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Development of Boundary Element Method in Polar Coordinate System for Elasticity Problems

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I. Introduction

The boundary element method [1, 2] is a helpful tool to solve the problems of computational mechanics. Many researchers and scientists use standard BEM, with the boundary approximation done by using linear segments (boundary elements), or standard BEM is improved by considering the conditions of a given problem [1-19]. The advantage of using linear boundary elements is the opportunity to analytically calculate the integrals, while with curvilinear elements generally, it is possible to do numerical integration [20].

The boundary element method, in particular, the fictitious load method formulated to solve the boundary value and boundary-contact problems of elasticity for a circular ring and its parts are improved in the present paper if considering that the circular segment of the boundary is divided into arcs instead of linear segments. This allows to describe the considered area more accurately and to arrive at a more accurate solution of the problem. So, when the considered area is limited with circles or their parts, i.e., with the coordinate axes of the polar coordinate system, then by dividing the circle into small arcs, we can formulate BEM in the polar coordinate system with all integrals solved analytically. In particular, a fictitious load method is considered in the polar coordinate system, whereas it was described in a Cartesian coordinate system by Crouch and Star field [1].

The article gives a fundamental solution written down in polar coordinate systems serving as a basis to obtain a numerical solution, and a problem of constant forces distributed along the arc is considered. A numerical procedure is presented and boundary coefficients of influence are written out.

Two test boundary-contact problems are solved:

1. Elastic equilibrium of an infinite area with a circular hole is studied when a circular ring inserts near the hole; normal constant stress is given on the internal surface of the ring, the body is free from stresses in the infinity, and the conditions of continuity of displacements and stresses are given on the contact line. Numerical values are obtained by using: a) analytical solution, b) standard BEM, i.e., when a circular boundary divides into linear segments, and c) PCSBEM, i.e., when the boundary divides into arcs, and the results obtained in all three cases are compared to one another.
2. A boundary-contact problem is solved for a double-layer circular ring when the internal circular boundary is loaded with a normal variable force, the outer boundary is not loaded, and conditions of a rigid contact are given for the contact line. The numerical results are obtained by using standard BEM and PCSBEM and are compared to one another. MATLAB software was used to obtain the relevant numerical values and graphs for both problems.

II. The Fundamental Solution in the Cartesian Coordinate System

Let us consider the problem shown in Fig. one known as Kelvin's problem of plane deformation [1].

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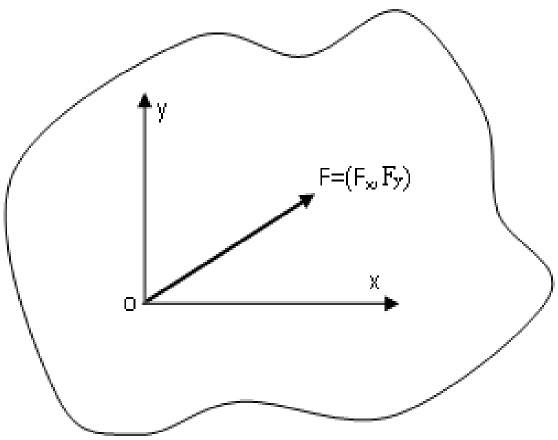

Figure 1: Kelvin's problem for plane deformation

$F=(F_x,F_y)$ forces in Fig. 1 are a line of the point force applied along axis z in the infinite elastic plane.

The solution to this problem is given by the following function [1]:

$$g(x,y)=-\frac{1}{4\pi(1-\nu)}\ln\sqrt{(x^2+y^2)}\ , \tag{1}$$

where ν is the Poisson's ratio.

The displacements will be written down as follows:

$$\begin{aligned} u_x(x,y)&=\frac{F_x}{2G}\left[(3-4\nu)g-xg_{,x}\right]+\frac{F_y}{2G}\left(-yg_{,x}\right),\\ u_y(x,y)&=\frac{F_x}{2G}\left(-xg_{,y}\right)+\frac{F_y}{2G}\left[(3-4\nu)g-yg_{,y}\right], \end{aligned} \tag{2}$$

where $G=\dfrac{E}{2(1+\nu)}$ is shear modulus, and E is Young's modulus.

For the plane deformation, the stresses for Kelvin's problem will be written down as follows:

$$\begin{aligned} \sigma_{xx}(x,y)&=F_x\left[2(1-\nu)g_{,x}-xg_{,xx}\right]+F_y\left(2\nu g_{,y}-yg_{,xx}\right),\\ \sigma_{yy}&=F_x\left(2\nu g_{,x}-xg_{,yy}\right)+F_y\left[2(1-\nu)g_{,y}-yg_{,yy}\right]\\ \sigma_{xy}(x,y)&=F_x\left[(1-2\nu)g_{,y}-xg_{,xx}\right]+F_y\left[(1-2\nu)g_{,x}-yg_{,xy}\right] \end{aligned} \tag{3}$$

As it can be seen from (1), (3), the stress at point $x=0,\quad y=0$ has the singularity. It can be shown that these stresses correspond to the point force at the origin of coordinates [21].

For the sake of simplicity, we mean that $F_i=(F_x,F_y)$ force is applied to the origin of coordinates.

III. The Fundamental Solution in the Polar Coordinate System

Let us write down formulae (1), (2) and (3) in polar coordinate system $r,\vartheta\ (0\le r<\infty, 0\le\vartheta<2\pi)$ [22]. Following certain algebraic transformations, we obtain the following expression for the function $g(x,y)$:

$$g(x,y)\equiv g_1(r,\vartheta)=-\frac{1}{4\pi(1-\nu)}\ln r.$$

For the components of a displacement vector, we will obtain the following equations:

$$\begin{aligned} \tilde{u}_x(r,\vartheta)&=\frac{F_x}{2G}\left[(3-4\nu)g_1-r\cos\vartheta g_{1,x}\right]+\frac{F_y}{2G}\left(-r\sin\vartheta g_{1,x}\right),\\ \tilde{u}_y(r,\vartheta)&=\frac{F_x}{2G}\left(-r\cos\vartheta g_{1,y}\right)+\frac{F_y}{2G}\left[(3-4\nu)g_1-r\sin\vartheta g_{1,y}\right] \end{aligned} \tag{4}$$

and for the components of the stress tensor we will obtain:

$$\begin{aligned} \tilde{\sigma}_{xx}(r,\vartheta)&=F_x\left[2(1-\nu)g_{1,x}-r\cos\vartheta g_{1,xx}\right]+F_y\left(2\nu g_{1,y}-r\sin\vartheta g_{1,xx}\right),\\ \tilde{\sigma}_{yy}(r,\vartheta)&=F_x\left(2\nu g_{1,x}-r\cos\vartheta g_{1,yy}\right)+F_y\left[2(1-\nu)g_{1,y}-r\sin\vartheta g_{1,yy}\right],\\ \tilde{\sigma}_{xy}(r,\vartheta)&=F_x\left[(1-2\nu)g_{1,y}-r\cos\vartheta g_{1,xx}\right]+F_y\left[(1-2\nu)g_{1,x}-r\sin\vartheta g_{1,xy}\right], \end{aligned} \tag{5}$$

where $g_{1,x}$, $g_{1,y}$, $g_{1,xx}$, $g_{1,yy}$, $g_{1,xy}$ in the polar coordinate system have following form:

$$g_{1,x} = -\frac{1}{4\pi(1-\nu)}\frac{\cos\vartheta}{r}, \quad g_{1,y} = -\frac{1}{4\pi(1-\nu)}\frac{\sin\vartheta}{r},$$

$$g_{1,xx} = \frac{1}{4\pi(1-\nu)}\frac{\cos(2\vartheta)}{r^2}, \quad g_{1,yy} = -\frac{1}{4\pi(1-\nu)}\frac{\cos(2\vartheta)}{r^2},$$

$$g_{1,xy} = \frac{1}{4\pi(1-\nu)}\frac{\sin(2\vartheta)}{r^2}.$$

By using the superposition principle, we can solve the problem for an infinite elastic body, with a set of point forces acting at any of its points. If distributing such forces continuously along some line of the plane, we will obtain a problem with the forces given along this line.

IV. Constant Forces Distributed Along the Curve

Let us consider the following problem: constant $t_r = P_r$ and $t_\vartheta = P_\vartheta$ forces are applied to the $\vartheta_1 \le \vartheta \le \vartheta_2$ arc of a circle with radius r in an infinite body. This problem can be solved by integrating a fundamental solution.

Let us divide arc MN into the elements with a length of $d\xi$ (See Fig. 2). Then, the sum of the forces acting on the arc element with its center at the point (r,ξ), equals $F_i(\xi) = P_i \cdot r d\xi$, where index i denotes r or ϑ. To solve this problem, let us insert the expressions of $F_r(\xi)$ and $F_\vartheta(\xi)$ forces in (4) and (5), change ϑ with $\vartheta - \xi$ and integrate the obtained expressions with ξ from ϑ_1 to ϑ_2. The following formulae are obtained for displacements:

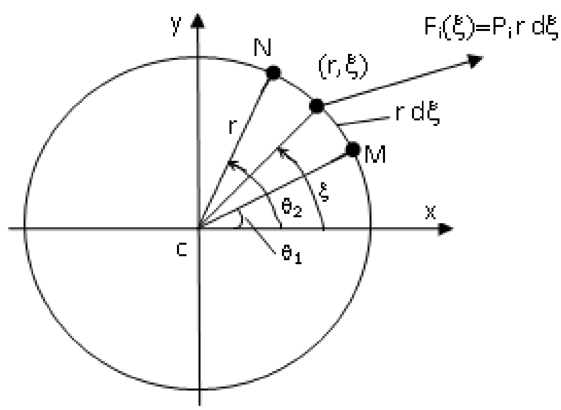

Figure 2: Integration of a fundamental solution

$$\begin{aligned}\bar{u}_x = \frac{r^2}{8G\pi(1-\nu)}\{P_\vartheta[(3-4\nu)(\vartheta_1-\vartheta_2)\ln r + 0.5(\vartheta_1-\vartheta_2) \\ + 0.5(\sin(2(\vartheta-\vartheta_1)) + \sin(2(\vartheta-\vartheta_2)))] \\ - 0.25P_r\cos(2(\vartheta-\vartheta_1)) - \cos(2(\vartheta-\vartheta_2))\},\end{aligned}$$

$$\begin{aligned}\bar{u}_y = -\frac{r^2}{8G\pi(1-\nu)}\{0.25P_\vartheta(\cos(2(\vartheta-\vartheta_1)) - \cos(2(\vartheta-\vartheta_2))) \\ + P_r[-(3-4\nu)(\vartheta_1-\vartheta_2)\ln r + 0.5(\vartheta_1-\vartheta_2) \\ + 0.25(\sin(2(\vartheta-\vartheta_1)) - \sin(2(\vartheta-\vartheta_2)))]\},\end{aligned} \tag{6}$$

and the following formulae are obtained for stresses:

$$
\begin{aligned}
\bar{\sigma}_{yy} = -\frac{r}{8\pi(1-\nu)}\Big\{ & P_{\vartheta}\Big[(4\nu-1)\big(\sin(\vartheta-\vartheta_1)-\sin(\vartheta-\vartheta_2)\big) \\
& -\frac{1}{3}\big(\sin(3(\vartheta-\vartheta_1))-\sin(3(\vartheta-\vartheta_2))\big)\Big] \\
& -P_r\Big[(5-4\nu)\big(\cos(\vartheta-\vartheta_1)-\cos(\vartheta-\vartheta_2)\big) \\
& -\frac{1}{3}\big(\cos(3(\vartheta-\vartheta_1))-\cos(3(\vartheta-\vartheta_2))\big)\Big]\Big\}, \\
\bar{\sigma}_{xx} = -\frac{r}{8\pi(1-\nu)}\Big\{ & P_{\vartheta}\Big[(5-2\nu)\big(\sin(\vartheta-\vartheta_1)-\sin(\vartheta-\vartheta_2)\big) \\
& +\frac{1}{3}\big(\sin(3(\vartheta-\vartheta_1))-\sin(3(\vartheta-\vartheta_2))\big)\Big] \\
& -P_r\Big[(4\nu-1)\big(\cos(\vartheta-\vartheta_1)-\cos(\vartheta-\vartheta_2)\big) \\
& +\frac{1}{3}\big(\cos(3(\vartheta-\vartheta_1))-\cos(3(\vartheta-\vartheta_2))\big)\Big]\Big\}, \\
\bar{\sigma}_{xy} = -\frac{r}{8\pi(1-\nu)} & P_{\vartheta}\Big[(4\nu-3)\big(\cos(\vartheta-\vartheta_1)-\cos(\vartheta-\vartheta_2)\big) \\
& +\frac{1}{3}\big(\cos(3(\vartheta-\vartheta_1))-\cos(3(\vartheta-\vartheta_2))\big)\Big] \\
& +P_r\Big[(5-4\nu)\big(\sin(\vartheta-\vartheta_1)-\sin(\vartheta-\vartheta_2)\big) \\
& -\frac{1}{3}\big(\sin(3(\vartheta-\vartheta_1))-\sin(3(\vartheta-\vartheta_2))\big)\Big]\Big\}.
\end{aligned}
\tag{7}
$$

Equations (6) and (7) are displacements and stresses in an infinite elastic body when constant $t_r = P_r$ and $t_{\vartheta} = P_{\vartheta}$ forces are applied to $\vartheta_1 \le \vartheta \le \vartheta_2$ arc of a circle with the radius r. These equations are the basis for the boundary element method considered later. The following peculiarity of the analytical solution given above is worth mentioning.

Displacements from the origin of coordinates to the infinitely distanced points are not limited because of the logarithm included in them. Therefore, equations (6) show only relative displacements. In any concrete case, we must choose a reference point and determine the displacement in respect of such a point.

V. Numerical Procedure

The analytical solution obtained above is the basis for the boundary element method used to obtain a numerical solution of the boundary value problem of the theory of elasticity. Let us explain the physical aspect of this method by using a specific example. Let us consider a boundary value problem for an infinite body with a hole (with a circular hole in our case). We will consider a plane deformation. Let us denote the boundary of the cut, which is a circle in our case, by C (See Fig. 3). At any point of the C curve, local s and n coordinates have the direction of a tangent and its perpendicular. Therefore, they change at different points along the border. We take these coordinates so that the direction of n should coincide with the direction of an outer normal at the same point as the border and s should coincide with the direction of the boundary line. In this case, the direction of the boundary line is anticlockwise. Let us assume that the same normal stress ($\sigma_n = -p$) acts at all points of the hole wall (i.e., there is compression) and tangential stress $\sigma_s = 0$. Let us calculate the displacements and stresses in the body caused by such a load of the boundary.

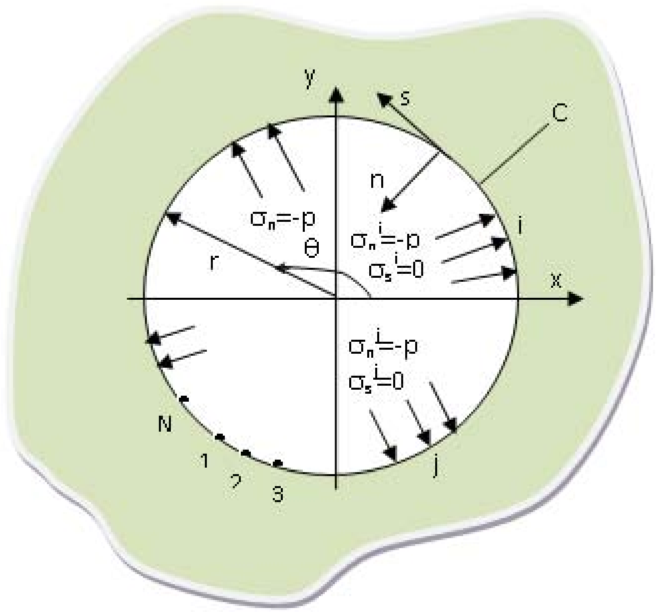

Figure 3: Illustration of the problem statement

The numerical solution of this problem can be obtained as follows: first, let us divide circle *C* into N small arcs (elements). As these elements are small, we can consider that normal $\sigma_n = -p$ stress acts along the whole length of every element, and the tangent is free from stress. In this case, the boundary conditions will be as follows:

$$\sigma_n^i = -p, \qquad \sigma_s^i = 0, \qquad (i = 1, \dots N).$$

Let us imagine that constant normal and tangent stresses act on every element of the circle, e.g., let us denote the normal and tangent stresses acting on the element j by P_n^j and P_s^j, respectively.

It should be noted that the real normal and tangent tresses acting on the element j do not equal to P_n^j and P_s^j, if stresses act on other elements, too. Therefore, there are two different kinds of stresses for every element. For example, for the element j, we have applied stresses P_n^j and P_s^j and real stresses σ_n^j and σ_s^j caused by the action of the stresses applied to all N elements.

By using (6) and (7), we can calculate real σ_n^i and σ_s^i stresses, $i = 1, \dots, N$ in the middle point of each element with the following formula:

$$\sigma_s^i = \sum_{j=1}^{N} \left(A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j \right),$$
$$\sigma_n^i = \sum_{j=1}^{N} \left(A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j \right), \quad i = 1, \dots, N, \tag{8}$$

where $A_{ss}^{ij}, \dots$ are the boundary coefficients of the influence of the stresses for the considered problem. For example, A_{sn}^{ij} is the real tangent stress in the center of the element i caused by the constant normal unit load ($P_n^j = 1$) applied to the element j.

By considering the boundary conditions, we will obtain the following equations:

$$\left. \begin{aligned} 0 &= \sum_{j=1}^{N} \left(A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j \right), \\ -p &= \sum_{j=1}^{N} \left(A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j \right), \end{aligned} \right\}, \quad i = 1, \dots, N, \tag{9}$$

which is a system of $2N$ linear algebraic equations with $2N$ P_n^j and P_s^j ($j = 1, \dots, N$) unknown values.

It should be noted that P_n^j and P_s^j stresses in these equations are fictitious values. They are introduced as an intermediate quantity to obtain the numerical value of the problem, and they have no physical essence. However, a linear combination of a fictitious load presented with formulae (8) has a physical essence in the considered problem, and is the basis to obtain a system of algebraic equations (9). After solving this system, we can express displacements and stresses at any point in a body with another combination of P_n^j and P_s^j, ($j=1,\dots N$) fictitious load.

The above-described boundary element method is called a fictitious load method [1].

VI. Influence Coefficients

Let us write down the expressions of the tangent and normal displacements and stresses in the middle point of the i-th element caused by fictitious loads P_n^j and P_s^j, $j=1,\dots N$ applied to the j-th element. For the displacements, we will have:

$$
\begin{aligned}
u_s^i = P_s^j\Big\{&\frac{\bar r^2}{8G\pi(1-\nu)}\big[(3-4\nu)(\vartheta_1-\vartheta_2)\ln\bar r+0.5(\vartheta_1-\vartheta_2)\\
&+0.25\sin(2(\bar\vartheta-\vartheta_1))-\sin(2(\bar\vartheta-\vartheta_2))\big]\Big\}\\
&+P_n^j\left\{\frac{-\bar r^2}{32G\pi(1-\nu)}\big[\cos(2(\bar\vartheta-\vartheta_1))-\cos(2(\bar\vartheta-\vartheta_2))\big]\right\},\\
u_n^i = P_s^j\Big\{&-\frac{\bar r^2}{G\pi(\ -\nu)}\big[\cos(2(\bar\vartheta-\vartheta_1))-\cos(2(\bar\vartheta-\vartheta_2))\big]\Big\}\\
&+P_n^j\Big\{\frac{\bar r^2}{8G\pi(1-\nu)}\big[(3-4\nu)(\vartheta_1-\vartheta_2)\ln\bar r-0.5(\vartheta_1-\vartheta_2)\\
&-0.25(\sin(2(\bar\vartheta-\vartheta_1))-\sin(2(\bar\vartheta-\vartheta_2)))\big]\Big\},
\end{aligned}
\tag{10}
$$

and for the stresses, the expressions will be as follows:

$$\sigma_n^i = P_s^j\left\{\frac{-\bar{r}}{8\pi(1-\nu)}\left[(4\nu-1)\left(\sin(\bar{\vartheta}-\vartheta_1)-\sin(\bar{\vartheta}-\vartheta_2)\right)\right.\right.$$
$$\left.\left.-\frac{1}{3}\left(\sin(3(\bar{\vartheta}-\vartheta_1))-\sin(3(\bar{\vartheta}-\vartheta_2))\right)\right]\right\}$$
$$+P_n^j\left\{\frac{\bar{r}}{8\pi(1-\nu)}\left[(5-4\nu)\left(\cos(\bar{\vartheta}-\vartheta_1)-\cos(\bar{\vartheta}-\vartheta_2)\right)\right.\right.$$
$$\left.\left.-\frac{1}{3}\left(\cos(3(\bar{\vartheta}-\vartheta_1))-\cos(3(\bar{\vartheta}-\vartheta_2))\right)\right]\right\},$$
$$\sigma_s^i = P_s^j\left\{\frac{-\bar{r}}{8\pi(1-\nu)}\left[(4\nu-3)\left(\cos(\bar{\vartheta}-\vartheta_1)-\cos(\bar{\vartheta}-\vartheta_2)\right)\right.\right.$$
$$\left.\left.+\frac{1}{3}\left(\cos(3(\bar{\vartheta}-\vartheta_1))-co(3(\bar{\vartheta}-\vartheta_2))\right)\right]\right\}$$
$$+P_n^j\left\{\frac{-\bar{r}}{8\pi(1-\nu)}\left[(3-4\nu)\left(\sin(\bar{\vartheta}-\vartheta_1)-\sin(\bar{\vartheta}-\vartheta_2)\right)\right.\right. \qquad (11)$$
$$\left.\left.-\frac{1}{3}\left(\sin(3(\bar{\vartheta}-\vartheta_1))-\sin(3(\bar{\vartheta}-\vartheta_2))\right)\right]\right\},$$
$$\sigma_t^i = P_s^j\left\{\frac{-\bar{r}}{\pi(\ -\nu)}\left[(5-2\nu)\left(\sin(\bar{\vartheta}-\vartheta_1)-\sin(\bar{\vartheta}-\vartheta_2)\right)\right.\right.$$
$$\left.\left.+\frac{1}{3}\left(\sin(3(\bar{\vartheta}-\vartheta_1))-\sin(3(\bar{\vartheta}-\vartheta_2))\right)\right]\right\}$$
$$+P_n^j\left\{\frac{\bar{r}}{8\pi(1-\nu)}\left[(4\nu-1)\left(\cos(\bar{\vartheta}-\vartheta_1)-\cos(\bar{\vartheta}-\vartheta_2)\right)\right.\right.$$
$$\left.\left.+\frac{1}{3}\left(\cos(3(\bar{\vartheta}-\vartheta_1))-\cos(3(\bar{\vartheta}-\vartheta_2))\right)\right]\right\},$$

where $\bar{r}$ and $\bar{\vartheta}$ are coordinates in the local coordinate system, with its center coinciding with the middle point of the i-th element. Generally, the displacements and stresses in the i-th element are functions of the P_s^j and P_n^j fictitious load on all N elements. So, by (10) and (11), we can write down:

$$u_s^i = \sum_{j=1}^{N}\left(B_{ss}^{ij}P_s^j + B_{sn}^{ij}P_n^j\right),$$

$$u_n^i = \sum_{j=1}^{N}\left(B_{ns}^{ij}P_s^j + B_{nn}^{ij}P_n^j\right),$$

$$\sigma_s^i = \sum_{j=1}^{N}\left(A_{ss}^{ij}P_s^j + A_{sn}^{ij}P_n^j\right),$$

$$\sigma_n^i = \sum_{j=1}^{N}\left(A_{ns}^{ij}P_s^j + A_{nn}^{ij}P_n^j\right).$$

In these equations, boundary influence $B_{ss}^{ij},\ldots$ and $A_{ss}^{ij},\ldots$ coefficients are calculated with the expressions in the curly braces of equations (10) and (11). For example, the A_{sn}^{ij} coefficient is calculated with the expression given in curly braces at P_n^j of the first equation of (10).

VII. Numerical Examples and Discussion

There are two test problems of using a fictitious load method given below. We have an exact solution to one problem. Therefore, in the case of dividing the boundary into segments and arcs, the numerical results obtained by using the boundary element method will be compared to the exact values. Another problem will compare the numerical values obtained by using the fictitious load method to one another in case of dividing the boundary into segments and into arcs on the other hand.

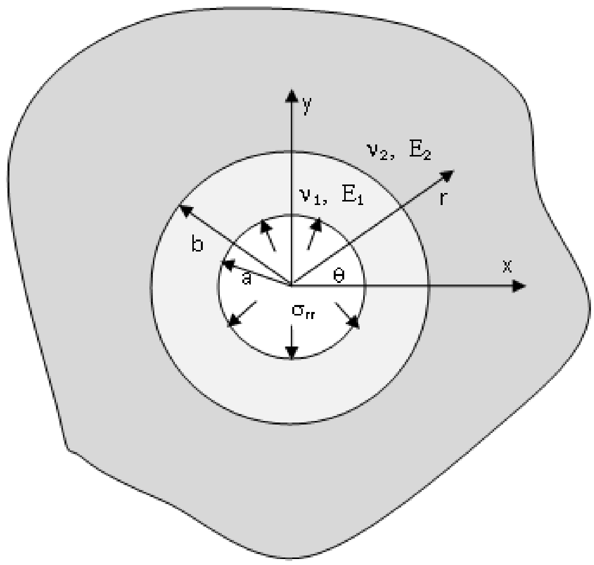

Figure 4: A circular ring in an infinite plate with a hole

a) Ring in an infinite plate with a circular hole

The study area consists of the $a \le r \le b$ ring with ν_1 and E_1 as its elastic characteristics and infinite area with a circular hole with $r = b$ radius, with ν_2 and E_2 as its elastic characteristics (See Fig. 4). $\sigma_{rr} = -p$ normal stress is given on the internal surface of the ring, while in the infinity, the body is free from stresses, and the continuity conditions of displacements and stresses are given on $r = b$ contact surface. So, we will have the following boundary conditions:

$$r = a: \ \sigma_{rr}^{(1)} = -p, \qquad \sigma_{r\vartheta}^{(1)} = 0,$$

$$r \to \infty: \ \sigma_{rr}^{(2)} = 0, \qquad \sigma_{r\vartheta}^{(2)} = 0,$$

and contact conditions:

$$r = b: \ \sigma_{rr}^{(1)} = \sigma_{rr}^{(2)}, \quad \sigma_{r\vartheta}^{(1)} = \sigma_{r\vartheta}^{(2)}, \quad u_r^{(1)} = -u_r^{(2)}, \quad u_\vartheta^{(1)} = -u_\vartheta^{(2)}.$$

The solution to this problem is obtained from standard formulae [23, 24] for a thick-wall cylinder. In particular, the radial and tangential stresses are calculated with the following formulae [1]:

$$\left.\begin{aligned} \sigma_{rr} &= \frac{1}{1-\dfrac{a^2}{b^2}}\left[\left(p\frac{a^2}{b^2} - p'\right) - (p - p')\frac{a^2}{r^2}\right] \\ \sigma_{\vartheta\vartheta} &= \frac{1}{1-\dfrac{a^2}{b^2}}\left[\left(p\frac{a^2}{b^2} - p'\right) + (p - p')\frac{a^2}{r^2}\right] \end{aligned}\right\} \quad \text{when} \quad a \le r \le b,$$

$$\left.\begin{aligned} \sigma_{rr} &= -p'\frac{b^2}{r^2} \\ \sigma_{\vartheta\vartheta} &= p'\frac{b^2}{r^2} \end{aligned}\right\} \quad \text{when} \quad r \ge b.$$

A numerical solution of this problem is obtained with a fictitious load method for the following parameter values: $\dfrac{a}{b} = \dfrac{1}{2}$, $\nu_1 = \nu_2 = 0.25$, $\dfrac{E_1}{E_2} = 2$, $\dfrac{p}{E_2} = 10^{-3}$. Because of the symmetry of the problem, one-fourth of the area, in particular, the space between $\vartheta = 0$ and $\vartheta = \dfrac{\pi}{2}$ is considered. Within this range, the $r = a$ boundary surface and both sides of the $r = b$ contact surface are divided into n=90 elements each, and the obtained visual and numerical results are presented in Fig. 5, Fig. 6, Table 1 and Table 2, while in cases shown in Fig. 7, Fig. 8, Table 3 and Table 4, they are divided into 180 elements each.

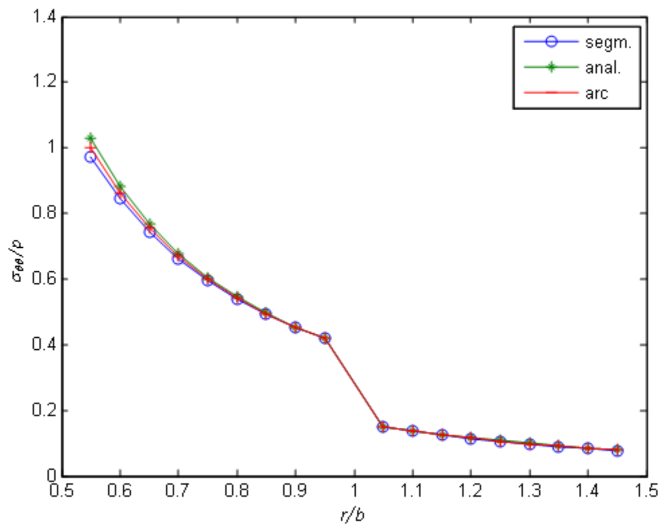

Figure 5: Shearing stress $\sigma_{\vartheta\vartheta}/p$ in the ring $0.5 \le \frac{r}{b} \le 1$ and infinite body $\frac{r}{b} \ge 1$ (n=90)

Table 1: Shearing stress $\sigma_{\vartheta\vartheta}/p$ in the ring $0.5 \le \frac{r}{b} \le 1$ and infinite body $\frac{r}{b} \ge 1$ (n=90)

No.	r/b	Exact solution	Approximate solution		Relative error, percent	
			In case of the division into segments	In case of the division into arcs	Segments	Arcs
1	0.5500	1.0294	0.9726	1.0010	5.5160	2.7574
2	0.6000	0.8827	0.8446	0.8637	4.3169	2.1543
3	0.6500	0.7686	0.7437	0.7561	3.2348	1.6174
4	0.7000	0.6780	0.6621	0.6701	2.3412	1.1706
5	0.7500	0.6049	0.5950	0.6000	1.6392	0.8196
6	0.8000	0.5451	0.5392	0.5422	1.0956	0.5478
7	08500	0.4956	0.4923	0.4939	0.6601	0.3300
8	0.9000	0.4540	0.4528	0.4534	0.2769	0.1395
9	0.9500	0.4189	0.4194	0.4191	0.1096	0.0548
10	1.0500	0.1512	0.1507	0.1509	0.3242	0.1621
11	1.1000	0.1377	0.1370	0.1374	0.5607	0.2803
12	1.1500	0.1260	0.1250	0.1255	0.7963	0.3981
13	1.2000	0.1157	0.1146	0.1151	1.0238	0.5119
14	1.2500	0.1067	0.1053	0.1060	1.2381	0.6192
15	1.3000	0.0986	0.0972	0.0979	1.4361	0.7181
16	1.3500	0.0914	0.0900	0.0907	1.6165	0.8083
17	1.4000	0.0850	0.0835	0.0843	1.7787	0.8894
18	1.4500	0.0793	0.0777	0.0785	1.9234	0.9714
Average					1.6605	0.8306

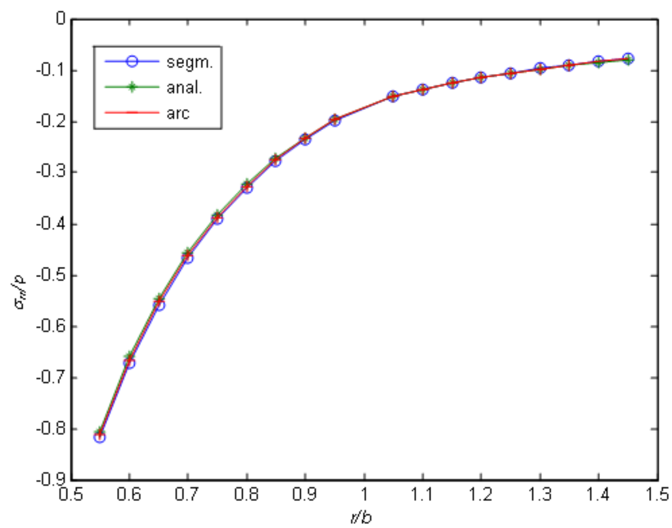

Figure 6: Normal stress σ_{rr} / p in the ring $0.5 \le \frac{r}{b} \le 1$ and infinite body $\frac{r}{b} \ge 1$ (n=90)

Table 2: Normal stress σ_{rr} / p in the ring $0.5 \le \frac{r}{b} \le 1$ and infinite body $\frac{r}{b} \ge 1$ (n=90)

No.	r/b	Exact solution	Approximate solution		Relative error, percent	
			In case of the division into segments	In case of the division into arcs	Segments	Arcs
1	0.5500	-0.8072	-0.8160	-0.8116	1.0951	0.5498
2	0.6000	-0.6605	-0.6725	-0.6665	1.8187	0.9093
3	0.6500	-0.5463	-0.5588	-0.5526	2.2884	1.1438
4	0.7000	-0.4558	-0.4671	-0.4615	2.4873	1.2545
5	0.7500	-0.3827	-0.3921	-0.3874	2.4410	1.2239
6	0.8000	-0.3229	-0.3300	-0.3265	2.1918	1.1097
7	08500	-0.2734	-0.2782	-0.2758	1.7851	0.8939
8	0.9000	-0.2318	-0.2347	-0.2333	1.2643	0.6365
9	0.9500	-0.1967	-0.1980	-0.1973	0.6700	0.3173
10	1.0500	-0.1512	-0.1521	-0.1516	0.6113	0.2834
11	1.1000	-0.1377	-0.1379	-0.1378	0.0929	0.1777
12	1.1500	-0.1260	-0.1256	-0.1258	0.3699	0.1777
13	1.2000	-0.1157	-0.1148	-0.1153	0.7785	0.3808
14	1.2500	-0.1067	-0.1055	-0.1061	1.1357	0.5313
15	1.3000	-0.0986	-0.0972	-0.0979	1.4453	0.7294
16	1.3500	-0.0914	-0.0899	-0.0907	1.7117	0.8195
17	1.4000	-0.0850	-0.0834	-0.0842	1.9396	0.9808
18	1.4500	-0.0793	-0.0776	-0.0784	2.1335	1.0984
Average					1.4589	0.7343

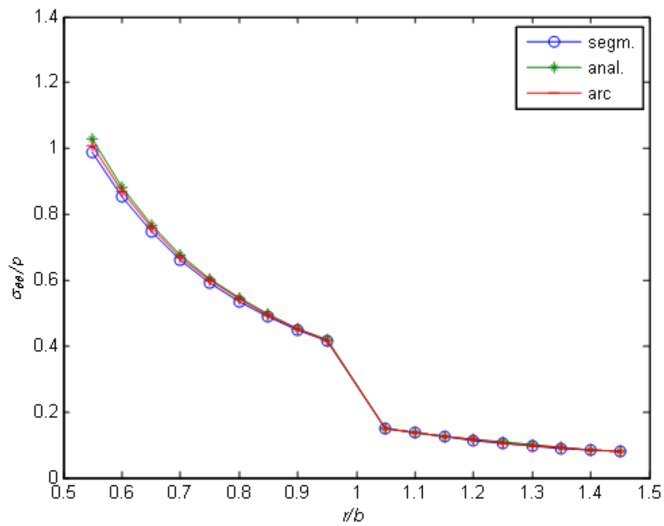

Figure 7: Shearing stress $\sigma_{\vartheta\vartheta}/p$ in the ring $0.5 \le \frac{r}{b} \le 1$ and infinite body $\frac{r}{b} \ge 1$ (n=180)

Table 3: Shearing stress $\sigma_{\vartheta\vartheta}/p$ in the ring $0.5 \le \frac{r}{b} \le 1$ and infinite body $\frac{r}{b} \ge 1$ (n=180)

No.	r/b	Exact solution	Approximate solution		Relative error, percent	
			In case of the division into segments	In case of the division into arcs	Segments	Arcs
1	0.5500	1.0294	0.9895	1.0095	3.8708	1.9317
2	0.6000	0.8827	0.8539	0.8683	3.2648	1.6331
3	0.6500	0.7686	0.7475	0.7580	2.7475	1.3757
4	0.7000	0.6780	0.6622	0.6701	2.3276	1.1659
5	0.7500	0.6049	0.5960	0.6005	1.4713	0.7274
6	0.8000	0.5451	0.5400	0.5445	0.9356	0.1101
7	08500	0.4956	0.4930	0.4940	0.5246	0.3228
8	0.9000	0.4540	0.4531	0.4536	0.1982	0.0881
9	0.9500	0.4189	0.4183	0.4185	0.1432	0.0955
10	1.0500	0.1512	0.1509	0.1510	0.1984	0.1323
11	1.1000	0.1377	0.1372	0.1375	0.3631	0.1452
12	1.1500	0.1260	0.1252	0.1256	0.6349	0.3175
13	1.2000	0.1157	0.1150	0.1154	0.6050	0.2593
14	1.2500	0.1067	0.1058	0.1062	0.8435	0.4686
15	1.3000	0.0986	0.0978	0.0983	0.3043	0.8114
16	1.3500	0.0914	0.0907	0.0912	0.7659	0.2188
17	1.4000	0.0850	0.0844	0.0847	0.7059	0.2353
18	1.4500	0.0793	0.0788	0.0791	0.6305	0.2522
Average					1.1408	0.5717

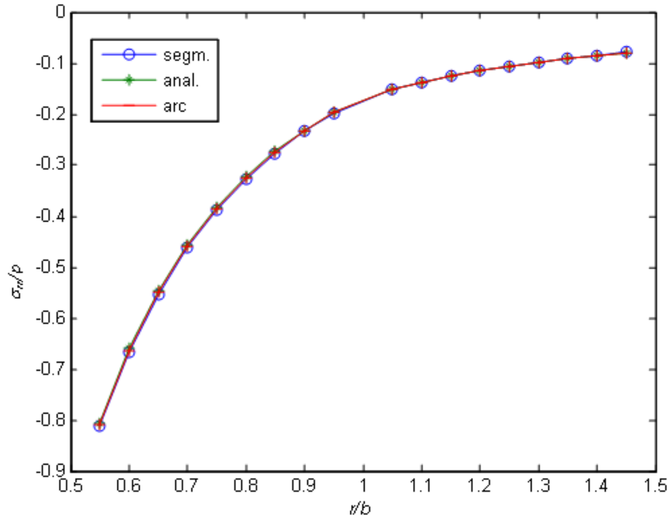

Figure 8: Normal stress σ_{rr}/p in the ring $0.5 \le \frac{r}{b} \le 1$ and infinite body $\frac{r}{b} \ge 1$ (n=180)

Table 4: Normal stress σ_{rr}/p in the ring $0.5 \le \frac{r}{b} \le 1$ and infinite body $\frac{r}{b} \ge 1$ (n=180)

No.	r/b	Exact solution	Approximate solution		Relative error, percent	
			In case of the division into segments	In case of the division into arcs	Segments	Arcs
1	0.5500	-0.8072	-0.8124	-0.8098	0.6519	0.3268
2	0.6000	-0.6605	-0.6675	-0.6640	1.0552	0.5308
3	0.6500	-0.5464	-0.5535	-0.5499	1.3036	0.6496
4	0.7000	-0.4558	-0.4622	-0.4590	1.4077	0.7060
5	0.7500	-0.3827	-0.3880	-0.3854	1.3926	0.7013
6	0.8000	-0.3229	-0.3271	-0.3250	1.2871	0.6452
7	08500	-0.2734	-0.2764	-0.2749	1.1177	0.5647
8	0.9000	-0.2318	-0.2339	-0.2329	0.9077	0.4640
9	0.9500	-0.1967	-0.1980	-0.1973	0.6761	0.3173
10	1.0500	-0.1512	-0.1522	-0.1517	0.6921	0.3496
11	1.1000	-0.1377	-0.1383	-0.1380	0.4447	0.1880
12	1.1500	-0.1260	-0.1263	-0.1262	0.2216	0.1397
13	1.2000	-0.1157	-0.1158	-0.1158	0.0207	0.0512
14	1.2500	-0.1067	-0.1065	-0.1066	0.1600	0.0625
15	1.3000	-0.0986	-0.0983	-0.0985	0.3226	0.1210
16	1.3500	-0.0914	-0.0910	-0.0912	0.4690	0.2728
17	1.4000	-0.0850	-0.08.45	-0.0848	0.6011	0.2752
18	1.4500	-0.0793	-0.0787	-0.0790	0.7203	0.3415
Average					0.7473	0.3726

b) *Double-layer circular ring*

Let us consider the boundary-contact problem shown in Fig. 9. This problem, too, is symmetrical to both coordinate axes and therefore, we will consider it for a one-fourth of a circular ring. So, the area to be considered is $\Omega = \Omega_1 + \Omega_2$,where

$$\Omega_1 = \left\{ a \le r \le b, \quad 0 \le \vartheta \le \frac{\pi}{2} \right\},$$

$$\Omega_2 = \left\{ b \le r \le c, \quad 0 \le \vartheta \le \frac{\pi}{2} \right\}.$$

The boundary conditions will be written down as follows:

$$r = a: \quad \sigma_{rr}^{(1)} = -p \sin^3(2\vartheta), \quad \sigma_{r\vartheta}^{(1)} = 0.$$

$$r = c: \quad \sigma_{rr}^{(2)} = 0, \quad \sigma_{r\vartheta}^{(0)} = 0.$$

The conditions of a rigid contact will be written down as follows:

$$r = b: \quad \sigma_{rr}^{(1)} = \sigma_{rr}^{(2)}, \quad \sigma_{r\vartheta}^{(1)} = \sigma_{r\vartheta}^{(2)}, \quad u_r^{(1)} = -u_r^{(2)}, \quad u_\vartheta^{(1)} = -u_\vartheta^{(2)}.$$

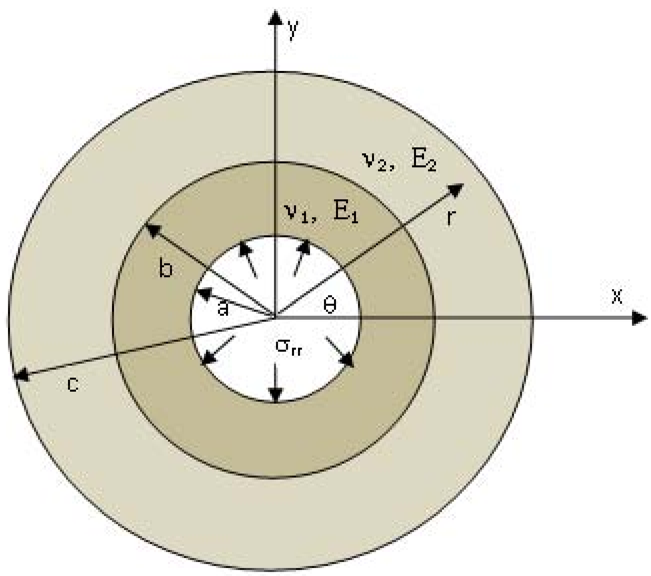

Figure 9: A double-layer circular ring

A numerical solution of the set problem is obtained by the boundary element method for the following data: $E_1 = 2*10^6 kg/cm^2$, , $\nu_1 = 0.3$, $\nu_2 = 0.46$., $a = 1cm$, $b = 1.5cm$, $c = 2cm$, $0 \le \vartheta \le \frac{\pi}{2}$ within the range $r = a$, $r = c$ boundary surfaces and both sides of the $r = b$ boundary surface is divided into n=50 elements each.

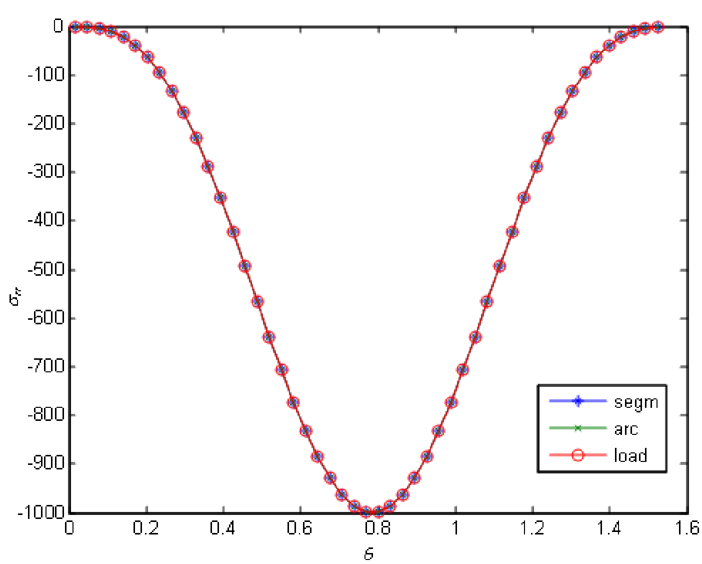

Figure 10: Stress σ_{rr} on the circle $r = a$

Table 5: Values of stress σ_{rr} on the circle $r=a$

ϑ	In case of the division into segments	In case of the division into arcs	Load	Relative error, percent	
				Segments	Arcs
0.0157	-3.09909783×10^{-2}	-3.09909790×10^{-2}	-3.09909789×10^{-2}	2.253×10^{-6}	2.674×10^{-7}
0.2042	-6.264072573×10	-6.26407257×10	-6.26407257×10	7.184×10^{-10}	2.235×10^{-10}
0.3927	$-3.53553391\times10^{+2}$	$-3.53553391\times10^{+2}$	$-3.53553390\times10^{+2}$	5.656×10^{-11}	1.414e-10
0.6440	$-8.85548231\times10^{+2}$	$-8.85548231\times10^{+2}$	$-8.85548230\times10^{+2}$	1.128×10^{-11}	6.776×10^{-11}
0.7697	$-9.98520411\times10^{+2}$	$-9.98520411\times10^{+2}$	$-9.98520411\times10^{+2}$	1.001×10^{-11}	4.007×10^{-11}
0.8953	$-9.29476324\times10^{+2}$	$-9.29476324\times10^{+2}$	$-9.29476325\times10^{+2}$	1.075×10^{-11}	3.227×10^{-11}
1.1467	$-4.22062458\times10^{+2}$	$-4.22062458\times10^{+2}$	$-4.22062458\times10^{+2}$	2.369×10^{-11}	4.738×10^{-11}
1.3352	-9.35707896×10	-9.35707897×10	-9.35707896×10	7.481×10^{-11}	1.175×10^{-10}
Average				2.818e-07	3.351e-08

c) Discussion

The error of the numerical solutions of the problem considered in paragraph 7.1 obtained by using PCSBEM and BEM, in particular: a) shearing stress $\sigma_{\vartheta\vartheta}/p$ (See Fig. 5, Table 1, when n=90 and Fig.7, Table 3, when n=180) and b) normal stress σ_{rr}/p in ring $0.5\le\frac{r}{b}\le1$ and infinite body $\frac{r}{b}\ge1$ (See Fig. 6, Table 2, when n=90 and Fig.8, Table 4, when n=180), is almost twice as less in terms of percents. It should be noted that in case of dividing the boundary into very small elements, e.g., when n=180, the error is more, as the arithmetic operations with very small numbers results in additional errors (counter error).

Paragraph 7.2 considers the boundary-contact problem for a double-layer circular ring with a normal load of a special kind given on its internal boundary ($\sigma_{rr}^{(1)}=-p\sin^3(2\vartheta)$), while the external boundary is free from loads. Here, the numerical values of σ_{rr} stress on the internal circle by using PCSBEM and BEM and numerical values of the special load are obtained. All three of them are given in Fig. 10 and Table 5, where one can see that they coincide with one another quite exactly.

VIII. Conclusion

The article develops BEM, in particular, the fictitious load method in the polar coordinate system (PCSBEM) to solve the boundary value and boundary-contact problems of the theory of elasticity for the areas limited by the coordinate axes of a polar coordinate system. The bodies relevant to such areas are quite frequent in practice, e.g., in building the underground structures (tunnels), in mechanical engineering, etc. Consequently, the above-described method (PCSBEM) is one of the means to obtain the adjusted solutions of the problems of computational mechanics, as the boundary of the considered area is divided not into small segments, like in case of a standard boundary element method (BEM), but into small arcs. In this case, the boundary of the considered area can be described more accurately, and consequently, the solution to the problem will be more accurate. To illustrate this case, two test boundary-contact problems are solved by using standard BEM and PCSBEM. The obtained numerical results given as tables and graphs are analyzed in paragraph 7.3.

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