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On the Squeezing Flow of Nanofluid Through a Porous Medium With Slip Boundary and Magnetic Field: A Comparative Study of Three Approximate Analytical Methods

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Abstract- This paper presents a comparative study of approximate analytical methods is carried out using differential transformation, homotopy perturbation and variation parameter methods for the analysis of a steady two-dimensional axisymmetric flow of nanofluid under the influence of a uniform transverse magnetic field with slip boundary condition. Also, parametric studies are carried out to investigate the effects of fluid properties, magnetic field and slip parameters on the squeezing flow. It is revealed from the results that the velocity of the fluid increases with increase in the magnetic parameter under the influence of slip condition while an opposite trend is recorded during no-slip condition. Also, the velocity of the fluid increases as the slip parameter increases but it decreases with increase in the magnetic field parameter and Reynold number under the no-slip condition.

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ONTHESQUEEZINGFLOWOFNANOFLUIDTHROUGHAPOROUSMEDIUMWITHSLIPBOUNDARYANDMAGNETICFIELDACOMPARATIVESTUDYOFTHREEAPPROXIMATEANALYTICALMETHODS

Strictly as per the compliance and regulations of:

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M. G. Sobamowo [α], L. O. Jayesimi [σ] & M. A. Whaeed [ρ]

Abstract- This paper presents a comparative study of approximate analytical methods is carried out using differential transformation, homotopy perturbation and variation parameter methods for the analysis of a steady two-dimensional axisymmetric flow of nanofluid under the influence of a uniform transverse magnetic field with slip boundary condition. Also, parametric studies are carried out to investigate the effects of fluid properties, magnetic field and slip parameters on the squeezing flow. It is revealed from the results that the velocity of the fluid increases with increase in the magnetic parameter under the influence of slip condition while an opposite trend is recorded during no-slip condition. Also, the velocity of the fluid increases as the slip parameter increases but it decreases with increase in the magnetic field parameter and Reynold number under the no-slip condition. The approximate analytical solutions are verified by comparing the results of the approximate analytical methods with the numerical method using Runge-Kutta coupled with shooting method. Although, very good agreements are established between the results, the results of variation parameter method provide excellent agreement with the results of numerical method.

Keywords: nanofluid; squeezing flow; slip boundary; differential transformation method; homotopy perturbation method; variation parameter method.

I. Introduction

The flow of nanofluid in a channel, between two contracting or expanding plates and also, over a stretching sheet have aroused research interests in recent times. Among the recent studies, the analysis of squeezing flow of nanofluid or viscous fluid between two parallel plates have increased tremendously due to its various industrial and biological applications. After the pioneer work on squeezing flow by Stefan [1], there have been improved works on the flow phenomena. However, the earlier studies [1-3] on squeezing flow were based on Reynolds equation. Jackson [4] and Usha and Sridhar an [5] pointed out the insufficiencies of the Reynolds equation for some cases of flow situations. Consequently, there have been several attempts and renewed research interests by different researchers to properly analyze and understand the squeezing flows using different analytical and numerical methods [5-26]. Also, effects of magnetic field, flow characteristics and fluid properties on the squeezing flow have been widely investigated under no slip conditions [27–42]. However, in many cases of fluid and flow problems such as polymeric liquids, thin film problems, nanofluids, rarefied fluid problems, fluids containing concentrated suspensions, and flow on multiple interfaces, slip condition prevails at the boundary of the flow process.

Therefore, Navier [43] proposed the general boundary condition which demonstrates the fluid slip at the surface. Such consideration of slip condition in the flow analysis of fluids is of great importance especially when fluids with elastic character are under consideration [44]. In a past study on slip effects on flow conditions of fluids, Ebaid [45] investigated the effects of magnetic field and wall slip conditions on the peristaltic transport in an asymmetric channel. The influence of slip on the peristaltic motion of third-order fluid in asymmetric channel was analyzed by Hayat *et al.* [46]. Also, Hayat and Abelman [47] presented a study on the effects of slip condition on the rotating flow of a third grade fluid in a nonporous medium. Abelman *et al.* [48] extended their work to a porous medium and obtained the numerical solutions for the steady magnetodrodynamics flow of a third grade fluid in a rotating frame. The past efforts in analyzing the squeezing flow problems have been largely based on the applications of various numerical and approximate analytical methods such as differential transformation method (DTM), Adomian Decomposition Method (ADM), homotopy analysis method (HAM), optimal homotopy asymptotic method (OHPM), variational iteration method (VIM). Moreover, most of the studies are based on viscous fluids. To the best of the authors' knowledge, a

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study on squeezing flow of nanofluid under the influences of magnetic field and slip boundary conditions using variation parameter method (VPM) has not been carried out in literature. Also, a comparative study of the three approximate analytical methods (differential transformation, homotopy perturbation and variation parameter methods) has presented in this paper has not been analyzed in past work. Therefore, in the paper, a comparative study of approximate analytical methods is carried out using differential transformation, homotopy perturbation and variation parameter method for the analysis of a steady two-dimensional axisymmetric flow of nanofluid under the influence of a uniform transverse magnetic field with slip boundary condition. The analytical solutions are used to investigate the effects of fluid properties, magnetic field and slip parameters on the squeezing flow.

II. Problem Formulation

Consider a squeezing flow of nanofluid squeezed between two parallel plates which are at distance $2h$ apart and they approach each other with slowly with a constant velocity under in the presence of a magnetic field as shown in Fig. 1. Assuming that the fluid is incompressible, the flow is laminar and isothermal, the governing equations of motion for the quasi steady flow of the nanofluid are given as:

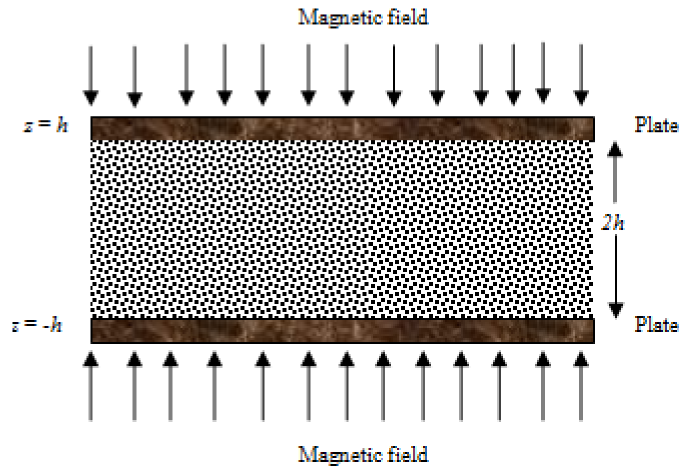

Fig. 1: Model of the MHD squeezing flow of nanofluid between two parallel paltes separated by distance 2h

Assume that the flow is quasi steady, and the Navier-Stokes equations governing such flow when inertial terms are retained are: The equations of motion governing the flow are:

$$\nabla.\tilde{v} = 0 \tag{1}$$

$$\left[\rho_f(1-\phi)+\rho_s\phi\right]\left[\frac{\partial\tilde{v}}{\partial t}+(\nabla.\tilde{v})\tilde{v}\right]=\left[\rho_f(1-\phi)+\rho_s\phi\right]f-\nabla.p+\left\{\frac{\mu_f}{(1-\phi)^{2.5}}\left(\nabla^2-\frac{1}{k}\right)-\sigma B_0^2\right\}\tilde{v} \tag{2}$$

Neglecting the body force, the continuity and Navier-Stokes' equation for the problem is given as

$$\nabla.\tilde{v} = 0 \tag{3}$$

$$-\left[\rho_f(1-\phi)+\rho_s\phi\right](\tilde{v}\times\tilde{w})+\tilde{\nabla}\left(\frac{\left[\rho_f(1-\phi)+\rho_s\phi\right]}{2}|\tilde{v}|^2+p\right)=\frac{\mu_f}{(1-\phi)^{2.5}}\left(\tilde{\nabla}\times\tilde{w}-\frac{1}{k}\tilde{v}\right)-\sigma B_0^2\tilde{v} \tag{4}$$

Introducing the stream function $\psi(r,z)$, vorticity function $\Omega(r,z)$ and a generalized pressure for the cylindrical coordinate system as follows:

$$u=\frac{1}{r}\frac{\partial\psi}{\partial z},\quad v=-\frac{1}{r}\frac{\partial\psi}{\partial r},\quad \Omega(r,z)=-\frac{1}{r}\lambda^2\psi,\quad p=\frac{\left[\rho_f(1-\phi)+\rho_s\phi\right]}{2}\left(u^2+v^2\right) \tag{5}$$

Eliminating the pressure term from Eqs. (3) and (4), we have

$$\left[\rho_f(1-\phi)+\rho_s\phi\right]\left[\frac{\partial\left(\psi,\lambda^2\psi/r^2\right)}{\partial(r,z)}\right]=-\frac{1}{r}\frac{\mu_f}{(1-\phi)^{2.5}}\lambda^4\psi+\frac{1}{r}\left(\frac{\mu_f}{k(1-\phi)^{2.5}}+\sigma B_0^2\right)\frac{\partial^2\psi}{\partial z^2} \tag{6}$$

where

$$\lambda^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{7}$$

The boundary conditions are given as

$$\begin{aligned} &z = 0, v = 0 \;\; and \; \frac{\partial v}{\partial z} = 0 \\ &z = H, \;\; v = -V_w \;\; and \; \frac{\partial v}{\partial z} = \beta v \end{aligned} \tag{8}$$

Applying a transformation $\psi(r,z) = r^2 f(z)$, the compatibility Eq. (6) reduces to Eq. (9) as

$$f^{iv}(z) - \left(\frac{1}{k} + \frac{\sigma B_0^2 (1-\phi)^{2.5}}{\mu_f}\right) f^{"}(z) + \frac{2\left[\rho_f (1-\phi) + \rho_s \phi\right](1-\phi)^{2.5}}{\mu_f} f(z) f^{'''}(z) = 0 \tag{9}$$

And the slip boundary conditions as

$$\begin{aligned} &f(0) = 0, \;\; f^{"}(0) = 0, \\ &f(h) = \frac{v}{2}, \;\; f^{'}(h) = \gamma f^{"}(h) \end{aligned} \tag{10}$$

Using the following dimensionless parameters in Eq. (11)

$$F^* = \frac{f}{v/2}, \; z^* = \frac{z}{h}, \; R = \frac{\rho_f H v}{\mu_f}, \; G = h\sqrt{\left(\frac{1}{k} + \frac{\sigma B_0^2}{\mu_{nf}}\right)} = \sqrt{\left(Da + m^2\right)}. \tag{11}$$

The dimensionless form of Eq. (9) is given as

$$F^{(iv)}(z) + R\left((1-\phi) + \phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} F(z) F^{'''}(z) - G^2 F^{"}(z) = 0 \tag{12}$$

And the dimensionless boundary conditions in Eq. (10) as

$$\begin{aligned} &F(0) = 0, \quad F''(0) = 0 \\ &F(1) = 1, \quad F'(1) = \gamma F''(1) \end{aligned} \tag{13}$$

where the asterisk, * has been omitted in Eqs. (12) and Eq. (13) for the sake of conveniences.

III. Approximate Analytical Methods Of Solution: Differential Transform Method

The differential transform method has widely been used to solve both singular and non-singular perturbed boundary values problems. It gives analytical solution to differential or integral solutions in the form of a polynomial by transforming each term in the differential equation or integral into a recursive form or relation of the equation which follows an iterative procedure for obtaining analytical series solutions of differential equation.

The basic definitions of the method is as follows:

If $u(x)$ is analytic in the domain T, then it will be differentiated continuously with respect to space *x*.

$$\frac{d^p u(x)}{dx^p} = \varphi(x,p) \quad \text{for} \quad \text{all } x \in T \tag{17}$$

For $x = x_i$, then $\varphi(x,p) = \varphi(x_i,p)$, where *p* belongs to the set of non-negative integers, denoted as the *p*-domain. Therefore Eq. (17) can be rewritten as

$$U(p) = \varphi(x_i,p) = \left[\frac{d^p u(x)}{dt^p}\right]_{x=x_i} \tag{18}$$

where U_p is called the spectrum of $u(x)$ at $x = x_i$

If $u(x)$ can be expressed by Taylor's series, the $u(x)$ can be represented as

$$u(x)=\sum_{p}^{\infty}\left[\frac{(x-x_i)^p}{p!}\right]U(p) \qquad (19)$$

where Eq. (19) is called the inverse of $U(k)$ using the symbol 'D' denoting the differential transformation process and combining Eq. (18) and Eq. (19), it is obtained that

$$u(x)=\sum_{p=0}^{\infty}\left[\frac{(x-x_i)^p}{p!}\right]U(p)=D^{-1}U(p) \qquad (20)$$

a) Operational properties of differential transformation method

If $u(x)$ *and* $v(x)$ are two independent functions with space (x) where $U(p)$ and $V(p)$ are the transformed function corresponding to $u(x)$ and $v(x)$, then it can be shown from the fundamental mathematics operations performed by differential transformation that.

i. If $z(x)=u(x)\pm v(x),$ then $Z(p)=U(p)\pm V(p)$

ii. If $z(x)=\alpha u(x),$ then $Z(p)=\alpha U(p)$

iii. If $z(x)=\dfrac{d^n u(x)}{dx^n},$ then $Z(p)=(p+1)\,(p+2)(p+3)...(p+n)U(p+n)$

iv. If $z(x)=u(x)v(x),$ then $Z(p)=\sum_{r=0}^{p}V(r)U(p-r)$

v. If $z(x)=u^m(x)$, then $Z(p)=\sum_{r=0}^{p}U^{m-1}(r)U(p-r)$

vi. If $z(x)=u(x)v(x),$ then $Z(p)=\sum_{r=0}^{p}(r+1)V(r+1)U(p-r)$

vii. If $z(x)=x^m\dfrac{d^n u(x)}{dx^n},$ then $Z(p)=\sum_{l=0}^{p}\delta(l-m-1)(p-l+1)(p-l+2)(p-l+3)...(p-l+n)U(p-l+n)$

viii. If $z(x)=\dfrac{d\ \ u(x)}{dx}\dfrac{d^3u(x))}{dx^3},$ then $Z(p)=\sum_{l=0}^{p}U(p-l)(l+1)(l+2)(l+3)U(l+3)$

ix. If $z(x)=\dfrac{d\ \ u(x)}{dx}\dfrac{d^2u(x))}{dx^2},$ then $Z(p)=\sum_{=}^{p}{}^{0}(p-l+1)U(p-l+1)(l+1)(l+2)U(l+2)$

x. If $z(x)=\left(\dfrac{du(x)}{dx}\right)^2$ then $Z(p)=\sum_{l=0}^{p}(p-l+1)U(p-l+1)(l+1)U(l+1)$

xi. If $z(x)=u\dfrac{du(x)}{dx},$ then $Z(p)=\sum_{l=0}^{p}U(p-l)(l+1)U(l+1)$

If $z(x)=\left[\dfrac{d^2u(x))}{dx^2}\right]^2,$ then $Z(p)=\sum_{l=0}^{p}(p-l+1)(p-l+2)U(p-l+2)(l+1)(l+2)U(l+2)$

IV. Application of the Differential Transform Method to the Present Problem

The differential transform of (15) and (16) is given by

$$\begin{aligned}&(k+1)(k+2)(k+3)(k+4)F[k+4]\\&R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\left(\sum_{l=0}^{k}(k-l+3)(k-l+2)(k-l+1)F[l]F[k-l+3]\right)\\&-G\left((k+1)(k+2)F[k+2]\right)+=0\end{aligned} \tag{21}$$

With differential transformed boundary conditions

$$\begin{aligned}&\tilde{F}[0]=0,\ \tilde{F}[1]=a,\ \tilde{F}[2]=0,\ \tilde{F}[3]=b,\\&\sum(k+1)F[k+1]=\gamma\sum(k+1)(k+2)F[k+2]\end{aligned} \tag{22}$$

Where *a* and *b* are unknowns to be determined later using the boundary conditions of Eq. (16b). Using Eqs. (21) and (22), the value of $\tilde{F}(i), i=1,2,3,4,5,\ldots 19,20.$ are

$$\tilde{F}[4]=0$$

$$\tilde{F}[5]=\frac{1}{20}\left(bG^2-abR\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)$$

$$\tilde{F}[6]=0$$

$$\tilde{F}[7]=\frac{1}{840}\left(\begin{aligned}&bG^4-6b^2R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}-4abG^2R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\\&+3a^2b\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2\end{aligned}\right)$$

$$\tilde{F}[8]=0$$

$$\tilde{F}[9]=\frac{1}{60480}\left\{\begin{aligned}&bG^6-72b^2G^2R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}-9abG^4R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\\&+96ab^2\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2+23a^2bG^2\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2\\&-15a^3bR\left(\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3\end{aligned}\right\}$$

$$\tilde{F}[10]=0$$

$$\tilde{F}[11]=\frac{1}{6652800}\left\{\begin{array}{l} bG^8-414b^2G^4R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}-16abG^6R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} \\ +1296b^3\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2+1716ab^2G\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2+ \\ 86a^2bG^4\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2-1446a^2b^2\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3 \\ -176a^3bG^2\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3+105a^4b\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^4 \end{array}\right\}$$

And so on

According to the definition of DTM, the solution is

$$\begin{aligned} F(z)=&\tilde{F}[0]+z\tilde{F}[1]+z^2\tilde{F}[2]+z^3\tilde{F}[3]+z^4\tilde{F}[4]+z^5\tilde{F}[5]+z^6\tilde{F}[6]+z^7\tilde{F}[7] \\ &+z^8\tilde{F}[8]+z^9\tilde{F}[9]+z^{10}\tilde{F}[10]+z^{11}\tilde{F}[11]+\ldots \end{aligned} \tag{23}$$

a) The basic idea of homotopy perturbation method

In order to establish the basic idea behind homotopy perturbation method, consider a system of nonlinear differential equations given as

$$A(U)-f(r)=0, \quad r\in\Omega \tag{24}$$

with the boundary conditions

$$B\left(u,\frac{\partial u}{\partial \eta}\right)=0, \qquad r\in\Gamma \tag{25}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω The operator A can be divided into two parts, which are L and N, where L is a linear operator, N is a non-linear operator. Eq.(24) can be therefore rewritten as follows

$$L(u)+N(u)-f(r)=0 \tag{26}$$

By the homotopy technique, a homotopy $U(r,p):\Omega\times[0,1]\rightarrow R$ can be constructed, which satisfies

$$H(U,p)=(1-p)\left[L(U)-L(U_o)\right]+p\left[A(U)-f(r)\right]=0, \quad p\in[0,1] \tag{27}$$

Or

$$H(U,p)=L(U)-L(U_o)+p\,L(U_o)+p\left[N(U)-f(r)\right]=0 \tag{28}$$

In the above Eqs. (27) and (28), $p\in[0,1]$ is an embedding parameter, u_o is an initial approximation of equation of Eq.(24), which satisfies the boundary conditions.

Also, from Eqs. (27) and (28), we will have

$$H(U,0)=L(U)-L(U_o)=0 \tag{29}$$

$$H(U,0)=A(U)-f(r)=0 \tag{30}$$

The changing process of p from zero to unity is just that of $U(r,p)$ from $u_o(r)$ to $u(r)$. This is referred to homotopy in topology. Using the embedding parameter p as a small parameter, the solution of Eqs. (27) and (28) can be assumed to be written as a power series in p as given in Eq. (28)

$$U=U_o+pU_1+p^2U_2+\ldots \tag{31}$$

It should be pointed out that of all the values of p between 0 and 1, $p=1$ produces the best result.

Therefore, setting $p=1$, results in the approximation solution of Eq.(24)

$$u=\lim_{p\rightarrow 1}U=U_o+U_1+U_2+\ldots \tag{32}$$

The basic idea expressed above is a combination of homotopy and perturbation method.

Hence, the method is called homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques. The series Eq.(32) is convergent for most cases.

b) Application of the homotopy perturbation method to the present problem

According to homotopy perturbation method (HPM), one can construct an homotopy for Eq. (16) as

$$H(z,p)=(1-p)\tilde{F}^{(iv)}+p\left[\tilde{F}^{(iv)}+RR\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}\tilde{F}'''-G^2\tilde{F}''\right] \quad (33)$$

Using the embedding parameter p as a small parameter, the solution of Eqs. (16) can be assumed to be written as a power series in p as given in Eq. (33)

$$\tilde{F}=\tilde{F}_o+p\tilde{F}_1+p^2\tilde{F}_2+p^3\tilde{F}_3+\ldots \quad (34)$$

On substituting Eqs. (34) and into Eq.(33) and expanding the equation and collecting all terms with the same order of p together, the resulting equation appears in form of polynomial in p. On equating each coefficient of the resulting polynomial in p to zero, we arrived at a set of differential equations and the corresponding boundary conditions as

$$p^0: \quad \tilde{F}_0^{(iv)}=0,$$

$$\tilde{F}_0(0)=0,\ \tilde{F}_0''(0)=0,\ \tilde{F}_0(1)=1,\ \tilde{F}_0'(1)=\gamma\tilde{F}_0''(1) \quad (35)$$

$$p^1: \quad \tilde{F}_1^{(iv)}-G^2\tilde{F}_0''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_0\tilde{F}_0'''=0, \quad (36)$$

$$\tilde{F}_1(0)=0,\ \tilde{F}_1''(0)=0,\ \tilde{F}_1(1)=0,\ \tilde{F}_1'(1)=\gamma\tilde{F}_1''(1)$$

$$p^2: \quad \tilde{F}_2^{(iv)}-G^2\tilde{F}_1''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_1\tilde{F}_0'''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_0\tilde{F}_1'''=0, \quad (37)$$

$$\tilde{F}_2(0)=0,\ \tilde{F}_2''(0)=0,\ \tilde{F}_2(1)=0,\ \tilde{F}_2'(1)=\gamma\tilde{F}_2''(1)$$

$$p^3:\tilde{F}_3^{(iv)}-G^2\tilde{F}_2''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_2\tilde{F}_0'''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_1\tilde{F}_1'''$$
$$+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_0\tilde{F}_2'''=0, \quad (38)$$

$$\tilde{F}_3(0)=0,\ \tilde{F}_3''(0)=0,\tilde{F}_3(1)=0,\ \tilde{F}_3'(1)=\gamma\tilde{F}_3''(1)$$

$$p^4: \quad \tilde{F}_4^{(iv)}-G^2\tilde{F}_3''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_3\tilde{F}_0'''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_3\tilde{F}_0'''$$
$$1+R\left((\ -\phi)+1\phi\frac{\rho_s}{\rho_f}\right)(\ -\phi)^{2.5}\tilde{F}_2\tilde{F}_1'''+R\left((\ -\phi)+1\phi\frac{\rho_s}{\rho_f}\right)(\ -\phi)^{2.5}\tilde{F}_1\tilde{F}_2''' \quad (39)$$
$$1+R\left((\ -\phi)+1\phi\frac{\rho_s}{\rho_f}\right)(\ -\phi)^{2.5}\tilde{F}_0\tilde{F}_3'''=0$$

$$\tilde{F}_4(0)=0,\ \ \tilde{F}_4''(0)=0, \tilde{F}_4(1)=0,\ \ \tilde{F}_4'(1)=\gamma\tilde{F}_4''(1)$$

$$\begin{aligned} p^5:\ \ &\tilde{F}_5^{(iv)}-G^2\tilde{F}_4''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_3\tilde{F}_0'''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_4\tilde{F}_0''' \\ &+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_3\tilde{F}_1'''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_2\tilde{F}_2''' \\ &+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_1\tilde{F}_3'''+R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\tilde{F}_0\tilde{F}_4'''=0 \end{aligned} \tag{40}$$

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$$\tilde{F}_5(0)=0,\ \ \tilde{F}_5''(0)=0, \tilde{F}_5(1)=0,\ \ \tilde{F}_5'(1)=\gamma\tilde{F}_5''(1)$$

On solving the above Eqs. (35-40), we arrived at

$$\tilde{F}_0(z)=\frac{3(2\gamma-1)z+z^3}{2(3\gamma-1)} \tag{41}$$

$$\tilde{F}_1(z)=\left\{\frac{3G^2}{3\gamma-1}+\frac{9R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}(2\gamma-1)}{2(3\gamma-1)^2}\right\}z^5+\frac{3R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}}{2(3\gamma-1)^2}z^7$$

$$-\frac{1}{3(2\gamma+1)}\left\{\begin{aligned}&\gamma\left(\frac{60G^2}{3\gamma-1}+\frac{90R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}(2\gamma-1)}{(3\gamma-1)^2}+\frac{63R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}}{(3\gamma-1)^2}\right)\\ &-\left(\frac{12G^2}{3\gamma-1}+\frac{3\ R6(1-\phi)+\phi\frac{\rho_s}{\rho_f}\Big)(1-\phi)^{2.5}(2\gamma-1)}{(3\gamma-1)^2}+\frac{9R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}}{(3\gamma-1)^2}\right)\end{aligned}\right\}z^3$$

$$+\frac{1}{3(2\gamma+1)}\left\{\begin{aligned}&\gamma\left(\frac{60G^2}{3\gamma-1}+\frac{90R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}(2\gamma-1)}{(3\gamma-1)^2}+\frac{63R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}}{(3\gamma-1)^2}\right)\\ &-\left(\frac{12G^2}{3\gamma-1}+\frac{3\ R6(1-\phi)+\phi\frac{\rho_s}{\rho_f}\Big)(1-\phi)^{2.5}(2\gamma-1)}{(3\gamma-1)^2}+\frac{9R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}}{2(3\gamma-1)^2}\right)\end{aligned}\right\}z$$

$$-\left\{\frac{3G^2}{3\gamma-1}+\frac{9R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}(2\gamma-1)}{(3\gamma-1)^2}+\frac{3R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}}{2(3\gamma-1)^2}\right\}z \tag{42}$$

In the same manner, the expressions for $\tilde{F}_2(z), \tilde{F}_3(z), \tilde{F}_4(z), \tilde{F}_5(z), \tilde{F}_6(z)...$ were obtained. However, they are too large expressions to be included in this paper.

Setting $p=1$, results in the approximation solution of Eq. (24)

$$F(z)=\lim_{p\to 1}\tilde{F}(z)=\tilde{F}_o(z)+\tilde{F}_1(z)+\tilde{F}_2(z)+\tilde{F}_3(z)+\tilde{F}_4(z)+... \tag{43}$$

c) The Procedure of Variation Parameter Method

The basic concept of VPM for solving differential equations is as follows: The general nonlinear equation is in the operator form

$$Lf(\eta)+Rf(\eta)+Nf(\eta)=g \tag{44}$$

The linear terms are decomposed into *L* + *R*, with *L* taken as the highest order derivative which is easily invertible and R as the remainder of the linear operator of order less than *L*. where *g* is the system input or the source term and *u* is the system output, *Nu* represents the nonlinear terms.

The VPM provides the general iterative scheme for Eq. (45) as:

$$f_{n+1}(\eta)= f_0(\eta)+\int_0^{\eta}\lambda(\eta,\xi)\left(-Rf_n(\xi)-Nf_n(\xi)-g(\xi)\right)d\xi \tag{45}$$

where the initial approximation $f_0(\eta)$ is given by

$$f_0(\eta)=\sum_{i=0}^{m}\frac{k_i f^i(0)}{i!} \tag{46}$$

m is the order of the given differential equation, k_i s are the unknown constants that can be determined by initial/boundary conditions and $\lambda(\eta,\xi)$ is the multiplier that reduces the order of the integration and can be determined with the help of Wronskian technique.

$$\lambda(\eta,\xi)=\sum_{i}^{m}\frac{(-1)^{i-1}\xi^{i-1}\eta^{m-1}}{(i-1)!(m-i)!}=\frac{(\eta-\xi)^{m-1}}{(m-1)!} \tag{47}$$

From the above, one can easily obtain the expressions of the multiplier for $Lf(\eta)= f^{n}(\eta)$

$$n=1,\quad \lambda(\eta,\xi)=1$$

$$n=2,\quad \lambda(\eta,\xi)=\eta-\xi$$

$$n=3,\quad \lambda(\eta,\xi)=\frac{\eta^2}{2!}-\eta\xi+\frac{\xi^2}{2!}$$

$$n=4,\quad \lambda(\eta,\xi)=\frac{\eta^3}{3!}-\frac{\eta^2\xi}{2!}+\frac{\eta\xi^2}{2!}-\frac{\xi^3}{3!}$$

$$n=5,\quad \lambda(\eta,\xi)=\frac{\eta^4}{4!}-\frac{\eta^3\xi}{3!}+\frac{\eta^2\xi^2}{2\cdot 2!}-\frac{\eta\xi^3}{3!}+\frac{\xi^4}{4!}$$

$$n=6,\quad \lambda(\eta,\xi)=\frac{\eta^5}{5!}-\frac{\eta^4\xi}{4!}+\frac{\eta^3\xi^2}{2\cdot 3!}-\frac{\eta^2\xi^3}{2\cdot 3!}+\frac{\eta\xi^4}{4!}-\frac{\xi^5}{5!}$$

$$n=7,\quad \lambda(\eta,\xi)=\frac{\eta^6}{6!}-\frac{\eta^5\xi}{5!}+\frac{\eta^4\xi^2}{2\cdot 4!}-\frac{\eta^3\xi^3}{6\cdot 3!}+\frac{15\eta^2\xi^4}{2\cdot 4!}-\frac{\eta\xi^5}{5!}+\frac{\xi^6}{6!}$$

$$n=8,\quad \lambda(\eta,\xi)=\frac{\eta^7}{7!}-\frac{\eta^6\xi}{6!}+\frac{\eta^5\xi^2}{2\cdot 5!}-\frac{\eta^4\xi^3}{6\cdot 4!}+\frac{\eta^3\xi^4}{6\cdot 4!}-\frac{\eta^2\xi^5}{2\cdot 5!}+\frac{\eta\xi^6}{6!}-\frac{\xi^7}{7!}$$

$$n=9,\quad \lambda(\eta,\xi)=\frac{\eta^8}{8!}-\frac{\eta^7\xi}{7!}+\frac{\eta^6\xi^2}{2\cdot 6!}-\frac{\eta^5\xi^3}{6!}+\frac{\eta^4\xi^4}{24\cdot 4!}-\frac{\eta^3\xi^5}{6!}+\frac{\eta^2\xi^6}{2\cdot 6!}-\frac{\eta\xi^7}{7!}+\frac{\xi^8}{8!}$$

$$n=10,\quad \lambda(\eta,\xi)=\frac{\eta^9}{9!}-\frac{\eta^8\xi}{8!}+\frac{\eta^7\xi^2}{2\cdot 7!}-\frac{\eta^6\xi^3}{36\cdot 5!}+\frac{\eta^5\xi^4}{24\cdot 5!}-\frac{\eta^4\xi^5}{24\cdot 5!}+\frac{\eta^3\xi^6}{36\cdot 5!}-\frac{\eta^2\xi^7}{2\cdot 7!}+\frac{\eta\xi^8}{8!}-\frac{\xi^9}{9!}$$

Consequently, an exact solution can be obtained when *n* approaches infinity.

Using the standard procedure of VPM as stated above, one can write the solution of Eq. (15) as

$$F_{n+1}(z)=k_1+k_2z+k_3\frac{z^2}{2}+k_4\frac{z^3}{6}$$
$$-\int_0^z\left(\frac{z^3}{3!}+\frac{z^2\xi}{2!}+\frac{z\xi^2}{2!}+\frac{\xi^3}{3!}\right)\left[R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}F_n(\xi)F_n'''(\xi)-G^2F_n''(\xi)\right]d\xi \tag{48}$$

Here, k_1, k_2, k_3, *and* k_4 are constants obtained by taking the highest order linear term of Eq. (15) and integrating it four times to get the final form of the scheme.

The above equation can also be written as

$$F_{n+1}(z)=F(0)+F'(0)z+F''(0)\frac{z^2}{2}+F'''(0)\frac{z^3}{6}$$
$$-\int_0^z\left(\frac{z^3}{3!}+\frac{z^2\xi}{2!}+\frac{z\xi^2}{2!}+\frac{\xi^3}{3!}\right)\left[R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}F_n(\xi)F_n'''(\xi)-G^2F_n''(\xi)\right]d\xi \tag{49}$$

From the boundary conditions in Eq. (16)

$$F(0)=0,\quad F''(0)=0$$

Using the above statement and inserting the boundary conditions of Eq. (16) into Eq. (49), we have

$$F_{n+1}(z)=k_1z+\frac{k_2z^3}{6}$$
$$-\int_0^\eta\left(\frac{z^3}{3!}+\frac{z^2\xi}{2!}+\frac{z\xi^2}{2!}+\frac{\xi^3}{3!}\right)\left[R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}F_n(\xi)F_n'''(\xi)-G^2F_n''(\xi)\right]d\xi \tag{50}$$

From the iterative scheme, it can easily be shown that the series solution is given as

$$F_0(z)=k_1z+\frac{k_2z^3}{6} \tag{51}$$

$$F_1(z) = k_1 z + \frac{k_2 z^3}{6} - \frac{k_1 k_2 R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} z^5}{120} + \frac{G^2 k_2 z^5}{120} - \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_2^2 z^7}{5040} \tag{52}$$

$$\begin{aligned}
F_2(z) = {} & k_1 z + \frac{k_2 z^3}{6} + \frac{k_1 k_2 R z^5}{120} + \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_1^2 k_2 z^7}{1680} - \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_2^2 z^7}{5040} \\
& - \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_1 k_2 G^2 z^7}{1260} + \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_2^2 k_1 z^9}{22680} \\
& - \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_2^2 G^2 z^9}{30240} + \frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2 k_2^3 z^{11}}{1108800} \\
& - \frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3 k_2^2 k_1^2 z^{11}}{1900800} - \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_2^2 G^4 z^{11}}{1900800} \\
& - \frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3 k_1 k_2^3 z^{13}}{38438400} + \frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2 k_2^3 G^2 z^{13}}{38438400} \\
& - \frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3 k_2^4 z^{15}}{396249600}
\end{aligned} \tag{53}$$

Similarly, the other iterations F z F z F z F F z F z are obtained. Therefore,

$$\begin{aligned}
F(z) = F_2(z) = {} & k_1 z + \frac{k_2 z^3}{6} + \frac{k_1 k_2 R z^5}{120} + \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_1^2 k_2 z^7}{1680} - \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_2^2 z^7}{5040} \\
& - \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_1 k_2 G^2 z^7}{1260} + \frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5} k_2^2 k_1 z^9}{22680}
\end{aligned}$$

$$-\frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}k_2^2G^2z^9}{30240}+\frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2k_2^3z^{11}}{1108800}$$
$$-\frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3k_2^2k_1^2z^{11}}{1900800}-\frac{R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}k_2^2G^4z^{11}}{1900800}$$
$$-\frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3k_1k_2^3z^{13}}{38438400}+\frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^2k_2^3G^2z^{13}}{38438400}$$
$$-\frac{\left(R\left((1-\phi)+\phi\frac{\rho_s}{\rho_f}\right)(1-\phi)^{2.5}\right)^3k_2^4z^{15}}{396249600}+\ldots \tag{54}$$

where the constants k_1 and k_2 are determined using the boundary conditions in Eq. (16) i.e.

$$F(1)=1,\quad F'(1)=\gamma F''(1)$$

The equations are solved for the corresponding values of k_1 and k_2 for the different values of γ.

V. Results and Discussion

The above analyses show the applications of three approximate analytical methods of differential transformation, homotopy perturbation and variation of parameters methods for the analysis of a steady two-dimensional axisymmetric flow of an incompressible viscous fluid under the influence of a uniform transverse magnetic field with slip boundary condition. Using VPM and DTM, closed form series solutions are obtained as they provide excellent approximations to the solution of the non-linear equation with higher accuracy than HPM. Also, the VPM and DTM shows to more convenient for engineering calculations compared to HPM as they appear more appealing than the HPM. However, higher accuracy and high rate of convergence was recorded in VPM than DTM as shown the table, the solution of VPM is used to carry out the parametric study shown in Figs. 2-7.

Table: Comparison of Results

	F(z)			
z	NM	VPM	DTM	HPM
0.00	0.000000	0.000000	0.000000	0.000000
0.10	0.075739	0.075739	0.075739	0.075738
0.20	0.152935	0.152935	0.152935	0.152935
0.30	0.233046	0.233046	0.233046	0.233045
0.40	0.317540	0.317540	0.317540	0.317540
0.50	0.407893	0.407893	0.407893	0.407892
0.60	0.505591	0.505591	0.505591	0.505592
0.70	0.612134	0.612134	0.612134	0.612134
0.80	0.729034	0.729034	0.729034	0.729035
0.90	0.857813	0.857813	0.857813	0.857813
1.00	1.000000	1.000000	1.000000	1.000000

Although, analytically, the VPM and DTM are somehow easier and straight-forward as compared to HPM, there is no search for Wronskian multiplier (as carried out in VPM) or the rigour of developing recursive relations or differential transforms coupled with the search for included unknown parameter that will satisfy second the boundary condition lead to additional computational cost in the generation of the solution to the problem using DTM. This drawback is not only peculiar to VPM and DTM, other approximate analytical methods such as HAM, ADM, VIM, DJM, TAM also required additional computational cost and time for the determination of included unknown parameter that will satisfy second the boundary condition. Also, the VPM and DTM have their own operational restrictions that severely narrow there functioning domains as they are limited to small domain. Using VPM or DTM for large or infinite domain is accompanied with either the application of before-treatment techniques such as domain transformation techniques, domain truncation techniques and conversion of the boundary value problems to initial value problems or the use of after-treatment techniques such as Pade-approximants, basis functions, cosine after-treatment technique, sine after-treatment technique and domain decomposition technique. This is because VPM and DTM were initially established for initial value problems. Amending the methods to boundary value problems especially for large or infinite domains boundary value problems leads to the inclusion of unknown parameters (that will satisfy second the boundary condition) in the solution. This drawback in the other approximation analytical methods is not experienced in HPM as such tasks of before- and after-treatment techniques might not necessarily be required in HPM. This is because HPM is easily applied to the boundary value problems without any included unknown parameter in the solution as found in VPM and DTM.

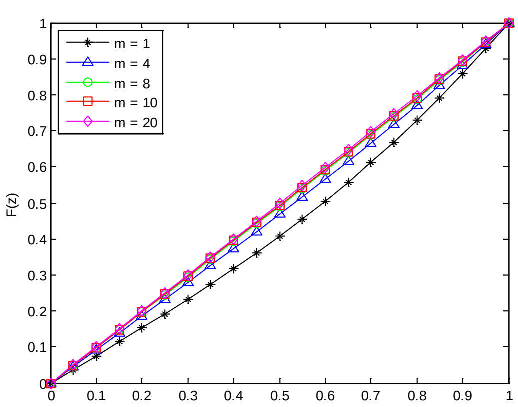

Figure 2: Effects of magnetic parameter on the flow of the fluid under the influence of slip condition behavior

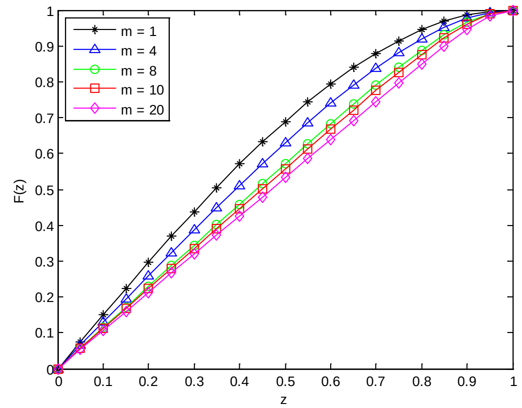

Figure 3: Effects of magnetic field parameter on the flow behavior of the fluid for no-slip condition

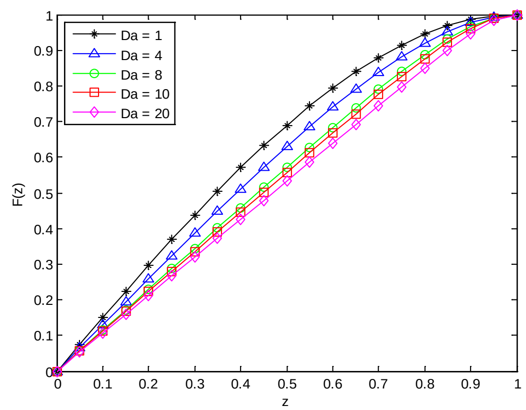

Figure 4: Effects of porous parameter on the flow behavior of the fluid under the influence of slip condition

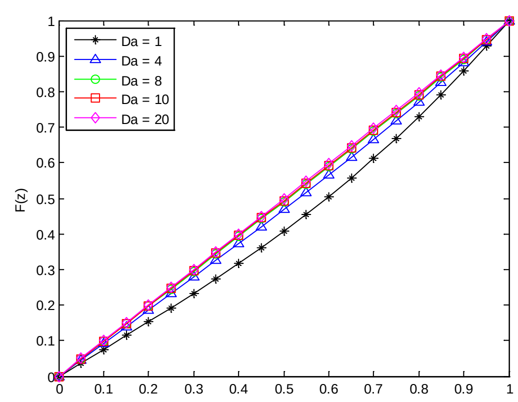

Figure 5: Effects of porous field parameter on the flow behavior of the fluid for no-slip condition

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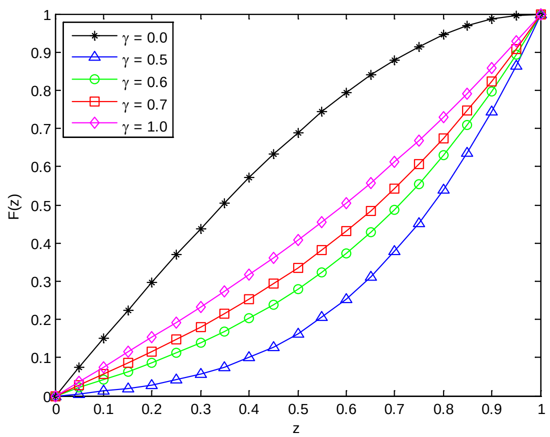

Figure 6: Effects of slip parameter on the flow behavior of the fluid

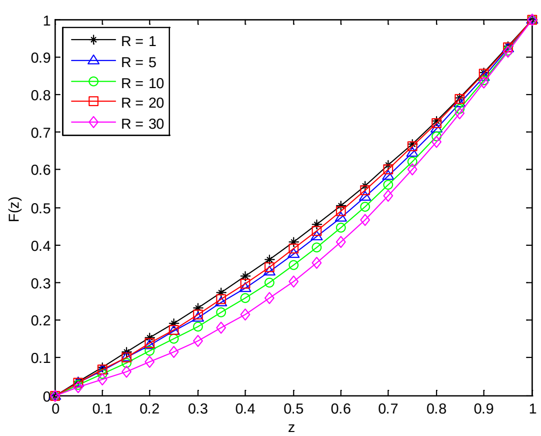

Figure 7: Effects of Reynolds number on the flow behavior of the fluid under the influence of slip condition

In order to get an insight into the problem, the effects of pertinent flow, magnetic field and slip parameters on the velocity profile of the fluid are investigated. Fig. 2 and 4 shows the effects of magnetic field and porous parameter on the velocity of the fluid under the influence of slip condition, while Fig. 3 and 5 depicts the influence of the porous and magnetic on the velocity of the fluid under no-slip condition. It could be inferred from the figures that the velocity of the fluid increases with increase in the porous-magnetic parameter under slip condition while an opposite trend was recorded during no-slip condition as the velocity of the fluid decreases with increase in the porous-magnetic parameter under the no slip condition. Fig. 6 shows the influence of the slip parameter γ on the fluid velocity. By increasing γ, it is observed that the velocity of the fluid increases. Fig. 7 presents the effects of Reynold's number on the velocity of the fluid. It is observed from the figure that by increasing the value R, the velocity of the fluid decreases.

VI. Conclusion

In this work, a comparative study of three approximate analytical methods have been carried out for the analysis of two-dimensional axisymmetric flow of an incompressible viscous fluid through porous medium under the influence of a uniform transverse magnetic field with slip boundary condition. From the analysis, it is established that VPM give higher accurate results than DTM and HPM with faster rate of convergence. Also, from the parametric study, it was established from the results that, the velocity of the fluid increases with increase in the porous-magnetic parameter under slip condition while the velocity of the fluid decreases with increase in the porous-magnetic parameter under no slip condition. By increasing the slip parameter, the velocity of the fluid increases, and the fluid velocity decreases as the Reynolds number increases. The approximate analytical solutions have been verified by comparing the results of the approximate analytical method with the numerical method using Runge-Kutta coupled with shooting method

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