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New Formulas for the Mutual Inductance and the Magnetic Force of the System: Thin Disk Coil (Pancake) with Inverse Radial Current Density and Thin Wall Solenoid with Constant Azimuthal Current Density

By Slobodan Babic & Cevdet Akyel

Abstract- This paper deals with two coaxial circular coils (thin disk coil and thin wall solenoid) for which we calculated the electromagnetic quantities such as the mutual inductance and the magnetic force. The disk coil (pancake) is with the nonlinear inverse radial current and the wall solenoid with the constant current in the azimuthal direction. The circular coils with the nonlinear inverse radial current are well known as the Bitter coils, and the circular coils with the azimuthal current are well known as the ordinary coils. Also, the coils with the azimuthal current can serve as the superconducting coils. These calculations give the semi-analytical and the analytical expressions respectively for these electromagnetic quantities. Also, we presented the improved filament method as the comparative method.

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Slobodan Babic α & Cevdet Akyel σ

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I. Introduction

The computation of the electromagnetic quantities (magnetic field, self-inductance, mutual inductance, magnetic force, etc.) for the conventional circular coaxial coils with the constant azimuthal current density has been presented in many papers, books, monographs and studies [1-19]. The analytical, the semi-analytical and the numerical methods have been used to calculate these electromagnetic quantities. These calculations are used in many electromagnetic applications (tubular linear motors, magnetically controllable devices and sensors, current reactors, cochlear implants, defibrillators, instrumented orthopedic implants, in magnetic resonance imaging (MRI) systems, superconducting coils, and tokamaks, etc.).

Also, there are the nonconventional circular coils with the nonlinear inverse radial density current which are used in many technical applications such as the superconducting coils, the electromagnets for the superconducting coils, the electromagnets for the production of the extremely powerful magnetic fields (Bitter coils) and the homopolar motors [20-36]. The calculation of the magnetic force and the mutual inductance for these coils is essential for the design of electromagnetic inductors. In this paper, we calculated these electromagnetic quantities for the coil's combination, the disk coil (pancake) with the nonlinear inverse radial current density (Bitter disk coil) and the wall solenoid with the constant azimuthal current density (superconducting wall solenoid). All expressions are obtained in the semi-analytical form (mutual inductance) and the closed form (magnetic force). Also, all singular case has been solved and given in the closed form. The results of these calculations are expressed over the elliptic integrals of the first kind and the Heuman's Lambda function and one simple friendly integral whose kernel function is the continuous function in all interval of the integration. We used the Gaussian numerical integration, [37-38]. The improved modified filament method for the presented configuration is given as the comparative method. We use the Matlab implementation to calculate the mutual inductance and the magnetic force by two independent methods.

II. Basic Expressions

The Bitter disk coil and the wall solenoid in the air are with the inverse radial current density and the uniform current density respectively [29-30], (See Fig. 1) as follow:

$$J_1 = \frac{N_1 I_1}{\ln \frac{R_2}{R_1}} \frac{1}{r_I} \tag{1}$$

$$J_2 = \frac{N_2 I_2}{(z_2 - z_1)} \tag{2}$$

Author α: Independent Researcher, Montréal, Berlioz 101, H3E 1N2 Canada. e-mail: slobobob@yahoo.com
Author σ: Département de Génie Électrique, École Polytechnique, C.P. 6079 Succ. Centre Ville, QC H3C 3A7, Montréal, Canada.
e-mail: cevdet.akyel@polymtl.ca

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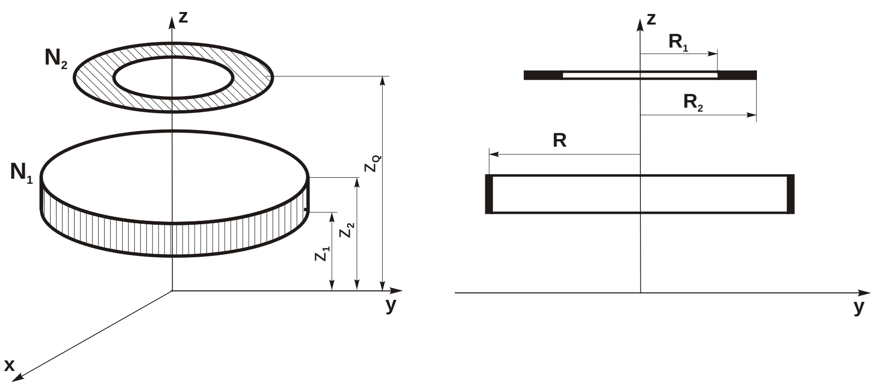

Figure 1: Bitter disk coil and thin solenoidthe mutual inductance and magnetic force between these coils, are respectively [29 30],

$$M=\frac{\mu_0 N_1 N_2 R}{(z_2-z_1)\ln\frac{R_2}{R_1}}\int_0^{\pi}\int_{R_1}^{R_2}\int_{z_1}^{z_2}\frac{\cos\theta dr_I dz d\theta}{r} \quad (3)$$

$$F=-\frac{\mu_0 N_1 N_2 I_1 I_2 R}{(z_2-z_1)\ln\frac{R_2}{R_1}}\int_0^{\pi}\int_{R_1}^{R_2}\int_{z_1}^{z_2}\frac{(z_Q-z_I)\cos\theta\, dr_I dz d\theta}{r^3} \quad (4)$$

where

$$r=\sqrt{(z_Q-z)^2+r_I^2+R^2-2r_I R\cos\theta}$$

Both configurations are in the air or a non-magnetic and non-conducting environment. We obtain the integral form to calculate these two physical quantities.

III. Calculation Method

After four analytical integration *M* and *F*are respectively:

$$M=\frac{\mu_0 N_1 N_2 R^2}{(z_2-z_1)\ln\frac{R_2}{R_1}}\sum_{n=1}^{n=4}(-1)^{n-1}T_n \quad (5)$$

$$F=\frac{\mu_0 N_1 N_2 I_1 I_2 R}{2(z_2-z_1)\ln\frac{R_2}{R_1}}\sum_{n=1}^{n=4}(-1)^{n-1}S_n \quad (6)$$

where

$$\rho_1=\rho_4=R_2,\ \rho_2=\rho_3=R_1, t_1=t_2=z_Q-z_1, t_3=t_4=z_Q-z_2$$

$$l_n=\frac{\rho_n}{R}, b_n=\frac{t_n}{R}, n=1,2,3,4$$

$$T_n=I_{0n}+\frac{\pi}{8}\mathrm{sign}(b_n)\mathrm{sign}(l_n-1)(l_n^2-3)[1-\Lambda_0(\varepsilon_n,k_n)]-\frac{\pi}{4}\mathrm{sign}(b_n)(b_n^2-1)V_n+$$

$$\frac{3k_n b_n}{8\sqrt{l_n}}[(1+l_n)^2+b_n^2]E(k_n)+\frac{k_n b_n}{8\sqrt{l_n}}[b_n^2-2-4l_n^2+\frac{(l_n-1)(l_n^2-3)}{l_n+1}-\frac{4(b_n^2-1)\sqrt{1+b_n^2}}{\sqrt{1+b_n^2}+1}]K(k_n)$$

$$S_n=\frac{k_n}{\sqrt{l_n}}[2\sqrt{1+b_n^2}-l_n^2-1-b_n^2]K(k_n)+\frac{k_n}{\sqrt{l_n}}[(1+l_n)^2+b_n^2]E(k_n)-\pi|b_n|V_n$$

$$V_n=1-\Lambda_0(\theta_{1n},k_n)+\mathrm{sgn}(\sqrt{R^2+b_n^2}-\rho_n)[1-\Lambda_0(\theta_{2n},k_n)]$$

$$k_n^2 = \frac{4l_n}{(1+l_n)^2 + b_n^2}, \quad h_n = \frac{4l_n}{(1+l_n)^2}, \quad m_n = \frac{2}{\sqrt{1+b_n^2}+1} \le 1$$

$$\theta_{1n} = \arcsin\frac{|b_n|}{\sqrt{1+b_n^2}+1}, \quad \theta_{2n} = \arcsin\sqrt{\frac{1-m_n}{1-k_n^2}}, \; k_n^2 \le m_n, \quad \varepsilon_n = \arcsin\sqrt{\frac{1-h_n}{1-k_n^2}}, \; k_n^2 \le h_n$$

$$I_{0n} = \int_0^{\pi/2} \sinh^{-1}\frac{b_n}{\sqrt{1+l_n^2+2l_n\cos 2\beta}}\, d\beta$$

From general cases (5) and (6) it is possible to obtain the special and singular cases.

The expression T_n is in a semi-analytical form where we need to solve the simple integral I_{0n} numerically by using the Gaussian integration for example.

The expression S_n is in the closed form.

Singular Cases

Singular cases are in the analytical form (5) and (6) respectively:

If $b_n = 0$ and $k_n^2 \neq 1$ or $b_n = 0$ and $k_n^2 = 0$.

If $b_n = 0$ and $k_n^2 \neq 1$.

$$S_n = \frac{k_n}{\sqrt{l_n}}[1-l_n^2]K(k_n) + \frac{k_n}{\sqrt{l_n}}(1+l_n)^2 E(k_n) \tag{7}$$

$$k_n^2 = \frac{4l_n}{(1+l_n)^2} = h_n$$

If $b_n = 0$ and $k_n^2 = 1$.

All expressions in (5), (6), (7), (8) and (9) are the complete elliptical integrals K, E and Λ_0, Heuman's Lambda function [37-38].

IV. Modified Filament Method

In this paper, we give the modified formulas for the mutual inductance and the magnetic force between two Bitter thick coils (See Fig. 2) using the filament method. Applying some modification in the mutual inductance calculation [30], we deduced the mutual inductance and the magnetic force between the Bitter disk and the wall solenoid as follows:

$$M = \frac{N_1 N_2 (R_2 - R_1)\sum_{g=-K}^{g=K}\sum_{l=-n}^{l=n}\frac{M(g,l)}{r_{II}(l)}}{(2K+1)(2n+1)\ln\frac{R_2}{R_1}} \tag{8}$$

$$F = \frac{N_1 N_2 I_1 I_2 (R_2 - R_1)\sum_{g=-K}^{g=K}\sum_{l=-n}^{l=n}\frac{F(g,l)}{r_{II}(l)}}{(2K+1)(2n+1)\ln\frac{R_2}{R_1}} \tag{9}$$

where

$$M(g,l) = \frac{2\mu_0\sqrt{R r_{II}(l)}}{k(g,l)}\left[\left(1-\frac{k^2(g,l)}{2}\right)K(k(g,l)) - E(k(g,l))\right]$$

$$F(g,l) = -\frac{\mu_0 I_1 I_2 z(g) k(g,l)}{4\sqrt{R r_{II}(l)}}\left[\frac{2-k^2(g,l)}{1-k^2(g,l)}E(k(g,l)) - 2K(k(g,l))\right]$$

$$r_{II}(l) = R_{II} + \frac{h_{II}}{2n+1}l \quad (l = -n, \ldots, 0, \ldots, n) \qquad z(g) = c - \frac{a}{2K+1}g, \quad g = -K, \ldots, 0, \ldots, K$$

$$R_{II} = \frac{R_3 + R_4}{2}, \quad h_{II} = R_4 - R_3 \qquad k^2(g,l) = \frac{4R r_{II}(l)}{(R + r_{II}(l))^2 + z(g)^2}$$

$$R_{11} = R_I = R, \quad h_{II} = R_4 - R_3$$

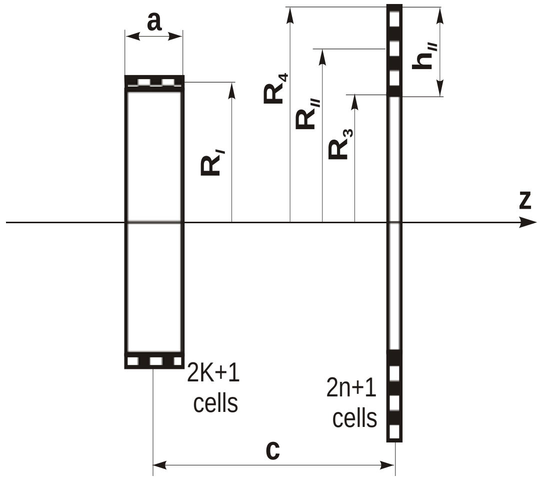

Figure 2: The Bitter disk coil and the thin solenoid (filament method)

Examples

To validate the new approach we present some examples, which cover either the regular or the singular cases. In these examples, all coils are with the unit currents. Also, we define the coil dimensions. For the comparative filament method, the number of subdivisions for each coil is also given. Our goal is to verify the accuracy of this method, so that we will fix the number of subdivisions ($K = n = 3000$) in the following examples without taking into consideration the computational time in the calculations. The number of turn in each coil is 100.

a) *Example1.*

Wall solenoid: $R = 2$ m, $z_1 = 0$ m, $z_2 = 1$ m.

Disk coil: $R_2 = 3$ m, $R_4 = 4$m, $z_Q = 2$ m.

From (5) and (6) we obtain:

M = 17.661179mH

F = 7.4710846mN

From (10) and (11) we obtain:

M = 17.661179mH

F = 7.4710845mN

b) *Example2.*

Wall solenoid: $R = 2$ m, m, $z_1 = 0$ m, $z_2 = 1$ m.

Disk coil: $R_2 = 3$ m, $R_4 = 4$m, $z_Q = 0.5$ m.

From (5) and (6) we obtain:

M = 26.158014mH

F = 0 N

From (10) and (11) we obtain:

M = 26.158014mH

F = 0N

c) *Example3.*

Wall solenoid: $R = 3$ m, $z_1 = 0$ m, $z_2 = 1$ m.

Disk coil: $R_2 = 3$ m, $R_4 = 4$m, $z_Q = 2$ m.

From (5) and (6) we obtain:

M = 36.827754mH

F = 20.338671mN

From (10) and (11) we obtain:

M = 36.827754mH

F = 20.338671mN

d) *Example4.*

Wall solenoid: $R = 3$ m, $z_1 = 0$ m, $z_2 = 1$ m.

Disk coil: $R_2 = 3$ m, $R_4 = 4$m, $z_Q = -2$ m.

From (5) and (6) we obtain:

M = 22.050066mH

F = -10.576130mN

From (10) and (11) we obtain:

M = 22.050066mH

F = -10.576130mN

e) *Example5.*

Wall solenoid: $R = 3$ m, m, $z_1 = 0$ m, $z_2 = 1$ m.

Disk coil: $R_2 = 3$ m, $R_4 = 4$m, $z_Q = 1$ m.

This case is the singular case.

From (5) and (6) we obtain:

M = 67.6203121mH

F = 45.309445mN

From (10) and (11) we obtain:

M = 67.620348mH

F = 45.309130mN

f) *Example6.*

Wall solenoid: $R = 4$ m, m, $z_1 = 0$ m, $z_2 = 1$ m.

Disk coil: $R_2 = 3$ m, $R_4 = 4$m, $z_Q = 1$ m.

This case is the singular case.

From (5) and (6) we obtain:

M = 83.323296mH

F = 49.855888mN

From (10) and (11) we obtain:

M =83.323370mH

F = 49.855568mN

g) *Example7.*

Wall solenoid: $R = 4$ m, m, $z_1 = 0$ m, $z_2 = 1$ m.

Disk coil: $R_2 = 3$ m, $R_4 = 4$m, $z_Q = 0$ m.

This case is the singular case.

From (5) and (6) we obtain:

M = 83.323296mH

F = -49.855888 NmN

From (10) and (11) we obtain:

M = 83.323370mH

F = -49.855568mN

h) Example8.

Wall solenoid: R= 4 m, m, z_1 = 0 m, z_2= 1 m.

Disk coil: R_2= 3 m, R_4= 5m, z_Q= 1 m.

This case is the singular case.

From (5) and (6) we obtain:

M =91.598922mH

F = 54.254023mN

From (10) and (11) we obtain:

M = 91.598976mH

F = 54.258225mN

i) Example9.

Wall solenoid: R= 3 m, m, z_1 = 0 m, z_2= 1 m, N_1=100.

Disk coil: R_2= 3 m, R_4= 5m, z_Q= 0.6 m, N_2=100.

This case is the singular case.

From (5) and (6) we obtain:

M = 65.436644mH

F = 5.7050033mN

From (10) and (11) we obtain:

M = 65.436923mH

F = 5.7050600 mN

By previous examples, we confirmed that all calculated results by two different methods are in an excellent agreement. The bold digits are significant with the same accuracy in both calculations.

V. Conclusion

The new accurate expressions for calculating two electromagnetic quantities such as the mutual inductance and the magnetic force are presented in this work. All expressions are in the semi-analytical and the closed form. We give the improved filament method as the comparative method. Results obtained by two different methods agree at least in five significant figures.

Nomenclature

I_1: Current imposed in the disk (pancake) in (m)

I_2: Current imposed in the superconducting solenoid in (m)

N_1: number of turns of the pancake

N_2: number of turns of the solenoid

$R_1$and R_2: Inner and outer radius of the pancake in (m)

R: The radius of the solenoid in (m)

z_Q: Axial position to the pancake in (m)

z_1: Axial position to the bottom of the wall solenoid in (m)

z_2: Axial position to the top of the wall solenoid in (m)

M: Mutual inductance in (H)

F: Magnetic force between coils in (N)

J_1 and J_2: Current densities at the pancake and the wall solenoid respectively in A/m

r_i: Radial positions along the pancake

r, θ, z: Cylindrical coordinates

$\mu_0 = 4\pi\times10^{-7}$ H/m: Magnetic permeability of free space

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