat{l}_k = \sum_{j=0}^K c_j v_{k-j} \quad (1)$$

The estimate l_k is quantized to the nearest information symbol.

Considerable research have been done to optimize the filter coefficients c_k . A measure of performance for digital communication system is the average probability of error. In this system this is highly non linear function of c_k . As we can see this method is computationally complex.

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1. INTRODUCTION

In this paper we explain the suboptimum channel equalization approach. This approach employ linear transversal filter that we will explain. This filter structure has computational complexity that is linear with the order.

This filter is shown in figure 1. Its input is the sequence v_k , its output is the estimate of the output sequence \hat{I}_k . The estimate might be expressed as

$$\hat{I}_k = \sum_{j=-K}^K c_j v_{k-j} \quad (1)$$

The estimate \hat{I}_k is quantized to the nearest information symbol.

Considerable research have been done to optimize the filter coefficients c_k . A measure of performance for digital communication system is the average probability of error. In this system this is highly non linear function of c_k . As we can see this method is computationally complex.

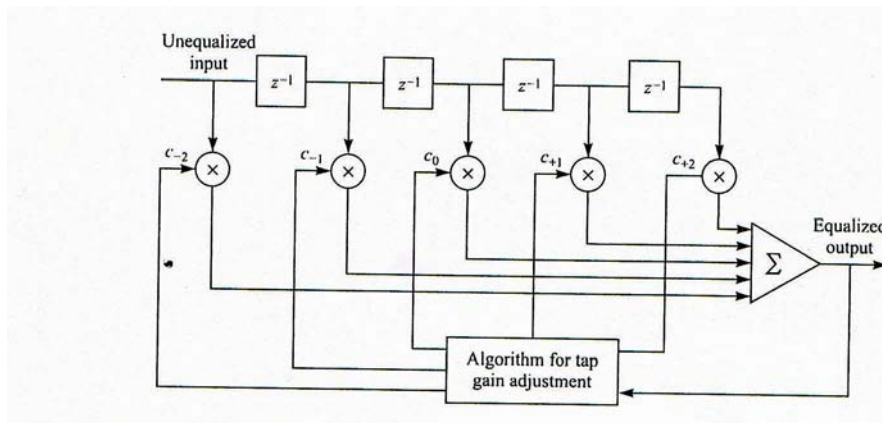


Figure 1

Two other methods are widely used. First, the peak distortion method. Second, the mean square error method.

a) The peak Distortion criterion

The peak distortion criterion is simply the worst case inter symbol interference at the output of the equalizer. It is the minimization of this performance index. First we assume that the equalizer has infinite taps. Second the case that this number is finite.

The cascade filter have an impulse response f_n . and the equalizer impulse response c_n . this can be represented by a single filter with impulse response

$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j} \quad (2)$$

Q_n is simply the convolution of c_n and f_n . the output of the k th sampling instant is

$$\hat{I}_k = q_0 I_k + \sum_{n \neq k} I_n q_{k-n} + \sum_{j=-\infty}^{\infty} c_j \eta_{k-j} \quad (3)$$

The first term of equation 3 is the desired symbol if we normalize q_0 to unity. The second term is inter symbol interference. The peak value of this interference is

$$\begin{aligned} \mathcal{D}(c) &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |q_n| \\ &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left| \sum_{j=-\infty}^{\infty} c_j f_{n-j} \right| \end{aligned} \quad (4)$$

$\mathcal{D}(c)$ is a function of the equalizer tap weights.

It is possible to have $D(c)$ i. e. $q_n = 0$ for all n except $n=0$. This way we will cancel all ISI. This can be expressed as

$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j} = \begin{cases} 1 & (n = 0) \\ 0 & (n \neq 0) \end{cases} \quad (5)$$

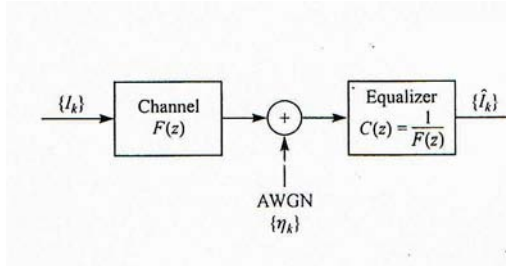


Figure 2

By taking the z transform we have

$$Q(z) = C(z)F(z) = 1 \quad (6)$$

Or simply

$$C(z) = \frac{1}{F(z)} \quad (7)$$

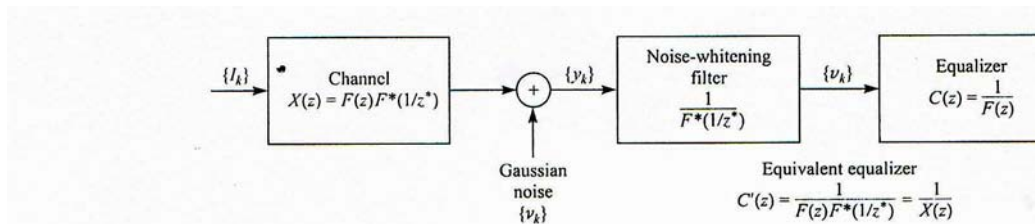


Figure 3

The performance can be expressed in terms of SNR at the output. We normalize the received signal energy to one. This means that $q_0 = 1$ and the expected value of I^2 is one. Then the SNR is the reciprocal of the noise variance at the output.

The value σ^2 can be determined by absorbing the input sequence of the equivalent equalizer which has a spectral density

$$S_{vv}(\omega) = N_0 X(e^{j\omega T}), \quad |\omega| \leq \frac{\pi}{T} \quad (10)$$

The noise sequence at the output of the equalizer has a power spectral density

$$S_{nn}(\omega) = \frac{N_0}{X(e^{j\omega T})}, \quad |\omega| \leq \frac{\pi}{T} \quad (11)$$

$C(z)$ is the z transform of c_j . Note that the equalizer $C(z)$ is simply the inverse of the linear filter $F(z)$. This will result in complete elimination of ISI. We call this a zero forcing filter. Figure 2 is the block diagram.

We use a noise whitening filter with transfer function $1/F(1/z)$ in cascade to the zero forcing function $1/F(z)$ which result in an equalizer with a transfer function

$$C'(z) = \frac{1}{F(z)F^*(1/z^*)} = \frac{1}{X(z)} \quad (8)$$

As in figure 3 the input is y_k . The output is the desired symbols. The impulse response is

$$\begin{aligned} c'_k &= \frac{1}{2\pi j} \oint C'(z) z^{k-1} dz \\ &= \frac{1}{2\pi j} \oint \frac{z^{k-1}}{X(z)} dz \end{aligned} \quad (9)$$

$X(z)$ is a polynomial with $2L$ roots. ($p_1, p_2, \dots, p_L, 1/p_1, 1/p_2, \dots, 1/p_L$). The closed contour of the integral can be the unit circle.

The variance of the noise at the output of the equalizer is

$$\begin{aligned} \sigma_n^2 &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} S_{nn}(\omega) d\omega \\ &= \frac{TN_0}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{d\omega}{X(e^{j\omega T})} \end{aligned} \quad (12)$$

The SNR for the zero forcing equalizer is

$$\gamma_\infty = \frac{1}{\sigma_n^2} = \left[\frac{TN_0}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{d\omega}{X(e^{j\omega T})} \right]^{-1} \quad (13)$$

The spectral characteristics of $X(\omega)$ is the Fourier transform of the samples x_n has a nice relationship to the filter $H(\omega)$.

$$x_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 e^{j\omega k T} d\omega \quad (14)$$

The equation 14 can be

$$x_k = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left[\sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right) \right|^2 \right] e^{j\omega k T} d\omega \quad (15)$$

The Fourier Transform of x_k is

$$X(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} x_k e^{-j\omega k T} \quad (16)$$

The inverse is

$$x_k = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} X(e^{j\omega T}) e^{j\omega k T} d\omega \quad (17)$$

From equation 15 and 17 we have

$$X(e^{j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right) \right|^2, \quad |\omega| \leq \frac{\pi}{T} \quad (18)$$

The right hand side of equation 18 is called the folded spectrum.

Using equation 13 and 18 mean that the SNR is

$$\gamma_{\infty} = \left[\frac{T^2 N_0}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{d\omega}{\sum_{n=-\infty}^{\infty} |H(\omega + 2\pi n/T)|^2} \right]^{-1} \quad (19)$$

We can see that if $H(\omega)$ has nulls takes small values the performance is poor. The ideal case is

$$\sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right) \right|^2 = T, \quad |\omega| \leq \frac{\pi}{T} \quad (20)$$

In this case the SNR is maximum and equals

$$\gamma_{\infty} = \frac{1}{N_0} \quad (21)$$

Let us say that the equalizer has $2K+1$ taps. The input is f_n . the output q_n is the convolution of f_n and c_n . If q_0 is normalized to unity the peak distortion is

$$\mathcal{D}(c) = \sum_{\substack{n=-K \\ n \neq 0}}^{K+L-1} |q_n| = \sum_{\substack{n=-K \\ n \neq 0}}^{K+L-1} \left| \sum_j c_j f_{n-j} \right| \quad (22)$$

The problem is to minimize $\mathcal{D}(c)$ with respect to c_j . It has been shown that $\mathcal{D}(c)$ has a global minimum. Minimization can be done using the method of steepest descent. One special case is

$$D_0 = \frac{1}{|f_0|} \sum_{n=1}^L |f_n| \quad (23)$$

Is less than unity. In this case the inter symbol interference is not severe. In this case we select $q_0 = 1$ and $q_n = 0$ for n not equal to zero.

b) *Mean Square Error (MSE) criterion*

In this method c_k is adjusted to minimize the mean square value of the error

$$\varepsilon_k = I_k - \hat{I}_k \quad (24)$$

The performance index J is

$$J = E|\varepsilon_k|^2 = E|I_k - \hat{I}_k|^2 \quad (25)$$

J is a quadratic function of the equalizer coefficients c_j . we consider the minimization of equation 25.

The estimate of I is equal to

$$\hat{I}_k = \sum_{j=-\infty}^{\infty} c_j v_{k-j} \quad (26)$$

Substituting 26 in 25 we will have the quadratic function of the coefficients c_k . this function can be minimized with respect to c_k to get c_k optimum. We use the orthogonality principle in MSE. That is we select c_k so that the error is orthogonal to the signal sequence v_{k-l} .

$$E \left[\left(I_k - \sum_{j=-\infty}^{\infty} c_j v_{k-j} \right) v_{k-l}^* \right] = 0 \quad (27)$$

Substituting for the error

$$\sum_{j=-\infty}^{\infty} c_j E(v_{k-j} v_{k-l}^*) = E(I_k v_{k-l}^*), \quad -\infty < l < \infty \quad (28)$$

This mean

$$E(v_{k-j}v_{k-l}^*) = \sum_{n=0}^L f_n^* f_{n+l-j} + N_0 \delta_{lj} \\ = \begin{cases} x_{l-j} + N_0 \delta_{lj} & (|l-j| \leq L) \\ 0 & (\text{otherwise}) \end{cases} \quad (29)$$

And

$$E(I_k v_{k-l}^*) = \begin{cases} f_{-l}^* & (-L \leq l \leq 0) \\ 0 & (\text{otherwise}) \end{cases} \quad (30)$$

Taking the z transform

$$C(z)[F(z)F^*(1/z^*) + N_0] = F^*(1/z^*) \quad (31)$$

This mean

$$C(z) = \frac{F^*(1/z^*)}{F(z)F^*(1/z^*) + N_0} \quad (32)$$

And

$$C'(z) = \frac{1}{F(z)F^*(1/z^*) + N_0} \\ = \frac{1}{X(z) + N_0} \quad (33)$$

We see that the deference of the two methods is the factor N_0 . This mean that if $N_0=0$ the two methods give the same results. The first method is to minimize the peak distortion $D(c)$ and the second method is to minimize the MSE performance index j .

$J = E(e^2)$. To minimize this we use the fact that the error e is equal to I minus the estimate of I . then we use the orthogonality condition to get

$$J_{\min} = E(\varepsilon_k I_k^*) \\ = E|I_k|^2 - \sum_{j=-\infty}^{\infty} c_j E(v_{k-j} I_k^*) \\ = 1 - \sum_{j=-\infty}^{\infty} c_j f_{-j} \quad (34)$$

If we take equation 34 to the frequency domain it will be more informative. This mean

$$B(z) = C(z)F(z) \\ = \frac{F(z)F^*(1/z^*)}{F(z)F^*(1/z^*) + N_0} \\ = \frac{X(z)}{X(z) + N_0} \quad (35)$$

The term b_0 is

$$b_0 = \frac{1}{2\pi j} \oint \frac{B(z)}{z} dz \\ = \frac{1}{2\pi j} \oint \frac{X(z)}{z[X(z) + N_0]} dz \quad (36)$$

This is equal to

$$b_0 = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{j\omega T})}{X(e^{j\omega T}) + N_0} d\omega \quad (37)$$

Substitution of 37 into 34 yields the minimum MSE in the form

$$J_{\min} = 1 - \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{j\omega T})}{X(e^{j\omega T}) + N_0} d\omega \\ = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{X(e^{j\omega T}) + N_0} d\omega \\ = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{T^{-1} \sum_{n=-\infty}^{\infty} |H(\omega + 2\pi n/T)|^2 + N_0} d\omega \quad (38)$$

For $X(\omega) = 1$ we have

$$J_{\min} = \frac{N_0}{1 + N_0} \quad (39)$$

The SNR is

$$\gamma_{\infty} = \frac{1 - J_{\min}}{J_{\min}} \quad (40)$$

Let us say we have $2K+1$ taps. The output of the equalizer

$$\hat{I}_k = \sum_{j=-K}^K c_j v_{k-j} \quad (41)$$

The MSE for the equalizer

$$J(K) = E|I_k - \hat{I}_k|^2 = E \left| I_k - \sum_{j=-K}^K c_j v_{k-j} \right|^2 \quad (42)$$

Minimization of $J(k)$ with respect to the weights c_j give us the set of equations

$$\sum_{j=-K}^K c_j \Gamma_{lj} = \xi_l, \quad l = -K, \dots, -1, 0, 1, \dots, K \quad (43)$$

Where

$$\Gamma_{lj} = \begin{cases} x_{l-j} + N_0 \delta_{lj} & (|l-j| \leq L) \\ 0 & (\text{otherwise}) \end{cases} \quad (44)$$

And

$$\xi_l = \begin{cases} f_{-l}^* & (-L \leq l \leq 0) \\ 0 & (\text{otherwise}) \end{cases} \quad (45)$$

We can express the set of linear equations in matrix form

$$\Gamma C = \xi \quad (46)$$

Where C is the coefficients and T is the covariance of the input and E is the desired times the input vector. C optimum is

$$C_{\text{opt}} = \Gamma^{-1} \xi \quad (47)$$

This C opt minimize J(k) with a minimum value

$$\begin{aligned} J_{\min}(K) &= 1 - \sum_{j=-K}^0 c_j f_{-j} \\ &= 1 - \xi^H \Gamma^{-1} \xi \end{aligned} \quad (48)$$

c) Performance Characteristics of the MSE Equalizer

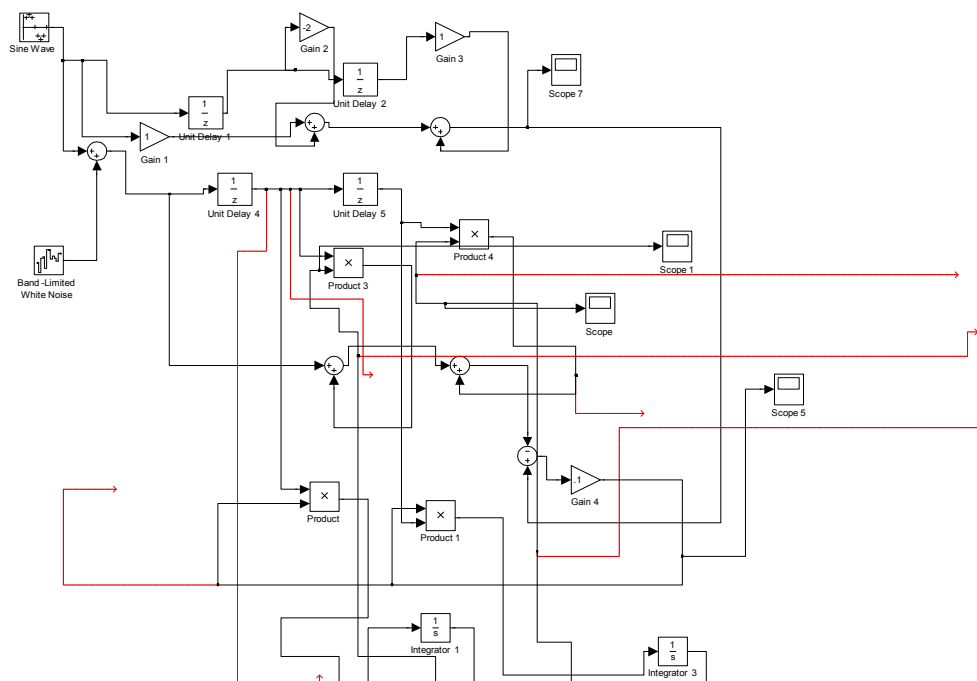


Figure 4 : The system used to measure performance of MSE. We have a wiener filter and an adaptive filter that try to pick it up. We might measure the performance with or without noise

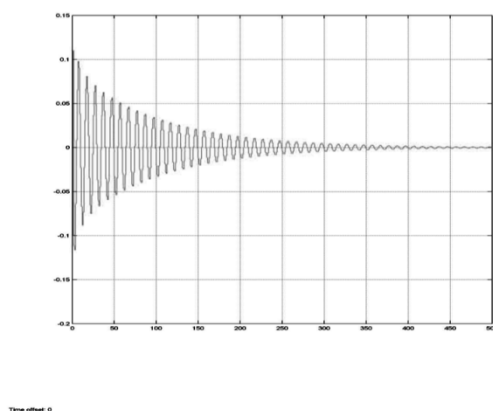
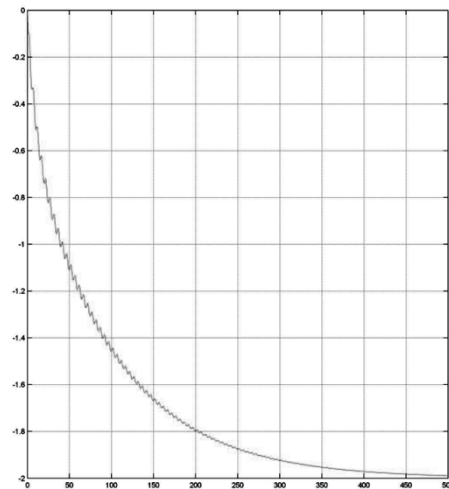
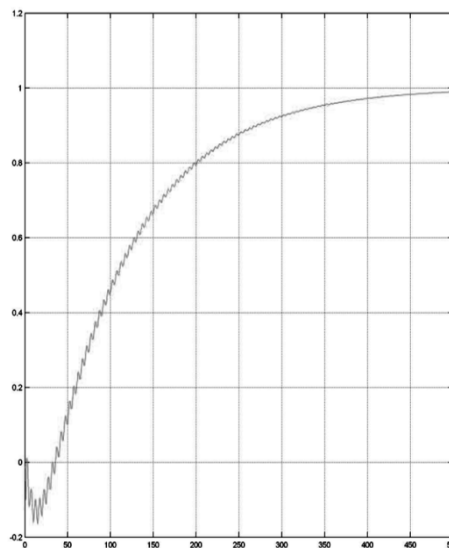


Figure 5 : The error goes to zero as the adaptive filter picks up the wiener filter. For this part the noise input is zero



Time offset: 0

Figure 6 : The first coefficient is one. This is the second coefficient as the adaptive filter pick up the wiener filter using MSE method. For this part the noise input is zero



Time offset: 0

Figure 7 : This is the third coefficient as the adaptive filter pick up the wiener filter using MSE method. For this part the noise input is zero

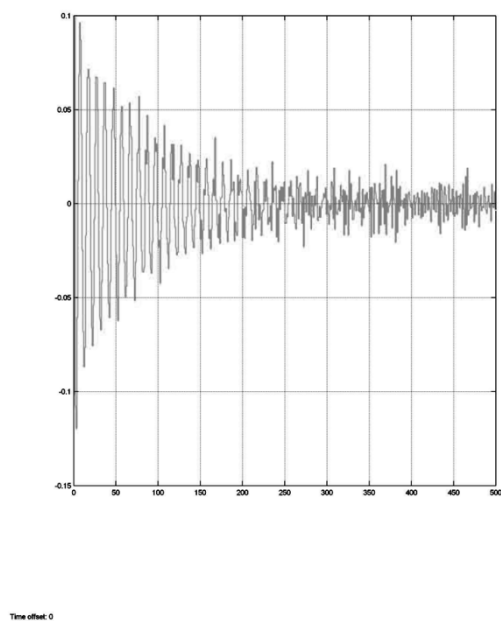


Figure 8 : The error goes to zero as the adaptive filter picks up the wiener filter. For this part the noise input is .001

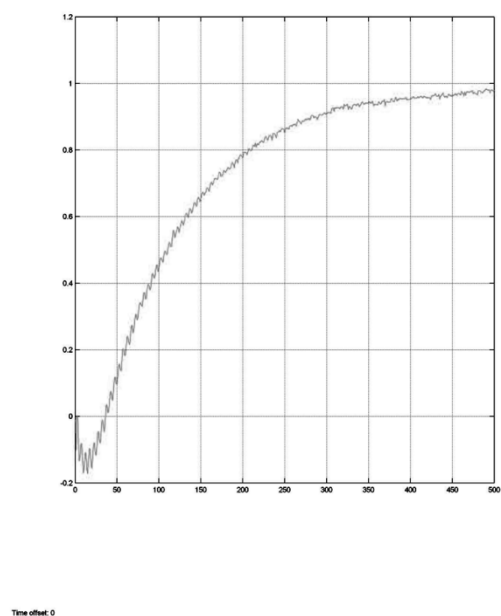


Figure 9 : This is the third coefficient as the adaptive filter pick up the wiener filter. For this part the noise input is .001

In this section we study the performance of the linear equalizer optimized using MSE criterion. Both the minimum MSE and probability of error are considered as performance measure. We begin by evaluating the minimum MSE and output SNR for some specific channels.

d) Probability of error performance of linear MSE equalizer

We studied the performance of linear equalizer in terms of minimum MSE and output SNR. There is no simple relation of these quantities and the probability of error.

The estimate of I_n is

$$\hat{I}_n = q_0 I_n + \sum_{k \neq n} I_k q_{n-k} + \sum_{j=-K}^K c_j \eta_{n-j} \quad (56)$$

q_n is the convolution of the impulse response of the equalizer and the channel

$$q_n = \sum_{k=-K}^K c_k f_{n-k} \quad (57)$$

The input to the equalizer is

$$v_k = \sum_{j=0}^L f_j I_{k-j} + \eta_k \quad (58)$$

The first term of 56 is the desired symbol and the second term is inter symbol interference and the third term is Gaussian noise of a variance

$$\sigma_n^2 = N_0 \sum_{j=-K}^K c_j^2 \quad (59)$$

For an equalizer with $2K+1$ taps and a channel response that spans $L+1$ symbols the number of symbols with inter symbol interference is $2K+L$. D is

$$\sigma_n^2 = N_0 \sum_{j=-K}^K c_j^2 \quad (60)$$

For a sequence of $2k+L$ information symbols, say a sequence I_j , the inter symbol interference $D=D_j$ is fixed. The probability of error is

$$\begin{aligned} P_e(D_j) &= 2 \frac{M-1}{M} P(N + D_j > q_0) \\ &= \frac{2(M-1)}{M} Q \left(\sqrt{\frac{(q_0 - D_j)^2}{\sigma_n^2}} \right) \end{aligned} \quad (61)$$

N is the noise. The average probability of error is obtained by averaging $P_e(D_j)$ over all possible I_j .

$$\begin{aligned} P_e &= \sum_{I_j} P_e(D_j) P(I_j) \\ &= \frac{2(M-1)}{M} \sum_{I_j} Q \left(\sqrt{\frac{(q_0 - D_j)^2}{\sigma_n^2}} \right) P(I_j) \end{aligned} \quad (62)$$

All the sequences are equally likely

$$P(I_j) = \frac{1}{M^{2K+L}} \quad (63)$$

The maximum D_j is

$$D_j^* = (M-1) \sum_{k \neq 0} |q_k|$$

$$P_e(D_j^*) = \frac{2(M-1)}{M} Q \left(\sqrt{\frac{q_0^2}{\sigma_n^2} \left(1 - \frac{M-1}{q_0} \sum_{k \neq 0} |q_k| \right)^2} \right) \quad (64)$$

An upper bound on the probability of error for equally likely sequences

$$P_e \leq P_e(D_j^*) \quad (65)$$

The computation of the exact probability of error (equation 62) is very difficult and the upper bound is too loose. One can use one of the methods to have a tight bound on P_e . these methods are explained on papers by Saltzberg (1968) and Lugannani (1969) and Yeh (1970) and Yao (1972).

Performance in the presence of severe inter symbol interference is shown in figure 10. This is the probability of error measured by Monte Carlo simulation for three channels of figure 11. To compare also included the graph with no interference. Figure 11 (a) is a good quality telephone channel. In contrast figure 11 (b) and (c) result in severe interference. The spectral characteristics of the three channels are in figure 12. This show that channel (c) has the worst characteristics. Next in performance is channel (b). The best performance is channel (a).

We can say that the performance is good in channels where the spectral characteristics are well behaved and do not have spectral nulls. In general spectral nulls result in noise enhancement at the output of the linear equalizer.

This equalizer has a problem with severe ISI which motivated research into non linear equalizer with low computational complexity. We will talk about the decision feed back equalizer next.

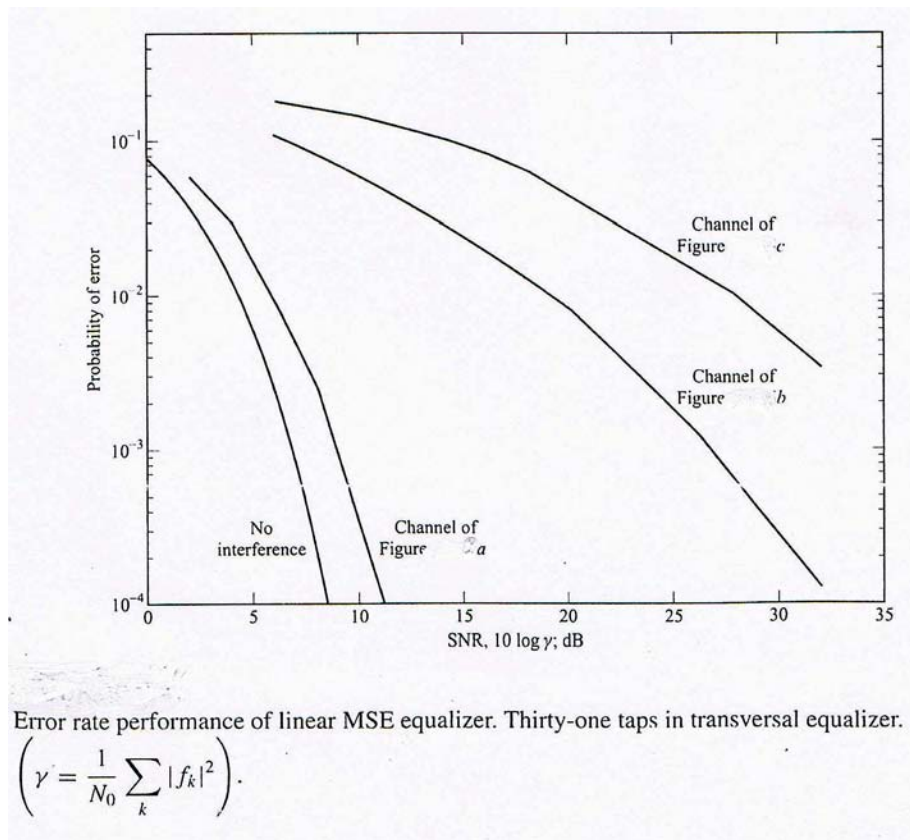


Figure 10

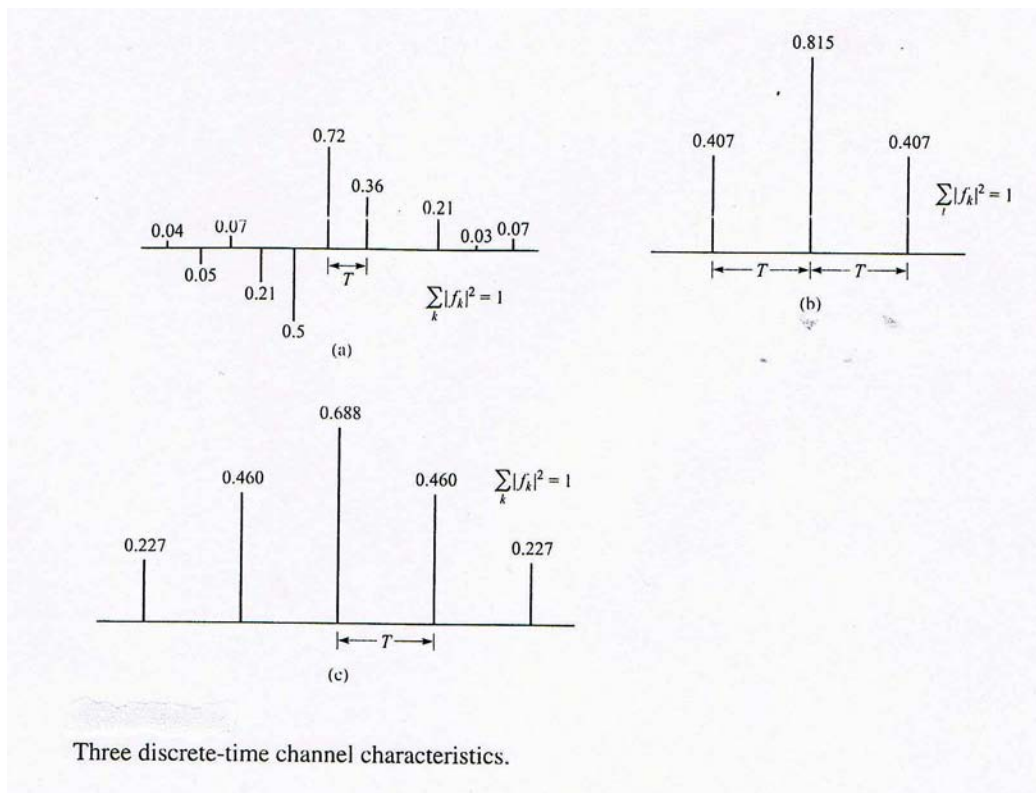


Figure 11

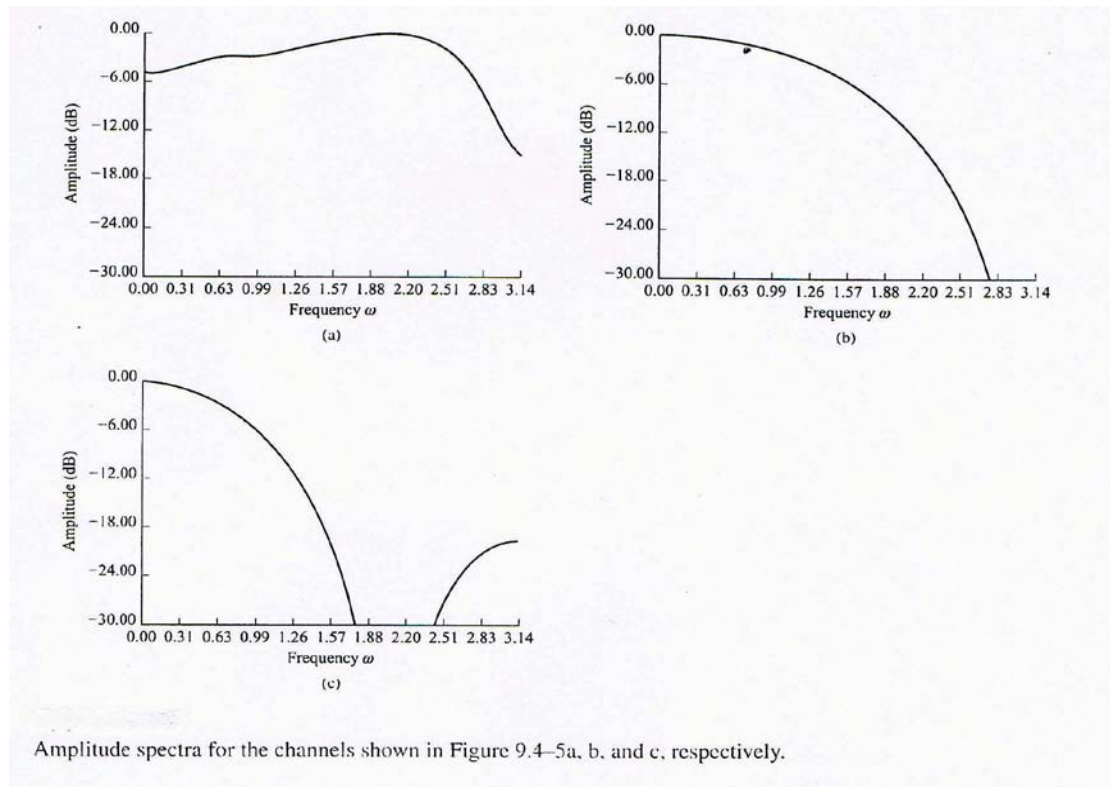


Figure 12

e) Decision feedback Equalizers

We have developed a model to receive information from a channel with ISI noise and additive noise. We say that the additive noise is colored. We simplified this model by using a whitening filter so the noise is AWGN. We reduced ISI corruption by using a linear transversal filter. We say this is the suboptimum solution.

In this part we use non linear equalizer which is also suboptimum, but its performance is better than the linear. This consist of two filters the first is feed forward and the second is feed back as shown in figure 13. This is called decision feedback equalizer. The input of the first filter is the received sequence. The feedback filter has its input the sequence of decisions on previously detected symbols. This mean that the feedback filter is used to remove ISI from the present symbol caused by previously detected symbols. Since the detector feeds hard decision to the feedback filter, the DFE is nonlinear. In our treatment we focus on finite duration impulse response filters and apply the MSE criterion to optimize the coefficients.

II. RECURSIVE LEAST SQUARES ALGORITHMS FOR ADAPTIVE EQUALIZATION

The LMS algorithm that we talked about for adaptively adjusting the coefficient of the equalizer is a

stochastic steepest descent algorithm in which the gradient is obtained from the data.

The advantage of the method of steepest decent is computational simplicity. However the convergence rate is slow specially if the autocorrelation matrix has large Lambda max / Lambda min. The only way to control this is delta. This is a fundamental limitation.

In order to have faster convergence we need more complex algorithm with more parameters. The matrix T has N Eigen values. The optimum selection of these parameters is in this section.

To get faster convergence we use the method of least squares. So we deal directly with the data and minimize the quadratic performance index. In the past we use to deal with the error and minimize the expected value of its square. This mean that the performance index is in time average not statistical average.

We need this in matrix form. So we need to introduce some notation and for the linear equalizer the estimate of the information symbol at time t.

By changing the index the estimate is

$$\begin{aligned}\hat{I}(t) &= \sum_{j=0}^{N-1} c_j(t-1)y(t-j) \\ &= \mathbf{C}_N^t(t-1)\mathbf{Y}_N(t)\end{aligned}\quad (1)$$

III. RECURSIVE LEAST-SQUARES (KALMAN) ALGORITHM

First we will do the formulation for RLS. In this method we want to minimize the time average weighted squared error

$$\mathcal{E}_N^{LS} = \sum_{n=0}^t w^{t-n} |e_N(n, t)|^2 \quad (2)$$

Where the error is

$$e_N(n, t) = I(n) - \mathbf{C}_N^t(t) \mathbf{Y}_N(n) \quad (3)$$

w is a weighting factor between zero and one which introduce exponential weighting to the past, this is good for time variant channel. Minimization of the error give the set of linear equations

$$\mathbf{R}_N(t) \mathbf{C}_N(t) = \mathbf{D}_N(t) \quad (4)$$

$\mathbf{R}(t)$ is the signal correlation matrix defined as

$$\mathbf{R}_N(t) = \sum_{n=0}^t w^{t-n} \mathbf{Y}_N^*(n) \mathbf{Y}_N^t(n) \quad (5)$$

$\mathbf{D}(t)$ is the cross correlation vector

$$\mathbf{D}_N(t) = \sum_{n=0}^t w^{t-n} I(n) \mathbf{Y}_N^*(n) \quad (6)$$

The solution is

$$\mathbf{C}_N(t) = \mathbf{R}_N^{-1}(t) \mathbf{D}_N(t) \quad (7)$$

The matrix \mathbf{R} is the autocorrelation matrix. \mathbf{D} is the cross correlation. \mathbf{R} may be ill conditioned so we initially add delta \mathbf{I} . With exponential weighting into the past the effect of adding delta \mathbf{I} dissipates with time.

Now suppose we have $\mathbf{C}(t-1)$ for time $(t-1)$. It is impractical to solve the set of N linear equations for each new signal component that is received to find $\mathbf{C}(t)$. To avoid this first we find $\mathbf{R}(t)$ may be corrupted in noise as.

$$\mathbf{R}_N(t) = w \mathbf{R}_N(t-1) + \mathbf{Y}_N^*(t) \mathbf{Y}_N^t(t) \quad (8)$$

Since the inverse of \mathbf{R} is needed we use the matrix inverse identity

$$\mathbf{R}_N^{-1}(t) = \frac{1}{w} \left[\mathbf{R}_N^{-1}(t-1) - \frac{\mathbf{R}_N^{-1}(t-1) \mathbf{Y}_N^*(t) \mathbf{Y}_N^t(t) \mathbf{R}_N^{-1}(t-1)}{w + \mathbf{Y}_N^t(t) \mathbf{R}_N^{-1}(t-1) \mathbf{Y}_N^*(t)} \right] \quad (9)$$

Where the inverse is computed recursively.

We define $\mathbf{P}(t) = \text{inverse } \mathbf{R}(t)$. The Kalman gain vector

$$\mathbf{K}_N(t) = \frac{1}{w + \mu_N(t)} \mathbf{P}_N(t-1) \mathbf{Y}_N^*(t) \quad (10)$$

Where $\mu(t)$ is a scalar defined as

$$\mu_N(t) = \mathbf{Y}_N^t(t) \mathbf{P}_N(t-1) \mathbf{Y}_N^*(t) \quad (11)$$

With this equation 11 becomes

$$\mathbf{P}_N(t) = \frac{1}{w} [\mathbf{P}_N(t-1) - \mathbf{K}_N(t) \mathbf{Y}_N^t(t) \mathbf{P}_N(t-1)] \quad (12)$$

Now we multiply both sides by $\mathbf{Y}(t)$

$$\begin{aligned} \mathbf{P}_N(t) \mathbf{Y}_N^*(t) &= \frac{1}{w} [\mathbf{P}_N(t-1) \mathbf{Y}_N^*(t) - \mathbf{K}_N(t) \mathbf{Y}_N^t(t) \mathbf{P}_N(t-1) \mathbf{Y}_N^*(t)] \\ &= \frac{1}{w} \{ [w + \mu_N(t)] \mathbf{K}_N(t) - \mathbf{K}_N(t) \mu_N(t) \} \\ &= \mathbf{K}_N(t) \end{aligned} \quad (13)$$

This is the Kalman gain vector $\mathbf{P}(t) \mathbf{Y}(t)$.

Now we use the matrix inversion identity to find $\mathbf{C}(t)$ from $\mathbf{C}(t-1)$.

$$\begin{aligned} \mathbf{C}_N(t) &= \frac{1}{w} [\mathbf{P}_N(t-1) - \mathbf{K}_N(t) \mathbf{Y}_N^t(t) \mathbf{P}_N(t-1)] [\mathbf{w} \mathbf{D}_N(t-1) + I(t) \mathbf{Y}_N^*(t)] \\ &= \mathbf{P}_N(t-1) \mathbf{D}_N(t-1) + \frac{1}{w} I(t) \mathbf{P}_N(t-1) \mathbf{Y}_N^*(t) \\ &\quad - \mathbf{K}_N(t) \mathbf{Y}_N^t(t) \mathbf{P}_N(t-1) \mathbf{D}_N(t-1) \\ &\quad - \frac{1}{w} I(t) \mathbf{K}_N(t) \mathbf{Y}_N^t(t) \mathbf{P}_N(t-1) \mathbf{Y}_N^*(t) \\ &= \mathbf{C}_N(t-1) + \mathbf{K}_N(t) [I(t) - \mathbf{Y}_N^t(t) \mathbf{C}_N(t-1)] \end{aligned} \quad (14)$$

Note that

$$I(t) = \mathbf{Y}_N^t(t) \mathbf{C}_N(t-1) \quad (15)$$

And

$$e_N(t, t-1) = I(t) - \hat{I}(t) \equiv e_N(t) \quad (16)$$

$\mathbf{C}(t)$ is updated recursively using the equation

$$\mathbf{C}_N(t) = \mathbf{C}_N(t-1) + \mathbf{K}_N(t) e_N(t) \quad (17)$$

The residual MSE from this optimization is

$$\mathcal{E}_{N \min}^{LS} = \sum_{m=0}^t w^{t-m} |I(m)|^2 - \mathbf{C}_N^t(t) \mathbf{D}_N^*(t) \quad (18)$$

To summarize we have $C(t-1)$ and $P(t-1)$. When new signal is received we have $Y(t)$. The recursive update of $C(t)$ and $P(t)$ is as follows:

- Compute output

$$e_N(t) = I(t) - \hat{I}(t)$$

- Compute error

$$K_N(t) = \frac{P_N(t-1)Y_N'(t)}{w + Y_N'(t)P_N(t-1)Y_N^*(t)}$$

- Compute Kalman gain vector
- Update the inverse of the correlation matrix

$$P_N(t) = \frac{1}{w} [P_N(t-1) - K_N(t)Y_N'(t)P_N(t-1)]$$

- Update coefficients

$$\begin{aligned} C_N(t) &= C_N(t-1) + K_N(t)e_N(t) \\ &= C_N(t-1) + P_N(t)Y_N^*(t)e_N(t) \end{aligned} \quad (19)$$

This is called the RLS direct form or Kalman algorithm. Note that the coefficients change by the error multiplied by the kalman gain. Each tap is controlled by one element of K . as a result we have rapid convergence. In contrast for the steepest descent we have

$$C_N(t) = C_N(t-1) + \Delta Y_N^*(t)e_N(t) \quad (20)$$

The only variable parameter is the step size delta.

Figure 13 show the convergence rate of the two algorithms for a channel with fixed parameters $f_0=.26$, $f_1=.93$, $f_2=.26$ and a linear equalizer with 11 taps. The Eigen value ratio for this channel $\lambda_{\max} / \lambda_{\min} = 11$. We start with zero coefficients. Delta for the steepest descent is .02. It is clear that the Kalman algorithm is great. This is important for time variant channel. For example the ionosphere high frequency radio channel is too fast to use steepest descent. Kalman algorithm adapt fast to track it.

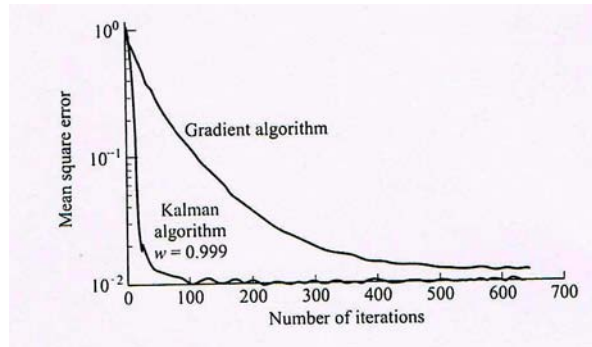


Figure 13

The kalman algorithm has a good performance. It has two disadvantages. First its complexity second the sensitivity to round off noise that accumulate due to the recursive computation. This may cause instability.

The number of computation to calculate C is proportional to N^2 . Most of them for updating P . Updating P may introduce round off noise and to solve this problem we use

$$P_N(t) = S_N(t)\Lambda_N(t)S_N'(t) \quad (21)$$

Where S is a lower triangular matrix with unity diagonal and Λ is a diagonal matrix. This is called square root factorization.

The square root algorithm is used in control systems with kalman algorithm involved.

It is also possible to develop fast RLS that has linear complexity with the number of coefficient. This was described by the paper of Kailath (1991).

IV. LINEAR PREDICTION AND THE LATTICE FILTER

We will develop a relation between linear FIR and a Lattice filter. This connection can be seen by considering the problem of linear prediction.

The prediction problem may be stated as follows: given a set of data predict the value of the next data point. The predictor of order p is

$$\hat{y}(t) = \sum_{k=1}^p a_{pk}y(t-k) \quad (22)$$

The MSE is

$$\begin{aligned} \mathcal{E}_p &= E[y(t) - \hat{y}(t)]^2 \\ &= E \left[y(t) - \sum_{k=1}^p a_{pk}y(t-k) \right]^2 \end{aligned} \quad (23)$$

Minimizing with respect to a_{pk} give the linear equations

$$\sum_{k=1}^p a_{pk} R(k-l) = R(l), \quad l = 1, 2, \dots, p \quad (24)$$

$$R(l) = E[y(t)y(t+l)]$$

This is the Yule Walker equations

The matrix R is a Toeplitz matrix and the Levinson Durbin algorithm is a good way to solve the linear equations recursively. Starting from the first order and going recursively to order p .

$$\begin{aligned} a_{11} &= \frac{R(1)}{R(0)}, & \mathcal{E}_0 &= R(0) \\ a_{mm} &= \frac{\phi(m) - \mathbf{A}_m^t \mathbf{R}_m^r}{\mathcal{E}_{m-1}} \\ a_{mk} &= a_{m-1k} - a_{mm} a_{m-1m-k} \\ \mathcal{E}_m &= \mathcal{E}_{m-1} (1 - a_{mm}^2) \end{aligned}$$

$$\begin{aligned} \mathbf{A}_{m-1} &= [a_{m-11} \ a_{m-12} \ \dots \ a_{m-1m-1}]^t \\ \mathbf{R}_{m-1}^r &= [R(m-1) \ R(m-2) \ \dots \ R(1)]^t \end{aligned} \quad (25)$$

The linear prediction FIR filter of order m has a transfer function

$$A_m(z) = 1 - \sum_{k=1}^m a_{mk} z^{-k} \quad (26)$$

The input is the data $y(t)$. The error e is $e(t) = y(t) - (\text{estimate } y(t))$. This filter may be realized in lattice form as we will show.

We start with the use of Levinson- Durbin algorithm to find a_{mk} in equation 29.

$$\begin{aligned} A_m(z) &= 1 - \sum_{k=1}^{m-1} (a_{m-1k} - a_{mm} a_{m-1m-k}) z^{-k} - a_{mm} z^{-m} \\ &= 1 - \sum_{k=1}^{m-1} a_{m-1k} z^{-k} - a_{mm} z^{-m} \left(1 - \sum_{k=1}^{m-1} a_{m-1k} z^k \right) \\ &= A_{m-1}(z) - a_{mm} z^{-m} A_{m-1}(z^{-1}) \end{aligned} \quad (30)$$

This mean we have an m order predictor in terms of $(m-1)$ order predictor.

Now we find G as

$$G_m(z) = z^{-m} A_m(z^{-1}) \quad (31)$$

Then equation 30 is

$$A_m(z) = A_{m-1}(z) - a_{mm} z^{-1} G_{m-1}(z) \quad (32)$$

The coefficients of $G_{m-1}(z)$ are exactly the same as $A_{m-1}(z)$ but in reverse order.

We can see more if we look at the output if the input is $y(t)$ and we take the z transform

$$A_m(z)Y(z) = A_{m-1}(z)Y(z) - a_{mm} z^{-1} G_{m-1}(z)Y(z) \quad (33)$$

The outputs of the filters are

$$\begin{aligned} F_m(z) &= A_m(z)Y(z) \\ B_m(z) &= G_m(z)Y(z) \end{aligned} \quad (34)$$

Now the equation 33 is

$$F_m(z) = F_{m-1}(z) - a_{mm} z^{-1} B_{m-1}(z) \quad (35)$$

In the time domain it is

$$f_m(t) = f_{m-1}(t) - a_{mm} b_{m-1}(t-1), \quad m \geq 1 \quad (36)$$

Where

$$f_m(t) = y(t) - \sum_{k=1}^{m-1} a_{mk} y(t-k) \quad (37)$$

$$b_m(t) = y(t-m) - \sum_{k=1}^{m-1} a_{mk} y(t-m+k) \quad (38)$$

$f_m(t)$ is the error for the m order predictor and $b_m(t)$ is the error for the m order backward predictor.

36 is one equation of two that specifies lattice filter. The second one from $G_m(z)$ is

$$\begin{aligned} G_m(z) &= z^{-m} A_m(z^{-1}) \\ &= z^{-m} [A_{m-1}(z^{-1}) - a_{mm} z^m A_{m-1}(z)] \\ &= z^{-1} G_{m-1}(z) - a_{mm} A_{m-1}(z) \end{aligned} \quad (39)$$

Now we multiply by $Y(z)$ and we get

$$B_m(z) = z^{-1} B_{m-1}(z) - a_{mm} F_{m-1}(z) \quad (40)$$

We go to the time domain to get the second equation

$$b_m(t) = b_{m-1}(t-1) - a_{mm} f_{m-1}(t), \quad m \geq 1 \quad (41)$$

And the initial conditions are

$$f_0(t) = b_0(t) = y(t) \quad (42)$$

The lattice by equation 36 and 41 is in figure 13. Each stage has its multiplication factor which defined by the Levinson Durbin algorithm. The f and b are called residuals and the mean square value of these residuals is

$$\mathcal{E}_m = E[f_m^2(t)] = E[b_m^2(t)] \quad (43)$$

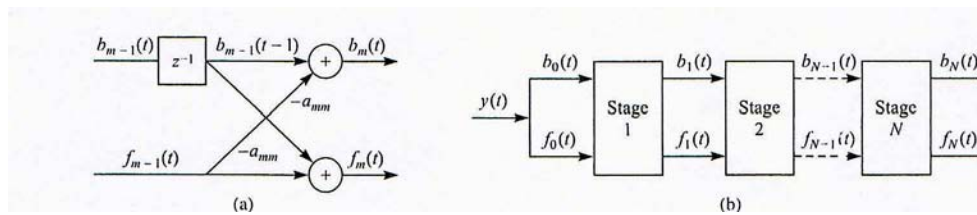


Figure 14

\mathcal{E}_m is given recursively by Levinson Durbin algorithm

$$\begin{aligned}\mathcal{E}_m &= \mathcal{E}_{m-1}(1 - a_{mm}^2) \\ &= \mathcal{E}_0 \prod_{i=1}^m (1 - a_{ii}^2)\end{aligned}\quad (44)$$

Where $\mathcal{E}_0 = R(0)$

f_m and b_m satisfy the orthogonality property as by Makhoul (1978)

$$\begin{aligned}E[b_m(t)b_n(t)] &= \mathcal{E}_m \delta_{mn} \\ E[f_m(t+m)f_n(t+n)] &= \mathcal{E}_m \delta_{mn}\end{aligned}\quad (45)$$

The cross correlation of f_m and b_m is

$$E[f_m(t)b_n(t)] = \begin{cases} a_{nn}\mathcal{E}_m & m \geq n \\ 0 & m < n \end{cases} \quad m, n \geq 0 \quad (46)$$

As a result of Makhoul work we can see that in the lattice filter we can remove the last stage without effecting the coefficient of the previous stages. This mean that with taking \mathcal{E}_m as a performance index we can stop when \mathcal{E}_m is satisfactory.

As we can see the linear prediction problem can be implemented by transverse filter or lattice filter. The lattice filter is order recursive. This mean the number of sections can be increased or decreased without effecting the parameters of the remaining sections. On the other hand if the order of the transverse filter is changed all coefficients to minimize the MSE will change. This mean that the Kalman filter is recursive in time but not order.

Let us study the computational complexity. The lattice computational complexity increase linearly with the order. A graph to compare deferent algorithms is Figure 14. The study of the RLS lattice was done by papers by Lee (1978) and Alexander (1979) and books by Proakis (2002) and Haykin (2002).

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