The Kinematics of a Puma Robot using Dual Quaternions

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Kinematics analysis studies the relative motions, such as, first of all, the displacement in space of the end effector of a given robot, and thus its velocity and acceleration, associated with the links of the given robot that is usually designed so that it can position its end-effector with a three degree-of-freedom of translation and three degree-of-freedom of orientation within its workspace.

First of all, examples of basic solid movements such as rotations, translations, their combinations and general screw motions are studied using both (4x4) rigid body transformations and dual quaternions so that the reader could compare and note the similarity of the results obtained using one or the other method.

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First of all, examples of basic solid movements such as rotations, translations, their combinations and general screw motions are studied using both (4x4) rigid body transformations and dual quaternions so that the reader could compare and note the similarity of the results obtained using one or the other method. Both dual quaternions technique as well as its counterpart the classical 'Denavit and Hartenberg parameters method' are finally applied to the first three degree of freedom of a Puma 560 robot. Finally, we and the reader, can observe that the two methods confirm exactly one another by giving us the same results for the considered application, while noting that the fastest, simplest more straightforward and easiest to apply method, is undoubtedly the one using dual quaternions. As a result this chapter may as well act as a beginners guide to the practicality of using dual-quaternions to represent the rotations and translations in character-based hierarchies.

We must emphasize the fact that the use of both Matlab software and quaternions and / or dual quaternions in the processing of 3D rotations and/or screw movements is and will always be the most efficient, fast and accurate first choice. Dual quaternion direct kinematics method could be generalised, in the future, to all kind of spatial and/ or industrial robots as well as to articulated and multibody systems.

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I. Introduction

Many research students have a great deal of trouble understanding essentially what quaternions are [1], [2], [3] and how they can represent rotation. So when the subject of dual-quaternions is presented, it is usually not welcomed with open arms. Dual-quaternions are a break from the norm (i.e., matrices) which we hope to entice the reader into supporting willingly to represent their rigid transforms.

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The reader should walk away from this analysis with a clear understanding of what dual-quaternions are and how they can be used [4]. First we begin with a short recent and related work that emphasises the power of dual-quaternions:

The dual-quaternion has been around since 1882 [5],[6],[7] but has gained less attention compared to quaternions alone; while the most recent work which has taken hold and has demonstrated the practicality of dual-quaternions, both in robotics and computer graphics can be resumed in: - Kavan [8] demonstrated the advantages of dual-quaternions in character skinning and blending. - Ivo [9] extended Kavan's work with dual-quaternions and q-tangents as an alternative method for representing rigid transforms instead of matrices, and gives evidence that the results can be faster with accumulated transformations of joints if the inferences per vertex are large enough. - Selig [10] address the key problem in computer games. - Vasilakis [11] discussed skeleton-based rigid-skinning for character animation. - Kuang [12] presented a strategy for creating real-time animation of clothed body movement.-Pham [13] solved linked chain inverse kinematic (IK) problems using Jacobian matrix in the dual-quaternion space. -Malte [14] used a mean of multiple computational (MMC) model with dual-quaternions to model bodies. - Ge [15] demonstrated dual-quaternions to be an efficient and practical method for interpolating three-dimensional motions. -Yang - Hsing [16] calculated the relative orientation using dual-quaternions. - Perez [17] formulated dynamic constraints for articulated robotic systems using dual-quaternions.- Further reading on the subject of dual numbers and derivatives is presented by Gino [18].

In the last three decades, the field of robotics has widened its range of applications, due to recent developments in the major domains of robotics like kinematics, dynamics and control, which leads to the sudden growth of robotic applications in areas such as manufacturing, medical surgeries, defense, space vehicles, under-water explorations etc.

To use robotic manipulators in real-life applications, the first step is to obtain the accurate kinematic model [19]. In this context, a lot of research has been carried out in the literature, which leads to the evolution of new modeling schemes along with the refinement of existing methodologies describing the kinematics of robotic manipulators.

Screw theory based solution methods have been widely used in many robotic applications. The elements of screw theory can be traced to the work of Chasles and Poinsot [20], [21], in the early 1800's and Whittaker [22]. Using the theorems of Chasles and Poinsot as a starting point, Robert S. Ball developed [23] a complete theory of screws which he published in 1900. Throughout the development of kinematics, numerous mathematic theories [24] and tools have been introduced and applied. The first pioneer effort for kinematic modeling of robotic manipulators was made by Denavit and Hartenberg in introducing a consistent and concise method to assign reference coordinate frames to serial manipulators, allowing the (4×4) homogeneous transformation matrices to be used (in 1955) [25], followed by Lie groups and Lie Algebra by J.M Selig and others, [26], [27], [28]) and quaternions and dual quaternions introduced by Yang and Freudenstein (1964) [29], see also Bottema and Roth (1979) [30] and McCarthy (1990) [31].The original D–H parameter method has many counterparts: Distal variant, proximal variant, …to name but a few. There even exist different options for these counterparts.

In this method, four parameters, popularly known as D–H parameters, are defined to provide the geometric description to serial mechanisms. Out of the four, two are known as link parameters, which describe the relative location of two attached axes in space. These link (See appendix 10,3,1.) parameters are: The link length (a_i) and the link twist (α_i).

The remaining two parameters are described as joint parameters, which describe the connection of any link to its neighboring link. These are the joint offset (d_i) and the joint angle (θ_i).

Modeling the movement of the rigid body by the theory of the helicoidal axis: a combination of an amount of rotation about and an amount of translation along a certain axis, hence the term helicoidal axis is used in various fields such as computer vision and biomechanics. The application of this theory in the field of robotics is taking more and more space. We can consider the motion of a joint segment as a series of finite displacements. In this case the movement is characterized by an angle of rotation about and an amount of translation along an axis defined in space by its position and its orientation. This axis is referred to as the finite helicoidal axis (FHA), because of the discretization of the movement into a series of displacements. On the other hand and by taking the continuity of the movement into account, this movement will be characterized by a rotational speed (angular velocity) about and translation speed along an axis defined by the instantaneous position and orientation in space. One speaks in this case of an instantaneous helicoidal axis (IHA).The application of the helicoidal theory with its two versions (FHA and IHA) is used to describe and understand the joint movement, and to study in biomechanics, for example, the different positioning techniques of prothèses. Thus there are several methods to estimate the helicoidal axis from a set of points representing a rigid body.

Any displacement of a rigid body is a helicoidal motion which may be decomposed into an angular rotational movement about and a linear translational movement along a certain axis in 3D space. The methods differ in the way of mathematically representing these two movements. These movements can be expressed using rotation matrices and translation vectors, homogeneous matrices, unit quaternions, dual quaternions, ...

The two representations; using (3x3) matrices or (4x4) homogeneous matrices and dual quaternions will be simultaneously used for all and each examples or applications studied so that comparisons for each case could be done.

II. DUAL QUATERNIONS

a) « Product type » dual quaternions

The dual quaternions have two forms thus two readings which are complementary and simultaneous: The first is the << product type >> description:

$$\hat{T}_G = \left\{ T_R + \varepsilon \frac{T_T.T_R}{2} \right\} \text{ With: } T_R = \left\{ \cos \frac{\psi}{2}, \text{n. } \sin \frac{\psi}{2} \right\} = \left\{ \cos \frac{\psi}{2}, \ \sin \frac{\psi}{2}.n_x, \sin \frac{\psi}{2}.n_y, \sin \frac{\psi}{2}.n_z \right\}$$

and $\quad T_T = (0, \{T_x, T_y, T_z\} = (0, \{T\})$

Then, the transformation is:

$$\hat{T}_G = \left\{ T_R + \varepsilon \frac{T_T.T_R}{2} \right\} = \left\{ \cos \frac{\psi}{2}, \text{n} \sin \frac{\psi}{2} \right\} + \ \varepsilon \left\{ 0, \ \frac{T}{2} \right\} . \left\{ \cos \frac{\psi}{2}, \text{n} \sin \frac{\psi}{2} \right\}$$

$$\hat{T}_G = \left\{ \cos \frac{\psi}{2}, \text{n} \sin \frac{\psi}{2} \right\} + \varepsilon \left\{ -\left(\frac{T.n}{2} \right) \sin \frac{\psi}{2}, \left\{ \frac{TXn}{2} \ \sin \frac{\psi}{2} + \frac{T}{2} \cos \frac{\psi}{2} \right\} \right\} \qquad << \text{product type} >> \quad (1)$$

b) « Dual type » dual quaternions

Indeed a general transformation, screw type, can be also described using dual angles and dual vectors and have therefore the following form << Dual type >>:

$$\hat{T}=\left\{\cos\frac{\hat{\theta}}{2},\ \sin\frac{\hat{\theta}}{2}\hat{w}\right\}=\left\{\cos\frac{\theta}{2},\sin\frac{\theta}{2}n\right\}+\varepsilon\left\{-\frac{d}{2}\sin\frac{\theta}{2},\left\{m\sin\frac{\theta}{2}+\frac{d}{2}n\cos\frac{\theta}{2}\right\}\right\}\ <<dual\ type>> \quad(2)$$

It is defined by the dual angle $\hat{\theta}$ and the dual vector $\hat{w}$ the rotation being represented by the angle θ around the axis n = (n_x, n_y, n_z) of norm 1, and a translation d along the same vector n.

The vector m = (m_x, m_y, m_z) is the moment of the vector n about the origin of reference (O, x, y, z); it is named the moment of the axis n, with: $\hat{\theta} = \theta + \varepsilon\, d$ with d being the amplitude of the translation along the dual vector $\hat{w}$ = n +ε m with m = p x n (the green vector see figure 1) that defines the vector according to

Plücker coordinates, $p,$ (the blue vecor), being the vector that gives the position of n ,(the red vector), using the vector OO$_1$ (see figure (1)).

The parameters of the transformation, the angle θ, the axis of rotation n, the magnitude of the translation d and the moment m are the four characteristics of all, any and every 3D rigid body transformation (4x4) matrix, a screw motion or a helicoidal movement of any kind (or type).
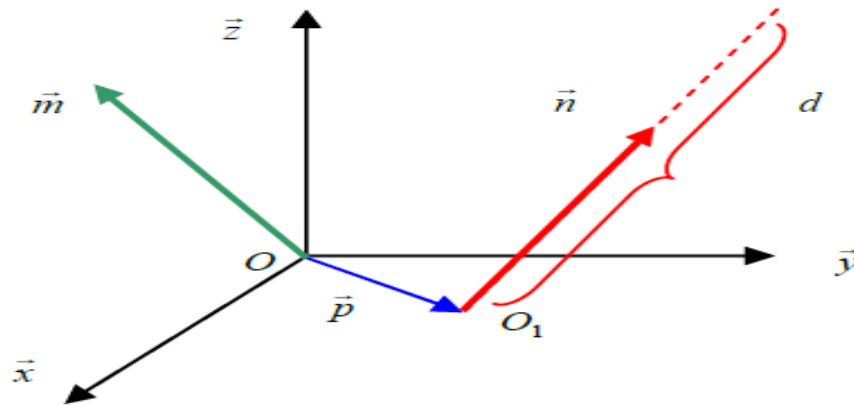

Figure 1: Helicoidal or screw motion

Note that this form resembles that used for classic quaternions; using the dual angle and the dual unitary vector instead of the classical ones.

And as a matter of fact: The screw displacement is the dual angle $\hat{\theta} = \theta + \varepsilon\, d$, along the

screw axis defined by the dual vector $\hat{l}$ or $\hat{s}$ or in our case $\hat{w}$ = n +ε m; such that we will obtain (respecting the rules of derivation and multiplication of dual numbers), dual vectors, quaternions and dual quaternions (see appendix 10,2. and eq (A15)):

$$\hat{T}=\left\{\cos\frac{\hat{\theta}}{2},\ \sin\frac{\hat{\theta}}{2}\hat{w}\right\}=\left[\cos\frac{\theta}{2}-\varepsilon\frac{d}{2}\sin\frac{\theta}{2},\ (\sin\frac{\theta}{2}+\varepsilon\frac{d}{2}\cos\frac{\theta}{2})\,(n+\varepsilon\, m\,)\right]=$$

$$\cos\frac{\theta}{2}-\varepsilon\frac{d}{2}\sin\frac{\theta}{2},\ n\sin\frac{\theta}{2}+\varepsilon\,(n\frac{d}{2}\cos\frac{\theta}{2}+\sin\frac{\theta}{2}m)=(\cos\frac{\theta}{2},n\sin\frac{\theta}{2})+\varepsilon(-\frac{d}{2}\sin\frac{\theta}{2},\sin\frac{\theta}{2}m+n\frac{d}{2}\cos\frac{\theta}{2})\quad(2)$$

The geometric interpretation of these quantities is related to the screw-type motion. The angle θ is the angle of rotation around n, the vector unit n represents the direction of the rotation axis. The element d is the translation or the displacement amplitude along the vector n, m being the vector moment of the vector axis n relative to the origin of the axes. The vector m is an unambiguous description of the position of an axis in space, in accordance with the properties of Plückér coordinates defining lines in space.

This form gives another interesting use: Whereas the classics quaternions can only represent rotations whose axes pass through the origin O of the coordinate system (O, x, y, z), the dual quaternions can represent rotations about arbitrary axes in space,

translations as well as any combination of both these two basic spatial motions.

These two forms << *product type* >> eq (1) or << *dual type* >> eq (2) represent the same motion that describe the same movement 'the screw motion':

III. EXAMPLE 1: ROTATIONS REPRESENTED BY QUATERNIONS

Let's apply two successive rotations to a rigid body: the first one of amplitude $\theta_1 = \frac{\pi}{2}$ around the axis Ox followed by a second rotation of the same amplitude $\theta_2 = \frac{\pi}{2}$ around the Oy axis:

Using quaternions the first rotation will be written; since $\frac{\theta_1}{2} = \frac{\pi}{4}$ then $\cos\frac{\theta_1}{2} = \sin\frac{\theta_1}{2} = \frac{\sqrt{2}}{2}$

$$q_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0) \; ; \; \text{having} \; \frac{\theta_1}{2} = \frac{\pi}{4} \; \text{then} \; \cos\frac{\theta_1}{2} = \sin\frac{\theta_1}{2} = \frac{\sqrt{2}}{2}$$

The second rotation will have the form: $q_2 = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0)$

The final composition of the two movements will be given by the quaternion q such that:

$$q = q_2.\,q_1 = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0).(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$$

Using quaternion's definition (A5) and quaternions properties:

$$q = ((\frac{1}{2}, \frac{\sqrt{3}}{2}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})) \; \text{or} \; ((\frac{1}{2}, -\frac{\sqrt{3}}{2}(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}))$$

It is then easy to extract both the amplitude and the resulting axis of the rotation from the result q:

$$\cos\frac{\theta}{2} = \frac{1}{2} \; \text{and} \; \sin\frac{\theta}{2} = \frac{\sqrt{3}}{2}; \; \text{wich implies the first solution} \; \theta = +120\,°, \text{around the unitary axis} \; (n) = \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\-1\end{Bmatrix}$$

or

$$\cos\frac{\theta}{2} = \frac{1}{2} \; \text{and} \; \sin\frac{\theta}{2} = -\frac{\sqrt{3}}{2}; \; \text{wich implies a second solution} \; \theta = -120\,°, \text{around the unit axis} \; (-n) = \frac{1}{\sqrt{3}}\begin{Bmatrix}-1\\-1\\1\end{Bmatrix}$$

In fact the two solutions represent the same and similar solution since for any q we have $q\,(\theta, n) = q\,(-\theta, -n)$.

Using our classical (3x3) rigid transformations we get:

$$R_{21} = R_2.R_1 = \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\-1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\0 & 0 & -1\\-1 & 0 & 0\end{pmatrix}$$

Here it is very important to note that unlike the quaternion method we cannot extract the needed results easily and straightforwardly but we must follow a long and sometimes complicated process (determinant, trace, antisimmetry, angle and axis of rotations signs,axis/angle (or conversions to Olinde Rodrigues (Axis, Angle) parameters) ...

Whichever used technique we will find: A rotation of $\theta = \frac{2\pi}{3} = 120\,°$ around the unit axis $n = \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\-1\end{Bmatrix}$

To show the anti commutativity of the product let's do the inverse and start by the second rotation instead:

$$q_i = q_1.\,q_2 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0).(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = ((\frac{1}{2}, \frac{\sqrt{3}}{2}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})) = q_i = ((\frac{1}{2}, -\frac{\sqrt{3}}{2}(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}))$$

and that will imply $\theta_i = 120\,°$ around the axis $n = \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$, or $\theta_i = -120\,°$ around the axis $(-n) = \frac{1}{\sqrt{3}}\begin{Bmatrix}-1\\-1\\-1\end{Bmatrix}$;

Which of course will imply that: $q_1.\,q_2 \neq q_2.\,q_1$

Using matrices : $R_i = R_1 R_2 = \begin{pmatrix}1 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\-1 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\1 & 0 & 0\\0 & 1 & 0\end{pmatrix} \neq R_{ii} = R_2 R_1$ which implies:

A rotation of $\theta = \frac{2\pi}{3} = 120°$ around the unit axis $n = \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$ equivalent to a rotation of $\theta = -\frac{2\pi}{3} = -120\,°$ around the

unit axis $n = -\frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$

Using MATLAB (See Appendix 10,1.) we can calculate easily both the two quaternions multiplications: $q = $ n1 = q2.q1 and $q_i = $ n2 = q1.q2 and the two equivalent product of matrices $R_{21} = R_2R_1$ and $R_i = R_1 R_2$.

IV. Important Notes: What about Translations?

We must recall that rotations act on translations, the reverse being not true; in fact when multiplying by blocks:

For a rotation followed by a translation: $\begin{pmatrix} I & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$; the rotation is not affected by the translation.

While for a translation followed by a rotation: $\begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & Rt \\ 0 & 1 \end{pmatrix}$; the translation is affected by the rotation.

When translations are performed first we can thus assume that the translation vector of the resulting matrix product; Rt acts as the translation vector t of a rotation followed by a translation .Or more generally speaking considering two six degree of freedom general rigid body transformations T_1 followed by T_2 we will have:

$$T_2 . T_1 = \begin{pmatrix} R_2 & t_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & t_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_2 R_1 & R_2 t_1 + t_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$$

The translation vector t of the product of the two transformations is $\begin{Bmatrix} t \\ 1 \end{Bmatrix} = R_2 t_1 + t_2 = \begin{pmatrix} R_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{Bmatrix} t_1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} t_2 \\ 1 \end{Bmatrix}$

The same analysis as the last one could then be done whatever the order and the number of the successive transformations being performed over the rigid body: The final result of the products of all the undertaken rigid body transformations will be finally the helicoidal, the helical or the screw motion given by the (4x4) matrix:

$$[T] = T_n \dots T_i \dots T_2 . T_1 = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \tag{3}$$

With T_i representing either a rotation, a translation, a rotation followed by a translation, a translation followed by a rotation or even simply a no movement (ie: the 4x4 identity matrix I).

V. Screw Motion

Any screw motion would be given by the following (4x4) matrix [T]:

$$\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R(\theta, n) & \frac{\theta p}{2\pi} n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & -u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R(\theta, n) & \frac{\theta p}{2\pi} n + (I - R(\theta, n)) u \\ 0 & 1 \end{pmatrix} = [T] \tag{3}$$

The middle matrix is a screw about a line through the origin; that is, a rotation of $\boldsymbol{\theta}$ radians around the axis n followed by a translation along n. The outer matrices conjugate the screw and serve to place the line at an arbitrary position in space. The parameter p is the pitch of the screw, it gives the distance advanced along the axis for every complete turn, exactly like the pitch on the thread of an ordinary nut or bolt. When the pitch is zero the screw is a pure rotation, positive pitches correspond to right hand threads and negative pitches to left handed threads.

To show that a general rigid motion is a screw motion, we must show how to put a general transformation into the form derived above. The unit vector in the direction of the line n is easy since it must be the eigenvector of the rotation matrix corresponding to the unit eigenvalue.(This fails if $R = I$, that is if the motion is a pure translation). The vector u is more difficult to find since it is the position vector of any point on the rotation axis. However we can uniquely specify u by requiring that it is normal to the rotation axis. So we impose the extra restriction that $n.u = 0$. So to put the

general matrix $\begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$ into the above form we must solve the following system of linear equations:

$$\frac{\theta p}{2\pi} n + (I - R) u = t$$

Now $n.Ru = n.u = 0$, since the rotation is about n. So we can dot the above equation with n to give: $0 = n.(t - \frac{\theta p}{2\pi})$ this enables us to find the pitch: $p = \frac{2\pi}{\theta} n.t$

All we need to do now is to solve the equation system: $(I - R)u = (t - (n.t) n)$;

This is possible even though $\det (I - R) = 0$, since the equations will be consistent.

This entire analysis established through this long paragraph concerning the helicoidal motion or rigid (4x4) transformation matrix [T] is contained in only one line enclosed in its counterpart dual quaternion $\hat{T}$ of the form:

$$\hat{T} = \left\{\cos\frac{\hat{\theta}}{2}, \sin\frac{\hat{\theta}}{2}\hat{w}\right\} = \hat{T}_n..\hat{T}_i..\hat{T}_2.\hat{T}_1 = \left\{\cos\frac{\theta}{2}, \sin\frac{\theta}{2}.n\right\} + \varepsilon\left\{-\frac{d}{2}\sin\frac{\theta}{2}, \left\{m\sin\frac{\theta}{2} + \frac{d}{2}.n\cos\frac{\theta}{2}\right\}\right\} \text{ or } \text{eq (2)} \equiv \text{eq (3)}$$

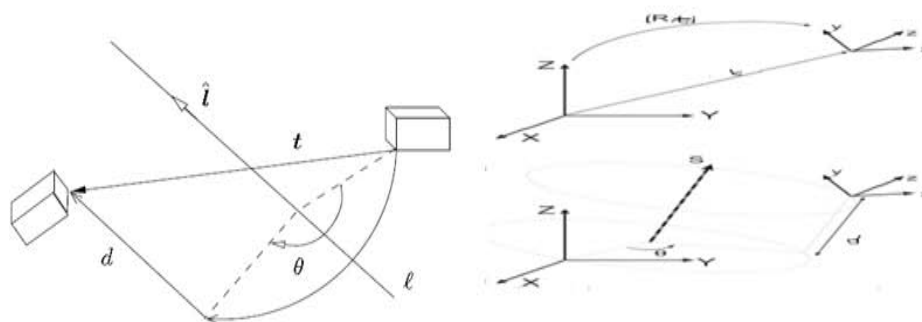These equations are best represented by the figures (2,1) or/and (2,2) :

Figure: (2, 1): A semi cubic solid performing simultaneously a rotation θ around the axis $\hat{l}$ and a displacement d along the same axis.

Figure: (2, 2): The same rigid (4D) transformation (R,t) represented by its screw axis S and displacement d.

VI. EXAMPLE 2: GENERAL MOVEMENT OR A SCREW MOTION

Let's apply two successive screw motions to a rigid body: the first one around the Oy axis of amplitude $\boldsymbol{\theta_1} = \frac{\pi}{2}$ and of pitch ($p = \frac{2\pi}{\theta}t = 4$) followed by a second one around the axis Ox and of the same amplitude $\boldsymbol{\theta_2} = \frac{\pi}{2}$ and same pitch p = 4 corresponding to a translation of 1 unit along any of the two chosen axes:

$$T_2.T_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

The rotation part of the product corresponds to that of the precedent example of successive rotations $R_i = R_1\,R_2$ with amplitude $\theta = \frac{2\pi}{3} = 120°$ around the unit axis $n = \frac{1}{\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$; its translation part being $t = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$

We can find its pitch $p = \frac{2\pi}{\theta}(n.t) = \frac{2\pi}{\frac{2\pi}{3}}\frac{1}{\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}.\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

The axis of rotation will keep its same original direction $n = \frac{1}{\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, it will go through a new centre C given by the shifting vector u which could be found by the linear equations system : $(I - R)\,u = t - \frac{\theta\,p}{2\pi}\,n$

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} - \frac{2\pi}{3.2\pi}\frac{6}{\sqrt{3}}\begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} - \frac{2}{\sqrt{3}}\begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{Bmatrix}$$

The vector translation T (or t) of the movement $\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$ is the sum of the two main perpendicular vectors $T_1 + T_2$ such as T_1 is to be chosen parallel to n while the rest T_2 is the translation vector part responsible for the shifting of the axis to its final position through the new center C as such we have:

$$T_1 = \begin{Bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{Bmatrix} \text{ and } T_2 = \begin{Bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{Bmatrix}; \quad T_1 \text{ being the translation part parallel to n while } T_2 \text{ being the perpendicular part with n.}$$

The solutions to the system of linear equations are: $u_x - u_z = \frac{1}{3}$; $-u_x + u_y = -\frac{2}{3}$; and $-u_y + u_z = \frac{1}{3}$ (5)

Choosing the centre C to belong to the plane (y-z); $u_x = 0$ or ($C_x = 0$) would imply the two coordinates representing the point C intersection of the shifted axis n with the (y-z) plane to be:

$$C_y = -\frac{2}{3} \quad \text{and} \quad C_z = -\frac{1}{3}.$$

For the (z-x) plane ; $u_y = 0$ or ($C_y = 0$) : $C_z = \frac{1}{3}$ and $C_x = \frac{2}{3}$.

And finally considering the (x-y) plane ; $u_z = 0$ or ($C_z = 0$): $C_x = \frac{1}{3}$ and $C_y = -\frac{1}{3}$

So that to confirm these results ; we can finally check the following conjugation matrices :

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \frac{-2}{3} \\ 0 & 1 & 0 & \frac{-1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & \frac{2}{3} \\ 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv (4) \text{ Or,}$$

$$\begin{pmatrix} 0 & 0 & 1 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & \frac{2}{3} \\ 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & \frac{-2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-1}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv (4)$$

Or finally; $\begin{pmatrix} 0 & 0 & 1 & \frac{1}{3} \\ 1 & 0 & 0 & \frac{-1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & \frac{2}{3} \\ 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & \frac{-1}{3} \\ 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv (4)$

Whenever necessary, Matlab was, throughout the chapter implemented, concerning all kinds of products or multiplication of quaternions or matrices.

VII. THE SAME GENERAL EXAMPLE USING DUAL QUATERNIONS

$$\hat{q} = q + \varepsilon q_\varepsilon = q_R + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k) \otimes q_R = R + \varepsilon \frac{TR}{2}$$

The two transformations T_1 and T_2 are basic centered helicoidal movements through the origin O of the axes, that can be written:

For the first movement around and along Oy: $\hat{q}_1 = q_1 + \frac{\varepsilon}{2} q_{\varepsilon 1} = \hat{q}_y = (c, 0, s, 0) + \frac{\varepsilon}{2}(-st_y, 0, ct_y, 0) =$

$(\cos\frac{\pi}{4}, 0, \sin\frac{\pi}{4}, 0) + \frac{\varepsilon}{2}(-\sin\frac{\pi}{4}.1, 0, \cos\frac{\pi}{4}.1, 0) = (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0) + \frac{\varepsilon}{2}(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0)$

followed by the second movement around and along Ox: $\hat{q}_2 = q_2 + \frac{\varepsilon}{2} q_{\varepsilon 2} = \hat{q}_z = (c, s, 0, 0) + \frac{\varepsilon}{2}(-st_z, ct_z, 0$

, 0) = ($\cos\frac{\pi}{4}$, $\sin\frac{\pi}{4}$, 0, 0) + $\frac{\varepsilon}{2}(-\sin\frac{\pi}{4}.1, \cos\frac{\pi}{4}.1$, 0, 0) = ($\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$, 0, 0) + $\frac{\varepsilon}{2}(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$, 0, 0)

The dual quaternion product of the two screw movements is:

$$\hat{q}_2.\hat{q}_1 = (q_2 + \frac{\varepsilon}{2} q_{\varepsilon 2}).(q_1 + \frac{\varepsilon}{2} q_{\varepsilon 1}) = q_2.q_1 + \frac{\varepsilon}{2}(q_2.q_{\varepsilon 1} + q_{\varepsilon 2}.q_1) =$$

$$[(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0) + \frac{\varepsilon}{2}(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 .0)].[(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0) + \frac{\varepsilon}{2}(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0)] =$$

$$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0).(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0) + \frac{\varepsilon}{2}[(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0).(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0) + (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 .0).(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0)] =$$

$$(\frac{1}{2},(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})) + \frac{\varepsilon}{2}[(-\frac{1}{2},(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2})) + (-\frac{1}{2},(\frac{1}{2}, -\frac{1}{2}.\frac{1}{2}))] =$$

$$(\frac{1}{2},\frac{\sqrt{3}}{2}(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})) + \frac{\varepsilon}{2}(-1,(0, 0, 1)) \tag{6}$$

Another way of doing it: We could get this same result starting from the (4x4) rigid transformation eq(4) matrix defined before: A rotation of amplitude $\theta = \frac{2\pi}{3} = 120°$ around the unit axis $n = \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$ followed by a translation $t = \begin{Bmatrix}1\\0\\1\end{Bmatrix}$ such that : $\hat{q} = q + \varepsilon q_\varepsilon = q_R + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k) \otimes q_R = R + \varepsilon\frac{TR}{2} =$

$$(\frac{1}{2},\frac{\sqrt{3}}{2}(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})) + \frac{\varepsilon}{2}[(0,1,0,1)(\frac{1}{2},(\frac{1}{2},\frac{1}{2},\frac{1}{2}))] =$$

$$(\frac{1}{2},\frac{\sqrt{3}}{2}(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})) + \frac{\varepsilon}{2}[(-1,(0,0,0)) + (0,(\frac{1}{2},0,\frac{1}{2})) + (0,(-\frac{1}{2},0,\frac{1}{2}))] =$$

$$(\frac{1}{2},\frac{\sqrt{3}}{2}(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})) + \frac{\varepsilon}{2}(-1,(0,0,1)) \qquad (6)$$

At this stage we know the complete integrality of informations concerning this movement thanks to our magic, rapid and powerful dual quaternion :The rotation part, as seen before, having amplitude $\theta = \frac{2\pi}{3} = 120°$ around the unit axis n; $n = \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$; the dual part will provide us gratefully with the translation along the axis of rotation; using eq (2): $\varepsilon\{-\frac{d}{2}\sin\frac{\theta}{2},\{m\sin\frac{\theta}{2}+\frac{d}{2}n\cos\frac{\theta}{2}\}\} = \frac{\varepsilon}{2}(-1,(0,0,1)) = \varepsilon(-\frac{1}{2},(0,0,\frac{1}{2}))$

We thus have the scalar part: $-\frac{d}{2}\sin\frac{\theta}{2} = -\frac{d}{2}\frac{\sqrt{3}}{2} = -\frac{1}{2}$ implying that $d = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ and pitch $p = 2\sqrt{3}$

We can also have the vector part: $\{m\sin\frac{\theta}{2}+\frac{d}{2}n\cos\frac{\theta}{2}\} = (0,0,\frac{1}{2})$ which implies:

$$m_x\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{3}\frac{1}{\sqrt{3}}\frac{1}{2} = m_x\frac{\sqrt{3}}{2}+\frac{1}{6} = 0 ; \quad m_y\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{3}\frac{1}{\sqrt{3}}\frac{1}{2} = m_y\frac{\sqrt{3}}{2}+\frac{1}{6} = 0 \quad \text{and} \quad m_z\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{3}\frac{1}{\sqrt{3}}\frac{1}{2} = m_z\frac{\sqrt{3}}{2}+\frac{1}{6} = \frac{1}{2}$$

We can then deduce the vector moment $m = \begin{Bmatrix}\frac{-1}{3\sqrt{3}}\\\frac{-1}{3\sqrt{3}}\\\frac{2}{3\sqrt{3}}\end{Bmatrix}$

Finally we can have the right position of the shifted axis u that have the same direction as the rotation axis n by defining the coordinates u_x, u_y and u_z of a point or a center C belonging to it so that: $m = u \wedge n$

Or $\begin{Bmatrix}\frac{-1}{3\sqrt{3}}\\\frac{-1}{3\sqrt{3}}\\\frac{2}{3\sqrt{3}}\end{Bmatrix} = \begin{Bmatrix}u_x\\u_y\\u_z\end{Bmatrix} \wedge \frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix} = \frac{1}{\sqrt{3}}\begin{Bmatrix}u_y-u_z\\u_z-u_x\\u_x-u_y\end{Bmatrix}$ implying that: $u_y - u_z = \frac{-1}{3}$; $u_z - u_x = \frac{-1}{3}$ and $u_x - u_y = \frac{2}{3}$

Which confirm the same obtained results eq (5) using the (4x4) rigid transformation matrix:

$$u_x - u_z = \frac{1}{3} ; \quad -u_x + u_y = -\frac{2}{3} ; \quad \text{and} \quad -u_y + u_z = \frac{1}{3} \qquad (5)$$

VIII. APPLICATION 2: KINEMATICS OF THE PUMA 560 ROBOT

The first three joints of this manipulator (Waist, Shoulder, Elbow) characterize for the first joint to be a rotation about a vertical axis , for the second and the third rotations about horizontal axes whose movements are identified by the variables q_1, q_2, and q_3. The last three joints, which constitute the wrist of the robot arm, are characterized by the rotations q_4, q_5, and q_6 whose axes intersect at the center of the wrist (See appendix 10,3. Figures (3),(4) and Table 1 for the forward kinematic solution using the Denavit and Hartenberg convention.

The elegant , most accurate , rapid and finally the best manner to get the forward kinematic solutions of this Puma 560 robot is to use the dual quaternions:

For the sake of comparaison let us choose the same home position for the robot with its geometry (a_i and d_i) given in table (1) and the same absolute home initial frame (x_0, y_0, z_0) with its origin O taken in link1 at the intersection of the base axis with the link1 axis (see figure (3)), assuming mobile frames at the centers of the six rotations: (x_n, y_n, z_n) which axes remain parallel to the

'home position' or initial axes (x_0, y_0, z_0). Let us begin, with the first two rotations using either equations A3 or A14 from appendix 10,2,1. to find the new vector position of the center O_3 (a_2, d_2, 0):

$$\hat{q}_1\,\hat{q}_2\,\hat{q}_v\,\overline{\hat{q}}_2^*\,\overline{\hat{q}}_1^* \;=\; \hat{q}_1\,(\hat{q}_2\,(\hat{q}_v)\,\overline{\hat{q}}_2^*\,)\overline{\hat{q}}_1^* \tag{7}$$

$$\hat{q}_2\,\hat{q}_v\,\overline{\hat{q}}_2^* \;=\; (c_2\,,0\,,s_2\,,0)\,[\,1+\varepsilon\,(a_2,\,d_2,\,0\,)\,]\,(c_2\,,0\,,-s_2\,,0)$$

Using correctly the rules for both quaternions eq (A1) and dual quaternions multiplications eq (A7) we have :

$$\hat{q}_2\,\hat{q}_v \;=\; (c_2\,,0\,,s_2\,,0) + \varepsilon\,(-s_2\,d_2\,,\,c_2\,a_2,\,c_2 d_2,\,-s_2\,a_2\,) \quad \text{and}$$

$$\hat{q}_2\,\hat{q}_v\,\overline{\hat{q}}_2^* \;=\; 1 + \varepsilon\,(0\,,\,a_2\cos\theta_2\,,\,d_2\,,-a_2\sin\theta_2) \quad \text{thus}$$

$$\hat{q}_1\,\hat{q}_2\,\hat{q}_v\,\overline{\hat{q}}_2^*\,\overline{\hat{q}}_1^* \;=\; (c_1\,,0\,,0\,,s_1)[\,1 + \varepsilon\,(0\,,\,a_2\cos\theta_2\,,\,d_2\,,-a_2\sin\theta_2)]\,(c_1\,,0\,,0\,,-s_1)$$

Performing the product and using the trigonometric properties we can have the new quaternion vector position:

$$1 + \varepsilon\,(0\,,\,a_2\cos\theta_2\cos\theta_1 - d_2\sin\theta_1,\,a_2\cos\theta_2\sin\theta_1 + d_2\cos\theta_1,\,-a_2\sin\theta_2\,)$$

or the three coordinates vector:
$$\begin{cases} a_2\cos\theta_2\cos\theta_1 - d_2\sin\theta_1 \\ a_2\cos\theta_2\sin\theta_1 + d_2\cos\theta_1 \\ -a_2\sin\theta_2 \end{cases} \tag{8}$$

This result is confirmed (see appendix 10,3,2.) by the fourth or last column of the matrix T_0^2 :

$$= T_0^1 T_1^2 = R_1 R_2 = \begin{pmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 c_2 & -c_1 s_2 & -s_1 & a_2 c_1 c_2 - d_2 s_1 \\ s_1 c_2 & -s_1 s_2 & c_1 & a_2 c_2 s_1 + d_2 c_1 \\ -s_2 & -c_2 & 0 & -a_2 s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The third rotation of the third link is around the axis $O_3 y_3$, with the center O_3 being displaced or shifted and thus having the position coordinates with respect to the asolute frame O_3 (a_2, d_2, 0).

Note: The conjugation (TRT^{-1}) technique could be used in its dual quaternion form or its (4x4) rigid transformation form.

The dual quaternion ($\hat{q}$) definition (2) may be used instead;

$$\hat{q} = \hat{T} = \left\{\cos\frac{\hat{\theta}}{2},\ \sin\frac{\hat{\theta}}{2}\hat{w}\right\} = \left\{\cos\frac{\theta}{2},\sin\frac{\theta}{2}n\right\} + \varepsilon\left\{-\frac{d}{2}\sin\frac{\theta}{2},\left\{m\sin\frac{\theta}{2}+\frac{d}{2}n\cos\frac{\theta}{2}\right\}\right\} \quad << dual\ type\ >> \tag{2}$$

So replacing $d = 0$ in eq (2), will give: $\hat{q}_3 = \left\{\cos\frac{\hat{\theta}}{2},\ \sin\frac{\hat{\theta}}{2}\hat{w}\right\} = [c_3,(0,s_3,0)] + \varepsilon\{0,\{m\,s_3\}\}$

The moment m, w- r - t the axis of rotation y_3 , is $m = \begin{Bmatrix} a_2 \\ d_2 \\ 0 \end{Bmatrix}\wedge\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ a_2 \end{Bmatrix}$ so that $\hat{q}_3 = [c_3,(0,s_3,0)] + \varepsilon\{0,\{(0,0,a_2 s_3)\}\}$

$$(\hat{q}_1\,\hat{q}_2\,)\hat{q}_3\,\hat{q}_v\,\overline{\hat{q}}_3^*\,(\overline{\hat{q}}_2^*\,\overline{\hat{q}}_1^*\,) \;=\; (c_1,0,0,s_1)\,(c_2,0,s_2,0)\,\hat{q}_3\,\hat{q}_v\,\overline{\hat{q}}_3^*\,(c_2\,,0\,,-s_2\,,0)\,(c_1\,,0,\,0,-s_1)$$

To find the new vector position of the wrist center O_4 :($a_2 + a_3$, $d_2 + d_3$, 0) = (A, D, 0) result of the three successives rotations we must start from the central operation namely :

$$\hat{q}_{v3} = \hat{q}_3\,\hat{q}_v\,\overline{\hat{q}}_3^* = [(c_3,0,s_3,0) + \varepsilon\,(0,0,a_2 s_3)]\,[(1+\varepsilon\,(A,D,0))]\,[\overline{\hat{q}}_3^*] =$$

$$[(c_3,0,s_3,0) + \varepsilon(0,0,a_2 s_3) + \varepsilon(-s_3 D,\,c_3 A\,,\,c_3 D, -s_3 A)]\,[\overline{\hat{q}}_3^*] =$$

$$[(c_3,0,s_3,0 + \varepsilon(-s_3 D,\,c_3 A\,,\,c_3 D\,,\,s_3(-A)]\,[(c_3,0,-s_3,0) + \varepsilon\,(0,0,a_2 s_3)] =$$

$$(c_3^2 + s_3^2,\,0,\,-c_3 s_3 + c_3 s_3\,,\,0) + \varepsilon(-c_3 s_3 D + s_3 c_3 D,\,s_3^2 a_2 + c_3^2 A + s_3^2\,(a_2 - A),\,s_3^2 D + c_3^2 D\,,\,c_3 s_3 a_2 - c_3 s_3 A + a_2 c_3 s_3 - c_3 s_3 A\,)$$

Using the basic trigonometric rules and properties we can write the solution vector:

$$\hat{q}_{v3} = \hat{q}_3\,\hat{q}_v\,\overline{\hat{q}}_3^* = 1 + \varepsilon\,(0\,,\,a_2 + a_3\cos\theta_3\,,\,d_2 + d_3\,,\,-a_3\sin\theta_3\,) = 1 + \varepsilon\,(a_2 + a_3\cos\theta_3\,,\,d_2 + d_3\,,\,-a_3\sin\theta_3\,)$$

For a better use of space we may adopt to write our result dual quaternions vectors under the form: $\begin{cases} \text{scalar part} \\ O_x \text{ coord.} \\ O_y \text{ coord.} \\ O_z \text{ coord.} \end{cases}$

So that the precedent result could be written $\hat{q}_{v3} = \begin{pmatrix} 0 \\ a_2 + a_3 \cos\theta_3 \\ d_2 + d_3 \\ -a_3 \sin\theta_3 \end{pmatrix}$ or simply as a vector $\begin{pmatrix} a_2 + a_3 \cos\theta_3 \\ d_2 + d_3 \\ -a_3 \sin\theta_3 \end{pmatrix}$

Following the second transformation we have: $(c_2, 0, s_2, 0)\, \hat{q}_{v3}\,(c_2, 0, -s_2, 0) =$

$$\hat{q}_{v2} = (c_2, 0, s_2, 0)[\, 1 + \varepsilon\,(0, a_2 + a_3 \cos\theta_3, d_2 + d_3, -a_3 \sin\theta_3\,)\,](c_2, 0, -s_2, 0) =$$

$$(c_2, 0, s_2, 0)\left[1 + \varepsilon \begin{pmatrix} 0 \\ a_2 + a_3 \cos\theta_3 \\ d_2 + d_3 \\ -a_3 \sin\theta_3 \end{pmatrix}\right][\bar{\hat{q}}_2^*] =$$

$$\left[(c_2, 0, s_2, 0) + \varepsilon \begin{pmatrix} -s_2(d_2 + d_3) \\ c_2(a_2 + a_3 \cos\theta_3) - s_2 a_3 \sin\theta_3 \\ c_2(d_2 + d_3) \\ -c_2 a_3 \sin\theta_3 - s_2(a_2 + a_3 \cos\theta_3) \end{pmatrix}\right](c_2, 0, -s_2, 0) =$$

$$(c_2^2 + s_2^2, 0, -c_2 s_2 + c_2 s_2, 0) + \varepsilon \begin{pmatrix} -c_2 s_2(d_2 + d_3) + c_2 s_2(d_2 + d_3) \\ c_2^2(a_2 + a_3 \cos\theta_3) - c_2 s_2 a_3 \sin\theta_3 - c_2 s_2 a_3 \sin\theta_3 - s_2^2(a_2 + a_3 \cos\theta_3) \\ c_2^2(d_2 + d_3) + s_2^2(d_2 + d_3) \\ -c_2^2 a_3 \sin\theta_3 - c_2 s_2(a_2 + a_3 \cos\theta_3) - c_2 s_2(a_2 + a_3 \cos\theta_3) + s_2^2 a_3 \sin\theta_3 \end{pmatrix} =$$

$$1 + \varepsilon \begin{pmatrix} 0 \\ c_2^2(a_2 + a_3 \cos\theta_3) - c_2 s_2 a_3 \sin\theta_3 - c_2 s_2 a_3 \sin\theta_3 - s_2^2(a_2 + a_3 \cos\theta_3) \\ c_2^2(d_2 + d_3) + s_2^2(d_2 + d_3) \\ -c_2^2 a_3 \sin\theta_3 - c_2 s_2(a_2 + a_3 \cos\theta_3) - c_2 s_2(a_2 + a_3 \cos\theta_3) + s_2^2 a_3 \sin\theta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \cos\theta_2(a_2 + a_3 \cos\theta_3) - a_3 \sin\theta_2 \sin\theta_3 \\ (d_2 + d_3) \\ -a_3 \cos\theta_2 \sin\theta_3 - \sin\theta_2(a_2 + a_3 \cos\theta_3) \end{pmatrix} =$$

We can finally get the transformed vector $\hat{q}_{v2}$:

$$\hat{q}_{v2} = 1 + \varepsilon\,[(a_3 \cos(\theta_2 + \theta_3) + a_2 \cos\theta_2,\ (d_2 + d_3),\ -a_3 \sin(\theta_2 + \theta_3) - a_2 \sin\theta_2\,)]$$

We can finally perform the first but last transformation given by the following dual quaternions products:

$$(c_1, 0, 0, s_1)\,(c_2, 0, s_2, 0)\,\hat{q}_3\,\hat{q}_v\,\bar{\hat{q}}_3^*\,(c_2, 0, -s_2, 0)(c_1, 0, 0, -s_1) =$$

$$(c_1, 0, 0, s_1)\,\hat{q}_{v2}\,(c_1, 0, 0, -s_1) =$$

$$(c_1, 0, 0, s_1)\,[1 + \varepsilon\,(a_3 \cos(\theta_2 + \theta_3) + a_2 \cos\theta_2,\ (d_2 + d_3),\ -a_3 \sin(\theta_2 + \theta_3) - a_2 \sin\theta_2\,)]\,\bar{\hat{q}}_1^*$$

$$(c_1, 0, 0, s_1)\left[1 + \varepsilon \begin{pmatrix} 0 \\ a_3 \cos(\theta_2 + \theta_3) + a_2 \cos\theta_2 \\ d_2 + d_3 \\ -a_3 \sin(\theta_2 + \theta_3) - a_2 \sin\theta_2 \end{pmatrix}\right][\bar{\hat{q}}_1^*] =$$

$$\left[(c_1, 0, 0, s_1) + \varepsilon \begin{pmatrix} a_3 s_1 \sin(\theta_2 + \theta_3) + a_2 s_1 \sin\theta_2 \\ a_3 c_1 \cos(\theta_2 + \theta_3) + a_2 c_1 \cos\theta_2 - s_1(d_2 + d_3) \\ c_1(d_2 + d_3) + a_3 s_1 \cos(\theta_2 + \theta_3) + a_2 s_1 \cos\theta_2 \\ -a_3 c_1 \sin(\theta_2 + \theta_3) - a_2 c_1 \sin\theta_2 \end{pmatrix}\right][\bar{\hat{q}}_1^*] =$$

$$\left[(c_1, 0, \quad 0, s_1) + \varepsilon \begin{pmatrix} a_3\, s_1 \sin\left(\theta_2 + \theta_3\right) + a_2 s_1 \sin\theta_2 \\ a_3\, c_1 \cos\left(\theta_2 + \theta_3\right) + a_2 c_1 \cos\theta_2 - s_1(d_2 + d_3) \\ c_1(d_2 + d_3) + a_3\, s_1 \cos\left(\theta_2 + \theta_3\right) + a_2 s_1 \cos\theta_2 \\ -a_3 c_1 \sin\left(\theta_2 + \theta_3\right) - a_2 c_1 \sin\theta_2 \end{pmatrix}\right] (c_1, 0, \ 0, -s_1) =$$

$$(c_1{}^2 + s_1{}^2, 0, -c_1 s_1 + c_1 s_1, 0) +$$

$$\varepsilon \begin{pmatrix} a_3\, c_1 s_1 \sin\left(\theta_2 + \theta_3\right) + a_2 c_1 s_1 \sin\theta_2 - a_3 c_1\, s_1 \sin\left(\theta_2 + \theta_3\right) - a_2 c_1\, s_1 \sin\theta_2 \\ a_3\, c_1{}^2 \cos\left(\theta_2 + \theta_3\right) + a_2 c_1{}^2 \cos\theta_2 - c_1 s_1(d_2 + d_3) - s_1 c_1(d_2 + d_3) - a_3\, s_1{}^2 \cos\left(\theta_2 + \theta_3\right) - a_2 s_1{}^2 \cos\theta_2 \\ c_1{}^2(d_2 + d_3) + a_3\, c_1 s_1 \cos\left(\theta_2 + \theta_3\right) + a_2 c_1 s_1 \cos\theta_2 + a_3\, s_1 c_1 \cos\left(\theta_2 + \theta_3\right) + a_2 s_1 c_1 \cos\theta_2 - s_1{}^2(d_2 + d_3) \\ -a_3 c_1{}^2 \sin\left(\theta_2 + \theta_3\right) - a_2 c_1{}^2 \sin\theta_2 - a_3\, s_1{}^2 \sin\left(\theta_2 + \theta_3\right) - a_2\, s_1{}^2 \sin\theta_2 \end{pmatrix} =$$

$$1 + \varepsilon \begin{pmatrix} 0 \\ a_3\, c_1{}^2 c_{23} + a_2 c_1{}^2 \cos\theta_2 - c_1 s_1(d_2 + d_3) - s_1 c_1(d_2 + d_3) - a_3\, s_1{}^2 c_{23} - a_2 s_1{}^2 \cos\theta_2 \\ c_1{}^2(d_2 + d_3) + a_3\, c_1 s_1 c_{23} + a_2 c_1 s_1 \cos\theta_2 + a_3\, s_1 c_1 c_{23} + a_2 s_1 c_1 \cos\theta_2 - s_1{}^2(d_2 + d_3) \\ -a_3 c_1{}^2 s_{23} - a_2 c_1{}^2 \sin\theta_2 - a_3\, s_1{}^2 s_{23} - a_2\, s_1{}^2 \sin\theta_2 \end{pmatrix}$$

With $c_{23} = \cos\left(\theta_2 + \theta_3\right)$ and $s_{23} = \sin\left(\theta_2 + \theta_3\right)$

The result vector is then:
$$\begin{pmatrix} \cos\theta_1\left(a_2 \cos\theta_2 + a_3\, c_{23}\right) - \sin\theta_1(d_2 + d_3) \\ \sin\theta_1(a_2 \cos\theta_2 + a_3\, c_{23}) + \cos\theta_1\left((d_2 + d_3)\right) \\ -\left(a_2 \sin\theta_2 + a_3\, s_{23}\right) \end{pmatrix}$$

Which is confirmed by the last column (see appendix (10, 3, 2.) of the matrix T_0^3 .

We can also, using the Denavit and Hartenberg formalism or the dual quaternions alike easily calculate the coordinates of the terminal element (or the end effector) and so the final positioning of our Puma 560 robot relative to the base or fixed absolute frame.

IX. CONCLUSION

We hope that the reader should not get us wrong: We never pretend that the D-H parameters method is wrong or obsolete and that it should be a thing of the past; recognising that this important classical method was the precursor that enlightened the path to modern robotics; we only say that there exist through the DQ parameters another short, free of singularities and easy to work with, when dealing with robot direct kinematics. On the light of the obtained results one has to say that the most perfect (not suffering singularities of any kind), easiest and rapid way to perform a 3D rigid transformation of any sort is to use the dual quaternion that caracterise that movement. Most of all we are free to use the 3D space, being sure that no loss of degree of freedom or guinball lock of any sort can never happen. Using a D-H parameters method or any of its counterparts means a choice of different sort of embarassing and somehow awkward three axes frames to be created and then allocated to each arm/ link; 'providing' our robot or mecanism with different direction axes and angles with very much complicated choice of signs (concerning the directions and the angles alike) to be chosen subject to some rules depending on the chosen method and model of robot.

Choosing to use dual quaternions we only need to know the constants or values that concern the construction or space geometry of the given robot (directions (orientations and axes) , rotations ,distances, lengths of links) to evaluate its kinematics without any threat to be lost in the maze or a jungle of choices .Most of all, it will prevent us from using the only other existing method, or one of its options, which is that of the Denavit and Hartenberg parameters that mainly consists of:

1. Choosing 3D frames attached to each link upon certain conditions /conventions,
2. Schematic of the numbering of bodies and joints in a robotic manipulator, following the convention for attaching reference frames to the bodies, this will help to create:
3. A table for exact definition of the four parameters, a_i, α_i, d_i, and θ_i, that locate one frame relative to another,
4. The (4x4) rigid transformation matrix that will have the given form : T_{i-1}^i. (See 10,3.)

This chapter provided a taste of the potential advantages of dual-quaternions, and one can only imagine the further future possibilities that they can offer. For example, there is a deeper investigation of the mathematical properties of dual-quaternions (e.g., zero divisions). There is also the concept of dual-dual-quaternions (i.e., dual numbers within dual numbers) and calculus for multi-parametric objects for the reader to pursue if he desires.

We should emphasize on the fact that Matlab software was used, throughout this work and whenever necessary, concerning all kinds of products or multiplication of quaternions or rigid transformation matrices.

Finally we hope all efforts should be conjugated to create a common 'PROJECT MATLAB QUATERNION/MATRIX platform' to be used for the straightforward calculations and manipulation of Quaternions and / or Dual Quaternions as well as conversions from or into 3D or 4D rigid body matrices.

X. APPENDICES

a) Quaternion-Matlab Implementation Class:

>> % See paragraph 3; Example 1: Rotations represented by Quaternions

>> % A first rotation of angleπ/2 around the x -axis ,q1 , followed by a rotation of angleπ/2 around the y -axis , q2 will result in a rotation given by the product n1 = q2.q1 :

>> q1 =[cos(pi/4) sin(pi/4) 0 0];

q2 =[cos(pi/4) 0 sin(pi/4) 0];

>> n1 = quatmultiply (q2,q1)

n1 = 0.5000   0.5000   0.5000   -0.5000

>> % If the order is inversed the result will be given by the quaternion n2 = q1.q2

>> n2 = quatmultiply (q1,q2)

n2 = 0.5000   0.5000   0.5000   0.5000

>> % Using 3*3 matrices; if the rotation R1 is performed first the rotation product is R2*R1:

R1 = [1 0 0;0 0 -1;0 1 0];

R2 = [0 0 1; 0 1 0;-1 0 0];

prod1 = R2*R1

prod1 =

   0   1   0

   0   0 -1

 -1   0   0

>> % if the order is inversed the multiplication will be R1*R2:

prod2 = R1*R2

prod2 =

   0   0   1

   1   0   0

   0   1   0

b) Quaternions and Dual Quaternions (Dq)

i. Quaternions or rotation representation

Quaternions were first discovered and described by the Irish mathematician Sir Rowan Hamilton in 1843. Indeed quaternion's representation and axis-angle representation are very similar.

Both are represented by the four dimensional vectors. Quaternions also implicitly represent the rotation of a rigid body about an axis. It also provides better means of key frame interpolation and doesn't suffer from singularity problems.

The definition of a quaternion can be given as (s, m) or (s, q_x, q_y, q_z) where m is a 3D vector, so quaternions are like imaginary (complex) numbers with the real scalar part s and the imaginary vector part m.

Thus it can be also written as: $s + q_x i + q_y j + q_z k$.

There are conversion methods between quaternions, axis-angle and rotation matrix.

Common operations such as addition, inner product etc can be defined over quaternions.
Given the definition of q_1 and q_2 :

$$q_1 = s_1 + q_{x1} i + q_{y1} j + q_{z1} k \text{ or } q_1 = (s_1, m_1)$$

$$q_2 = s_2 + q_{x2} i + q_{y2} j + q_{z2} k \text{ or } q_2 = (s_2, m_2)$$

Addition operation is defined as:
$$q_1 + q_2 = (s_1 + s_2, m_1 + m_2) = (s_1 + s_2) + (q_{x1} + q_{x2})i + (q_{y1} + q_{y2})j + (q_{z1} + q_{z2})k$$

dot (scalar, inner): product operation(.) as:

$$q_1 . q_2 = s_1 . s_2 + m_1 . m_2$$

Quaternion multiplication is non commutative, but it is associative.
Multiplication identity element is defined as: (1, (0, 0, 0))

We can also perform the multiplication in the imaginary number domain using the definitions:

$$i^2 = j^2 = k^2 = -1; \quad i.j = k, \quad j.k = i, \quad k.i = j ;$$
$$j.i = -k, \quad k.j = -i, \quad i.k = -j$$

Equations (A1) to (A15) state the definitions, rules and properties of dual quaternion algebra.
Quaternion multiplication ($\otimes$) is defined as:

$$q_1 \otimes q_2 = (s_1 . s_2 - m_1 . m_2, s_1 . m_2 + s_2 . m_1 + m_1 \wedge m_2)$$
$$\text{(A1)}$$

Each quaternion has a conjugate q^* (except zero quaternion) defined by:

$$q^* = (s, -m) \qquad \text{(A2)}$$

and an inverse $q^{-1} = (\frac{1}{|q|})^2 q^*$; $(q \neq 0)$ Where $|q|^2 = s^2 + q_x{}^2 + q_y{}^2 + q_z{}^2 = q \otimes q^* = q^* \otimes q$

Rotations are defined by unit quaternions. Unit quaternions must satisfy $|q| = 1$. Since multiplication of two unit quaternions will be a unit quaternion, N

rotations can be combined into one unit quaternion $q_R = q_{R1} \cdot q_{R2} \cdot q_{R3} \cdots q_{RN}$

It is also possible to rotate a vector directly by using quaternion multiplication. To do this, we must define a 3D vector $V = (v_x, v_y, v_z)$ that we want to rotate in quaternion definition as $q_v = (0, v) = 0 + v_x i + v_y j + v_z k$. The rotated vector $V' = (v_x', v_y', v_z')$ can be defined as $q_{v'} = (0, v') = 0 + v_x' i + v_y' j + v_z' k$

Noting that, in quaternion rotation $q^{-1} = q^*$ (For unit quaternion). So, rotation of q_v by quaternion q can be calculated as:

$$q_{v'} = q \otimes q_v \otimes q^{-1} = q \otimes q_v \otimes q^* \qquad (A3)$$

And, assuming another quaternion rotation p, two rotations can be applied to the vector V such as:

$$q_{v'} = p \otimes (q \otimes q_v \otimes q^{-1}) \otimes p^{-1} = (p \otimes q) \otimes q_v \otimes (q^{-1} \otimes p^{-1}) = C \otimes q_v \otimes C^{-1} \qquad (A4)$$

Providing that quaternion $C = (p \otimes q)$ is a combinaison of the precedent quaternions q and p.

The equation implies that vector V is first rotated by the rotation represented by q followed by the rotation p.

A quaternion q that defines a rotation about (around) the axis n denoted by the unit vector (n_x, n_y, n_z) of an angle θ could be written as :

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(n_x i + n_y j + n_z k) \qquad (A5)$$

This same quaternion represents a rotation of amplitude $(-\theta)$ around the opposite axis $(-n)$

ii. *Dual quaternions*

Dual Quaternions (DQ) were proposed by William Kingdom Clifford in 1873. They are an extension of quaternions. They represent both rotations and translations whose composition is defined as a rigid transformation.

They are represented by the following eight dimensional vector:

$$\hat{q} = (\hat{s}, \hat{m}) = (s, q_x, q_y, q_z, q_{\varepsilon_s}, q_{\varepsilon_x}, q_{\varepsilon_y}, q_{\varepsilon_z}) = (\hat{s}, \hat{x}, \hat{y}, \hat{z}) \qquad (A6)$$

Such that: $\hat{q} = q + \varepsilon q_\varepsilon = s + q_x i + q_y j + q_z k + \varepsilon(q_{\varepsilon_s} + q_{\varepsilon_x} + q_{\varepsilon_y} + q_{\varepsilon_z})$

Dual quaternion multiplication is defined by:

$$\hat{q}_1 \otimes \hat{q}_2 = q_1 \otimes q_2 + \varepsilon(q_1 \otimes q_{2_\varepsilon} + q_{1_\varepsilon} \otimes q_2) \qquad (A7)$$

With $\varepsilon^2 = 0$; ε being the second order nilpotent dual factor.
The dual conjugate (analogous to complex conjugate) is denoted by:

$$\bar{\hat{q}} = q - \varepsilon q_\varepsilon \qquad (A8)$$

This conjugate operator can lead to the definition of the inverse of $\hat{q}$ which is:

$\hat{q}^{-1} = \frac{1}{\hat{q}} = \frac{\bar{\hat{q}}}{\hat{q}} \frac{1}{\hat{q}} = \frac{1}{q} - \varepsilon\frac{q_\varepsilon}{q^2}$; which means that a pure dual number $(ie: q = 0)$ does not have an inverse)

$$\hat{q} = = \hat{q} \otimes \hat{q}^{-1} = (q + \varepsilon q_\varepsilon)(\frac{1}{q} - \varepsilon\frac{q_\varepsilon}{q^2}) = \frac{q}{q} - \varepsilon\frac{qq_\varepsilon}{q^2} + \frac{\varepsilon q_\varepsilon}{q} = \frac{q}{q} - \varepsilon\frac{q_\varepsilon}{q} + \frac{\varepsilon q_\varepsilon}{q} = 1 - 0 = 1$$

A second conjugation operator is defined for DQs. It is the classical quaternion conjugation and is denoted by: $\hat{q}^* = q^* + \varepsilon q_\varepsilon^*$

Combining these two conjugation operators will lead to the formalization of DQ transformation on 3D points. Use of both conjugations on $\hat{q}$ can be denoted $\bar{\hat{q}}^*$. Using definitions (A2), (A6) and (A8) we finally have:

$$\bar{\hat{q}}^* = (s, -q_x, -q_y, -q_z, -q_{\varepsilon_s}, q_{\varepsilon_x}, q_{\varepsilon_y}, q_{\varepsilon_z}) \qquad (A9)$$

It is well know that we can use dual quaternions to represent a general transformation subject to the following constraints:

The DQ screw motion operator $\hat{q} := (q, q_\varepsilon)$ must be of unit magnitude: $|\hat{q}| = (q + \varepsilon q_\varepsilon)^2 = 1$

This requirement means two distinct conditions or constraints:

$$s^2 + q_x^2 + q_y^2 + q_z^2 = 1 \quad \text{and}$$

$$s q_{\varepsilon_s} + q_x q_{\varepsilon_x} + q_y q_{\varepsilon_y} + q_z q_{\varepsilon_z} = 0 \qquad (A10)$$

Which imposed on the eight (8) parameters of a general DQ, effectively reduce the number of degree of freedom $(8 - 2) = 6$; equivalent to the degree of freedom of any free rigid body in 3-D space

iii. *Dual Quaternions or general 3D rigid transformation representation*

While equation (A5) defines completely and unambiguously (without any singularity like guimbal lock and other loss of degree of freedom) all 3D rotations in the physical space, dual quaternions can represent translations;

A DQ defined as: $\hat{q}_T = 1 + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k)$ corresponds to the translation vector $\vec{T} = (t_x, t_y, t_z)^t$

which could symbolically be noted T; so $\quad \hat{q}_T = 1 + \varepsilon \frac{T}{2}$

The translation T on the vector $\vec{v}$ can be computed by: $\hat{q}'_v = \hat{q}_T \otimes \hat{q}_v \otimes \bar{\hat{q}}^*_T$

So fortunately using def (A9), we have:

$$\bar{\hat{q}}^*_T = \hat{q}_T = 1 + \varepsilon \frac{T}{2}, \text{ then } \hat{q}'_v = \hat{q}_T \otimes \hat{q}_v \otimes \bar{\hat{q}}^*_T = \hat{q}_T \otimes \hat{q}_v \otimes \hat{q}_T = [1 + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k)] \otimes [1 + \varepsilon(v_x i + v_y j + v_z k)] \otimes [1 + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k)] =$$

$$1 + \varepsilon[(v_x + t_x)i + (v_y + t_y)j + (v_z + t_z)k)]$$

Which correspond to the transformed vector: $\vec{v}' = (v_x + t_x)i + (v_y + t_y)j + (v_z + t_z)k$

iv. *Combining rotation and translation*

Transformations represented by DQs can be combined into one DQ (similar to quaternions combination Assuming: $\hat{q}$ and then $\hat{p}$, two DQ transformations applied successively and in that order to a DQ position vector $\hat{q}_v$; Their combined DQ transformation $\hat{C}$ applied to $\hat{q}_v$ gives:

$$\hat{q}'_v = \hat{p} \otimes (\hat{q} \otimes \hat{q}_v \otimes \bar{\hat{q}}^*) \otimes \bar{\hat{p}}^* = (\hat{p} \otimes \hat{q}) \otimes \hat{q}_v \otimes (\bar{\hat{q}}^* \otimes \bar{\hat{p}}^*) = \hat{C} \otimes \hat{q}_v \otimes \bar{\hat{C}}^* \qquad (A11)$$

It is very important to notice that the most inner transformation of the equation is applied first with an inside to outside manner.
In eq (22), $\hat{q}$ is the first transformation followed by the second one $\hat{p}$.

The successive composition or combination of unit DQ rotation $\hat{q}_R = R$ followed by a DQ translation $\hat{q}_T = 1 + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k)$
will give:

$$\hat{q}_T \otimes \hat{q}_R = (1 + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k)) \otimes q_R = q_R + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k) \otimes q_R = R + \varepsilon \frac{TR}{2} \qquad (A12)$$

Its inverse being: $(R + \varepsilon \frac{TR}{2})^{-1} = R^* - \frac{R^*T}{2}$

If the translation is applied first:

$$\hat{q}_R \otimes \hat{q}_T = \hat{q}_R \otimes (1 + \frac{\varepsilon}{2}(t_x i + t_y j + t_z k)) = q_R + \hat{q}_R \otimes \frac{\varepsilon}{2}(t_x i + t_y j + t_z k) q_R = R + \varepsilon \frac{RT}{2} \qquad (A13)$$

Its inverse being: $(R + \varepsilon \frac{RT}{2})^{-1} = R^* - \frac{TR^*}{2}$

v. *Several transformations*

Suppose that the vector V in its dual quaternion form $\hat{q}_v = 1 + \varepsilon v$ is under a sequence of rigid transformations represented by the dual quaternions $\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_n$. The resulting vector is encapsulated in the dual quaternion:

$$1 + \varepsilon v' = \hat{q}_n \otimes (\hat{q}_{n-1} \otimes \ldots \otimes (\hat{q}_1 \otimes (1 + \varepsilon v) \otimes \bar{\hat{q}}^*_1) \otimes \ldots \otimes \bar{\hat{q}}^*_{n-1}) \otimes \bar{\hat{q}}^*_n \qquad (A14)$$

$$= (\hat{q}_n \otimes \ldots \otimes \hat{q}_1) \otimes (1 + \varepsilon v) \otimes (\bar{\hat{q}}^*_1 \otimes \ldots \otimes \bar{\hat{q}}^*_n)$$

We denote the product dual quaternion as $\hat{q} = \hat{q}_n \otimes \ldots \otimes \hat{q}_1$. The effect is equivalent to a single rigid transformation represented by $\hat{q}$; namely,

$$1 + \varepsilon v' = \hat{q} \otimes (1 + \varepsilon v) \otimes \bar{\hat{q}}^*.$$

Using dual numbers and plucker coordinates and introducing the following dual angle and dual vector we can write:

$$\hat{\theta} = \theta + \varepsilon d \quad \text{and}$$

$$\hat{l} = l + \varepsilon m$$

It can be easily shown that:

$$\cos \frac{\theta + \varepsilon d}{2} = \cos \frac{\theta}{2} - \varepsilon \frac{d}{2} \sin \frac{\theta}{2} \qquad \text{and}$$

(A15)

$$\sin \frac{\theta + \varepsilon d}{2} = \sin \frac{\theta}{2} + \varepsilon \frac{d}{2} \cos \frac{\theta}{2}$$
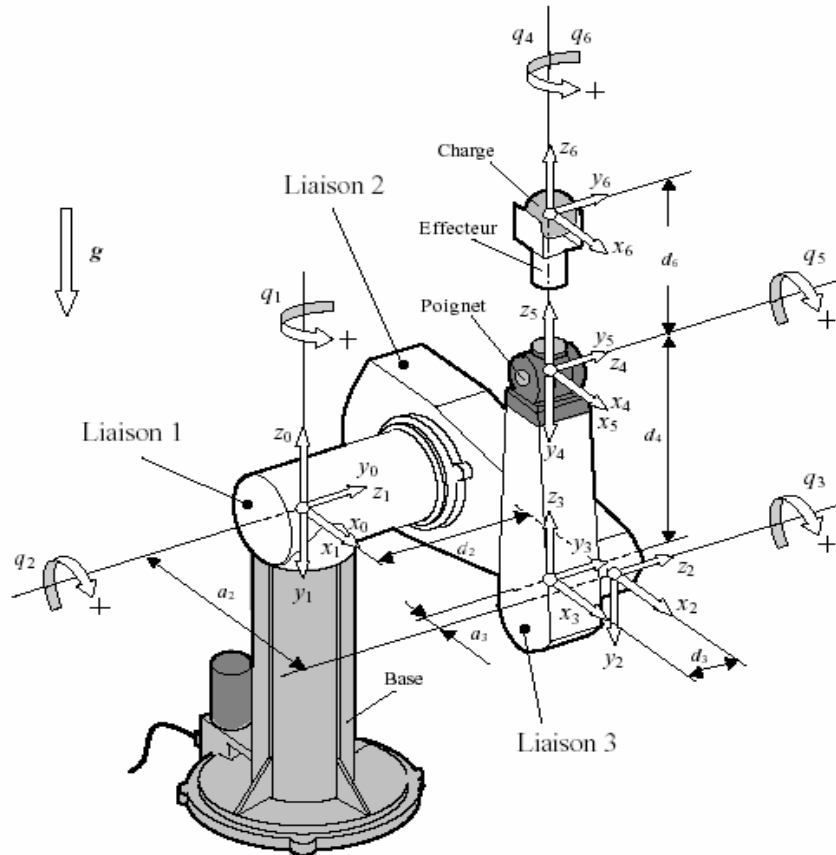
vi. D-H Parameters For The Puma 560 Robot

Figure 3: System of connections coordinates and parameters of joints for the PUMA 560 robot arm according to the Denavit and Hartenberg convention

vii. Parameters of Denavit and Hartenberg

The Denavit and Hartenberg Convention is a systematic method. It allows the passage between adjacent joints of a robotics system. It relates to the open kinematic chains where the joint possesses only one degree of freedom, and the adjacent surfaces remain in contact. For this aspect the use of hinges or slides is indispensable. The choice of the frames for the links facilitates the calculation of the DH homogeneous matrices and makes it possible to rapidly express information of the terminal element towards the base or the reverse.

The steps for this technique are as follows:

1. Numbering of the constituent segments of the manipulator arm from the base to the terminal element. The zero referential is associated with the base of it, and the order n to the terminal element (end effector);

2. Definition of the main axes of each segment:
   - If z_i and z_{i-1} do not intersect we choose x_i so as to be the parallel with the axis perpendicular to z_i and z_{i-1}.
   - If z_i and z_{i-1} are collinear, x_i is chosen in the plane perpendicular to z_{i-1}.

3. Fix the four geometric parameters: d_i, θ_i, a_i, α_i (see Figure(4)) for each joint such as:
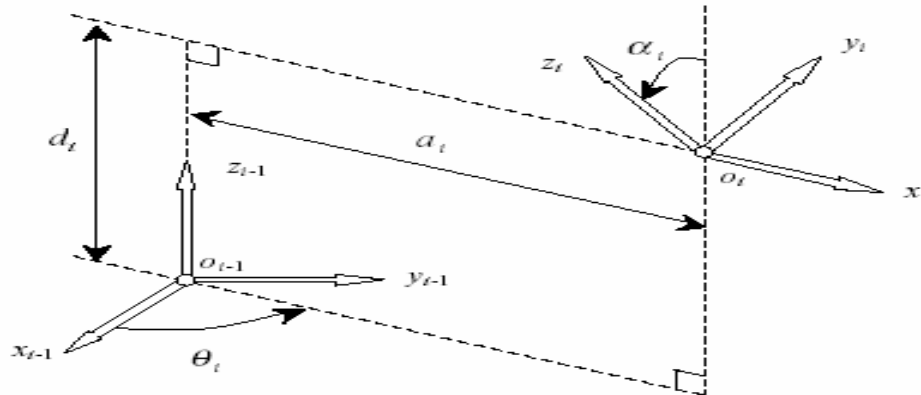
Figure 4: Coordinate systems and parameters of Denavit and Hartenberg

- d_i coordinate of the origin O_i on the axis z_{i-1} For a slide d_i is a variable and for a hinge d_i is a constant.
- θ_i is the angle obtained by screwing x_{i-1} to x_i around the axis z_{i-1}.For a slide q_i is a constant and for a hinge q_i is a variable.
- a_i is the distance between the axes z_i and z_{i-1} measured on the axis x_i negative from its origin up to the intersection with the axis z_{i-1}.

- α_1 is the angle between z_i et z_{i-1} obtained by screwing z_{i-1} to z_i around x_i.

Finally, the homogeneous DH displacement matrix $[T^i_{i-1}]$ which binds together the rotation and the translation is formed . Its left upper part defines the rotation matrix R^i_{i-1} and on its right the translation vector

$$[T^i_{i-1}] : \begin{bmatrix} R^i_{i-1} & d^i_{i-1} \\ 0\ 0\ 0 & 1 \end{bmatrix} \tag{9}$$

With

$$R^i_{i-1} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix} \tag{10}$$

And

$$d^i_{i-1} = \begin{cases} a_i \cos\theta_i \\ a_i \sin\theta_i \\ d_i \end{cases} \tag{11}$$

Figure (4) represents the Denavit and Hartenberg parameters for a two successive frames $(x_{i-1}, y_{i-1}, z_{i-1})$ and (x_i, y_i, z_i).

And finally the (4x4) rigid transformation matrix will have the form: T^i_{i-1}

$$\begin{pmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

The definition of the frames associated with the links according to the Denavit and Hartenberg convention is as follows:

Link1: Frame (x_0, y_0, z_0) ;The origin O is taken in link1 at the intersection of the base axis with the link1 axis. z_0 axis of rotation, $+ z_0$ upwards.$+ y_0$ coincides with the axis of the link 1 and the axis $+ z_1$.y_1 is parallel to the link 2.

Link 2: Frame (x_1, y_1, z_1) ;The origin coincides with the origin of the frame $(x_0, y_0, z_0,)$ .z_1 axis of rotation, $+ z_1$ is perpendicular to the link 2 and parallel to the axis $+$

z_2.$+ y_1$ downwards, superimposed with the axis of the base and parallel with y_2.$+ x_1$ is parallel to the link 2.

Link 3: Frame (x_2, y_2, z_2) ;The origin is taken in link 2 at the intersection of the axis of the link 2 with the axis of the joint 3.z_2 axis of rotation, $+ z_2$ is perpendicular to link 2 and axis z_3.$+ y_2$ downwards, opposite with $+ z_3$.$+ x_2$ is parallel to the link 2.

Link 4: Frame (x_3, y_3, z_3); The origin is taken in link 3.z_3 axis of rotation, $+ z_3$ towards the wrist and perpendicular to $+z_4$.$+ y_3$ is perpendicular to the link 2, and parallel to $+z_4$.$+ x_3$ is parallel to the link 2.

Link 5: Frame (x_4, y_4, z_4) ;The origin is taken at the center of the wrist.z_4 axis of rotation, $+ z_4$ is perpendicular to link 2 superposed with $+z_5$. $+ y_4$ is opposite to $+ z_5$. $+ x_4$ is parallel to link 2.

Link 6: Frame (x_5, y_5, z_5) ;The origin coïncides with the origin of the link (x_4, y_4, z_4).z_5 axis of rotation, $+z_5$ towards the effector parallel to $+z_6$. $+ y_5$ coïncides with the axis of joint 5. $+ y_5$ is perpendicular to the axis of joint 5.

The end effector: Frame (x_6, y_6, z_6) ;The origin coïncides with the origins of the links (x_4, y_4, z_4) and (x_5, y_5, z_5).

$+z_6$ is parallel to $+z_5$. and $+y_6$ is parallel to $+ y_5$. $+x_6$ is parallel to $+x_5$.

By respecting the original position of the robot and the definition of the links and correspondant frames presented in Figure (3), the parameters of the PUMA 560 robot arm given by the Denavit and Hartenberg Convention are shown in Table (1):

Table 1: Puma 560 Denavit and Hartenberg Parameters

i Numéro de la liaison	α_i (degrés)	θ_i Variable	a_i (mètres)	d_i (mètres)
1	−90	q_1	0	0
2	0	q_2	a_2	d_2
3	+90	q_3	a_3	d_3
4	−90	q_4	0	d_4
5	+90	q_4	0	0
6	0	q_5	0	−

The distance d_6 is not shown in Table (I)..This distance varies according to the effector used for the application (the effector is the tool attached to the wrist on the last articulation of the robot for the manipulation of the objects). In this application the distance between the end of the effector and the axis of the wrist is assumed to be null $d_6 = 0$.

The dynamics of the last three articulations is negligible compared to the first three. Therefore, we have been interested in studying the movement of the three first joints of the PUMA 560 robot arm fixing the others to the original position (i.e., wrist attached to the original position: $q_4 = q_5 = q_6 = 0$).

v. *D-H kinematics of the PUMA 560 ROBOT*

The appropriate transformations for the first three considered articulations are:

$$T_0^1 = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13}$$

$$T_1^2 = \begin{pmatrix} c_2 & -s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

Such that we will have : $T_0^2 = T_0^1 T_1^2$

$$= \begin{pmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1c_2 & -c_1s_2 & -s_1 & a_2c_1c_2 - d_2s_1 \\ s_1c_2 & -s_1s_2 & c_1 & a_2c_2s_1 + d_2c_1 \\ -s_2 & -c_2 & 0 & -a_2s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

We can also write:

$$T_2^3 = \begin{pmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_3 & 0 & s_3 & a_3c_3 \\ s_3 & 0 & -c_3 & a_3s_3 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

And finally write $T_0^3 = T_0^2 T_2^3 = \begin{pmatrix} c_1c_2 & -c_1s_2 & -s_1 & a_2c_1c_2 - d_2s_1 \\ s_1c_2 & -s_1s_2 & c_1 & a_2c_2s_1 + d_2c_1 \\ -s_2 & -c_2 & 0 & -a_2s_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_3 & 0 & s_3 & a_3c_3 \\ s_3 & 0 & -c_3 & a_3s_3 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$

$$T_0^3 = \begin{pmatrix} c_1 c_{23} & -s_1 c_1 s_{23} & c_1(a_2 c_2 + a_3 c_{23}) - (d_2 + d_3)s_1 \\ s_1 c_{23} & c_1 s_1 s_{23} & s_1(a_2 c_2 + a_3 c_{23}) + (d_2 + d_3)c_1 \\ -s_{23} & 0 & c_{23} & -(a_2 s_2 + a_3 s_{23}) \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (17)$$

With $\qquad c_i = \cos\theta_i, s_1 = \sin\theta_i, c_{ij} = \cos(\theta_i + \theta_j), s_{ij} = \sin(\theta_i + \theta_j)$

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