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Geometrical Characteristics of the Surfaces on Trapezium-Curved Plans

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GEOMETRICALCHARACTERISTICSOFTHESURFACESONTRAPEZIUMCURVEDPLANS

Geometrical Characteristics of the Surfaces on Trapezium-Curved Plans

Dr. V N Ivanov [α] & Imomnazarov T.S [σ]

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I. Introduction

Equation of the surface on the trapezium-curved plan, coefficients of quadratic forms of the surface.

The orthogonal curved system of coordinates at the plane there is formed by the system of the straight lines orthogonal to the plane base curve $\boldsymbol{r}_0(u)=x(u)\boldsymbol{i}+y(u)\boldsymbol{j}$ (Fig. 1).

So, the curved-orthogonal coordinates there are organized by the system of the equidistant curves parallel to the based curve and the system of the straight lines orthogonal to the system of the equidistant curves.

The equation of the curved coordinate system

$$\boldsymbol{r}(u,v)=\boldsymbol{r}_0(u)-v\boldsymbol{\nu}\ , \quad (1)$$

$\boldsymbol{\nu}$ is a normal to the base curve, v is the coordinate of the generating curves along the normal to the base curve.

The positive direction of the coordinate of straight lines there is taken to the side of the convexity of the base curve, because in the direction of the concavity the straight lines may to cross.

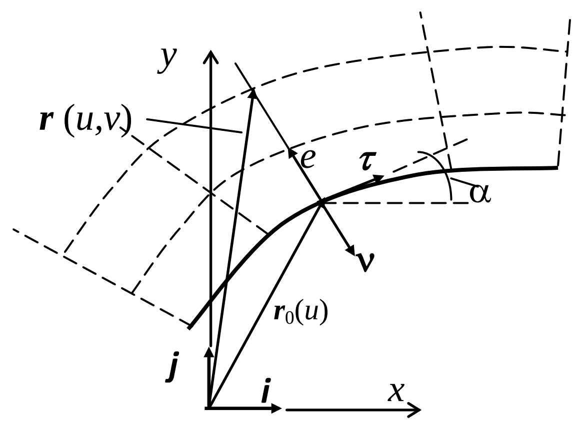

Fig. 1: Pseudo-polar coordinate system

Assigning some function of vertical coordinate $z(u,v)$, we receive the vector equation of the surface $\boldsymbol{\rho}(u,v)$ on the base curved-orthogonal coordinate system at the plane

$$\boldsymbol{\rho}(u,v)=\boldsymbol{r}_0(u)-v\boldsymbol{\nu}+z(u,v)\boldsymbol{k}\,. \quad (2)$$

For deduction the formulas of the coefficients it's necessary to receive the derivatives of the vector equation and to use the formulas of the classic differential geometry [2];

$$\boldsymbol{r}_0'=s'\boldsymbol{\tau}\,;\ \ s'=|\boldsymbol{r}_0'|\,;\ \ \boldsymbol{\tau}'=s'k\boldsymbol{\nu}=k_s\boldsymbol{\nu}\,;\ \ k_s=s'k\,;\ \ \boldsymbol{\nu}'=-k_s\boldsymbol{\tau}\,. \quad (3)$$

Then receive:

$$\boldsymbol{\rho}_u=(s'+vk_s)\boldsymbol{\tau}+z_u\boldsymbol{k}\,;\qquad \boldsymbol{\rho}_v=-\boldsymbol{\nu}+z_v\boldsymbol{k}\,. \quad (4)$$

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The coefficients of the first fundamental forms:

$$E = (\boldsymbol{\rho}_u \boldsymbol{\rho}_u) = (s' + vk_s)^2 + z_u^2; \quad G = (\boldsymbol{\rho}_v \boldsymbol{\rho}_v) = 1 + z_v^2; \quad F = (\boldsymbol{\rho}_v \boldsymbol{\rho}_v) = z_u z_v. \tag{5}$$

The unit normal vector of the surface

$$\boldsymbol{m} = \frac{1}{\Sigma}(\boldsymbol{\rho}_г \times \boldsymbol{\rho}_v) = \frac{1}{\Sigma}(z_u \boldsymbol{\tau} - (s' + vk_s)(z_v \mathbf{v} + \boldsymbol{k})), \tag{6}$$

$\Sigma = \sqrt{EG - F^2} = |(\boldsymbol{\rho}_г \times \boldsymbol{\rho}_v)| = \sqrt{(s' + vk_s)^2 (1 + z_v^2) + z_u^2}$ is a discriminant of the surface.

The second derivatives of the equation:

$$\boldsymbol{\rho}_{uu} = (s'' + vk_s')\boldsymbol{\tau} + (s' + vk_s)k_s \mathbf{v} + z_{uu}\boldsymbol{k}; \quad \boldsymbol{\rho}_{uv} = k_s \boldsymbol{\tau} + z_{uv}\boldsymbol{k}; \quad \boldsymbol{\rho}_{vv} = z_{vv}\boldsymbol{k}. \tag{7}$$

The coefficients of the second fundamental form:

$$L = (\boldsymbol{\rho}_{uu}\boldsymbol{m}) = \frac{(s'' + vk_s')z_u + (s' + vk_s)^2 k_s z_v - (s' + vk_s)z_{uu}}{\Sigma};$$

$$N = (\boldsymbol{\rho}_{vv}\boldsymbol{m}) = \frac{(s' + vk_s)z_{vv}}{\Sigma}; \quad M = (\boldsymbol{\rho}_{uv}\boldsymbol{m}) = \frac{-z_u k_s - (s' + vk_s)z_{uv}}{\Sigma}. \tag{8}$$

The curvatures of the surface:

$$k_u = \frac{L}{E} = \frac{(s'' + vk_s')z_u + (s' + vk_s)^2 k_s z_v - (s' + vk_s)z_{uu}}{\Sigma\left[(s' + vk_s)^2 + z_u^2\right]};$$

$$k_v = \frac{N}{G} = \frac{(s' + vk_s)z_{vv}}{\Sigma(1 + z_v^2)}; \quad k_{uv} = \frac{M}{\sqrt{EG}} = \frac{-z_u k_s - (s' + vk_s)z_{vv}}{\Sigma\left[(s' + vk_s)^2 + z_u^2\right]\sqrt{(1 + z_v^2)}}. \tag{9}$$

The coordinate system of the investigated surfaces isn't orthogonal and isn't conjugated in common, as the coefficients $F, M \neq 0$ and the coordinate system of the surfaces isn't the lines of principle curvatures of the surface.

The investigated system of the surfaces is related to the class of normal surfaces [4-6] – the surfaces with the system of plane coordinate lines (generating curves) at the normal plane of the directrix curve. At the works [4, 5] there was shown, that only for two kinds of normal surfaces the system of generating curves is the system of principle curvatures: 1- surfaces of rotation – directrix is a straight line, generating lines are circles; 2 – normal surfaces with the system of non-changed generating curve. This type of surfaces is related to the Monge's surfaces [5, 7-10].

If $z=z(v)$ – the generating curve doesn't change during moving in normal plane of the directrix ($z_u=z_{uu}=0$), there will be received the Monge's surfaces:

$$E = (s' + vk_s)^2; \quad G = (\boldsymbol{\rho}_v \boldsymbol{\rho}_v) = 1 + z_v^2; \quad F = 0; \quad \Sigma = (s' + vk_s)\sqrt{1 + z_v^2};$$

$$L = \frac{(s' + vk_s)k_s z_v}{\sqrt{1 + z_v^2}}; \quad N = \frac{z_{vv}}{\sqrt{1 + z_v^2}}; \quad M = 0;$$

$$k_1 == \frac{k_s z_v}{(s' + vk_s)\sqrt{1 + z_v^2}}; \quad k_2 = \frac{z_{vv}}{(1 + z_v^2)^{3/2}}. \tag{10}$$

The coordinate system of the Monge's surfaces is lines of principle curvatures of the surface.

If the generating curve will be a straight line $z = vtg\theta$ (θ is an angle of slope of generating strait line to the plane of base curve), then there will be received the torus surface of constant slope [10-12]. Then we'll receive:

$$z_v = tg\theta\,;\;\; z_{vv} = 0\,;\;\; 1 + z_v^2 = \frac{1}{\cos^2\theta}\,;\;\; \Sigma = \frac{s' + vk_s}{\cos\theta}\,;$$

$$E = (s' + vk_s)^2\,;\;\; G = \frac{1}{\cos^2\theta}\,;\;\; L = (s' + vk_s)k_s \sin\theta\,;\;\; N=0;\;\; k_1 == \frac{k_s z_v \sin\theta}{s' + vk_s}\,;\;\; k_2=0. \tag{11}$$

If the angle of slope of the generating strait line $\theta=0$, $z=0$, then will be received the trapezium- curved plate:

$$E = (s' + vk_s)^2\,;\;\; G = 1\,;\;\; L=N=0;\;\; k_1=k_2=0. \tag{12}$$

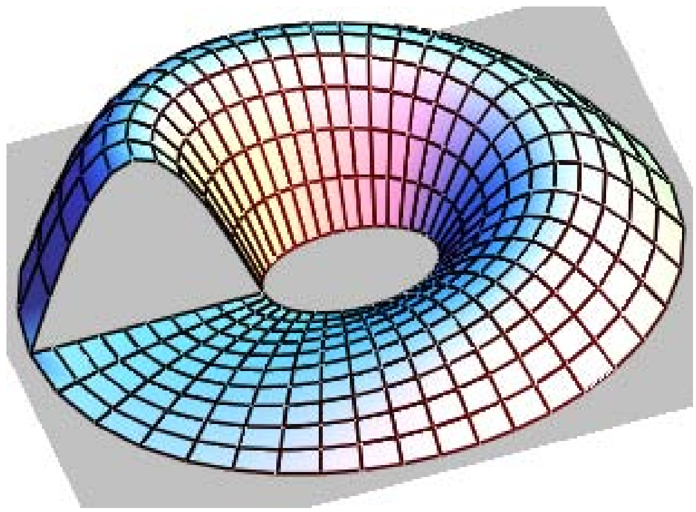

Fig. 2: Surface with ellipse directrix and sine generating curve

The geometric characteristics of surfaces with concrete directrix and generating curves will be received on the base of the common formulas of coefficient of the surfaces on trapezium-curved plans (3-11). On the fig. 2 there is shown the surface with ellipse as directrix and generating sine with linier change of its amplitude:

$$\mathbf{r}_0(u) = X(u)\mathbf{i} + Y(u)\mathbf{j}\,;\;\; X(u) = a\cos u\,;\;\; Y(u) = b\sin u\,;\;\; z(u,v) = c\frac{u}{2\pi}\sin\pi\frac{v}{d}\,;$$

$$u = 0 \div 2\pi\,;\;\; v = 0 \div d\,,$$

c is maximum amplitude of sine curve; d is the width of trapezium curved plan.

Determine parameters of the directrix ellipse and derivatives of generative curve:

$$s' = \sqrt{X'^2 + Y'^2} = a\sqrt{\eta}\,;\;\; \eta = \sin^2 u + \varepsilon^2\cos^2 u\,;\;\; \varepsilon = \frac{b}{a}\,;\;\; s'' = \frac{a}{2}\frac{\eta'}{\sqrt{\eta}}\,;\;\; \eta' = (1+\varepsilon^2)\sin 2u\,;$$

$$k = \frac{X'Y'' - X''Y'}{s'^3} = \frac{\varepsilon}{a\eta^{3/2}}\,;\;\; k_s = s'k = \frac{\varepsilon}{\eta}\,;\;\; k_s' = -\frac{\varepsilon}{\eta^2}\eta' = -\varepsilon(1+\varepsilon^2)\frac{\sin 2u}{\eta^2}\,;$$

$$z_u = \frac{c}{2\pi}\sin\pi\frac{v}{d}\,;\;\; z_{uu} = 0\,;\;\; z_{uv} = \frac{c}{2d}\cos\pi\frac{v}{d}\,;\;\; z_v = \frac{c}{2d}u\cos\pi\frac{v}{d}\,;\;\; z_{vv} = -\frac{c\pi}{2d^2}u\sin\pi\frac{v}{d}\,.$$

Coefficients of the fundamental forms:

$$E=\left(a\sqrt{\eta}+v\frac{\varepsilon}{\eta}\right)^2+\frac{c^2}{4\pi^2}\sin^2\pi\frac{v}{d}\quad;\quad G=1+\frac{c^2}{4d^2}u^2\cos^2\pi\frac{v}{d};\quad F=\frac{c^2}{8\pi d}u\sin 2\pi\frac{v}{d};$$

$$\Sigma=\sqrt{\left(a\sqrt{\eta}+v\frac{\varepsilon}{\eta}\right)^2\left(1+\frac{c^2}{4d^2}u^2\cos^2\pi\frac{v}{d}\right)+\frac{c^2}{4\pi^2}\sin^2\pi\frac{v}{d}}\ -$$

$$L=(\boldsymbol{\rho}_{uu}\boldsymbol{m})=c\frac{\frac{1+\varepsilon^2}{\pi}\left(\frac{a}{2\sin 2u}+\varepsilon\frac{v}{\eta^2}{}_s\right)\sin 2u\sin\pi\frac{v}{d}+\left(a\sqrt{\eta}+v\frac{\varepsilon}{\eta}\right)^2\frac{\varepsilon}{d}\frac{u}{\mu}\cos\pi\frac{v}{d}}{2\Sigma};$$

$$N=-\frac{c\pi}{2d^2}\frac{\left(a\sqrt{\eta}+v\frac{\varepsilon}{\eta}\right)u\sin\pi\frac{v}{d}}{\Sigma};\quad M=-c\frac{\frac{1}{\pi}\frac{\varepsilon}{\eta}\sin\pi\frac{v}{d}+\frac{1}{d}\left(a\sqrt{\eta}+v\frac{\varepsilon}{\eta}\right)\cos\pi\frac{v}{d}}{2\Sigma}. \tag{12}$$

On the fig. 2 there is shown the Monge's surface with evolvent of the circle as directrix:

$X(u)=a(\cos u+u\sin u)$; $Y(u)=a(\sin u-u\cos u)$ and generating sine $z=b\sin\frac{v}{d}$; $u=(1\div 5)\pi$;

$v=0\div d$;

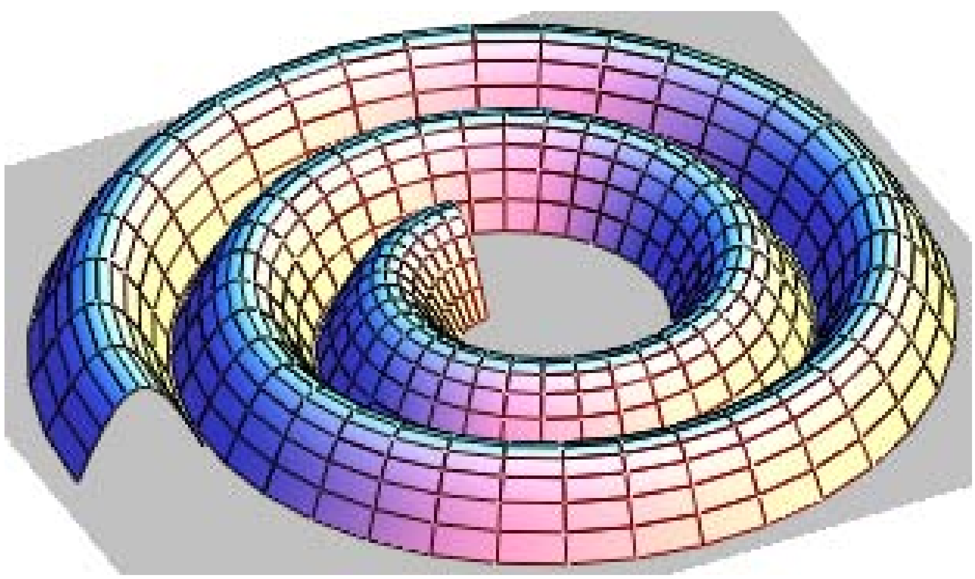

Fig. 3: Evolvent-sine Monge's surface

b is an amplitude of the sine, *d* is width of the sine (surface).
Parameters of directrix and generating curve:

$$s'=au;\quad s''=a;\quad k=\frac{1}{au};\quad k_s=1;\quad k_s'=0;$$

$$z_v=\frac{b}{d}\cos\frac{v}{d};\quad z_{vv}=-\frac{b}{d^2}\sin\frac{v}{d}.$$

Coefficients of the fundamental forms:

$$E=(au+v)^2;\quad G=(\rho_v\rho_v)=1+\frac{b^2}{d^2}cos^2\frac{v}{d};\ \Sigma=(au+v)\sqrt{1+\frac{b^2}{d^2}cos^2\frac{v}{d}};$$

$$L=\frac{b(au+v)cos\frac{v}{d}}{d\sqrt{1+\frac{b^2}{d^2}cos^2\frac{v}{d}}};\quad N=-\frac{b\,sin\frac{v}{d}}{d^2\sqrt{1+\frac{b^2}{d^2}cos^2\frac{v}{d}}};$$

$$k_1==\frac{b\,cos\frac{v}{d}}{d(au+v)\sqrt{1+\frac{b^2}{d^2}cos^2\frac{v}{d}}};\ k_2=-\frac{b\,sin\frac{v}{d}}{d^2\left(1+\frac{b^2}{d^2}cos^2\frac{v}{d}\right)^{3/2}}. \quad (13)$$

On fig. 4 there is shown the torus surface of constant slope with Bernoulli's lemniscate as directrix:

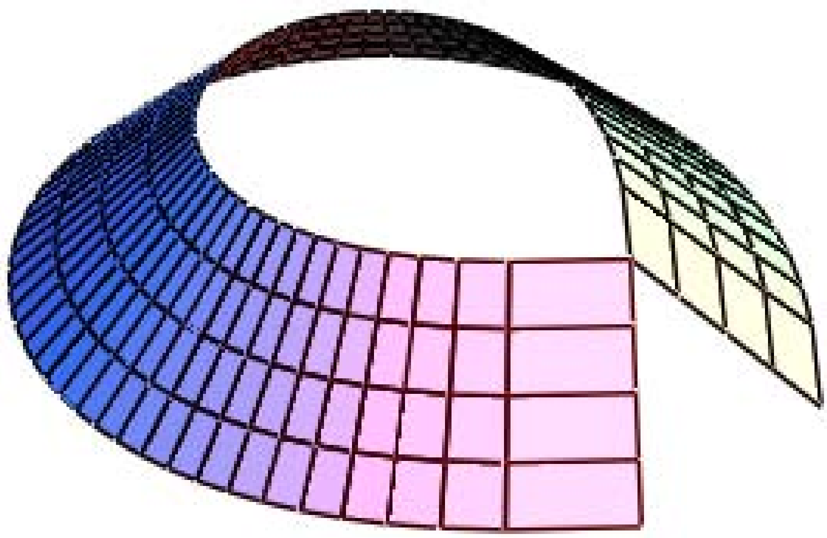

Fig. 4: Lemniscate surface of constant slope

$$X(u)=aR(u)cos\,u;\quad Y(u)=aR(u)sin\,u;$$

$$R(u)=\sqrt{2\,cos\,2u},\qquad u=(-1\div1)\pi/4.$$

$$s'=2\frac{a}{R(u)};\quad k=\frac{3}{2}\frac{R(u)}{a};\quad k_s=3.$$

The coefficients of fundamental forms and the curvatures of the surface:

$$E=\left(\frac{2a}{R(u)}+3v\right)^2;\quad G=\frac{1}{cos^2\theta};\quad L=3\left(\frac{2a}{R(u)}+3v\right)sin\,\theta;\quad k_1=3\frac{R(u)sin\,\theta}{2a+3R(u)v};\quad k_2=0. \quad (14)$$

Let us consider the linier surfaces which aren't surfaces of constant slope.
On fig. 5 there are shone wavy linear surfaces with different directrix curves.

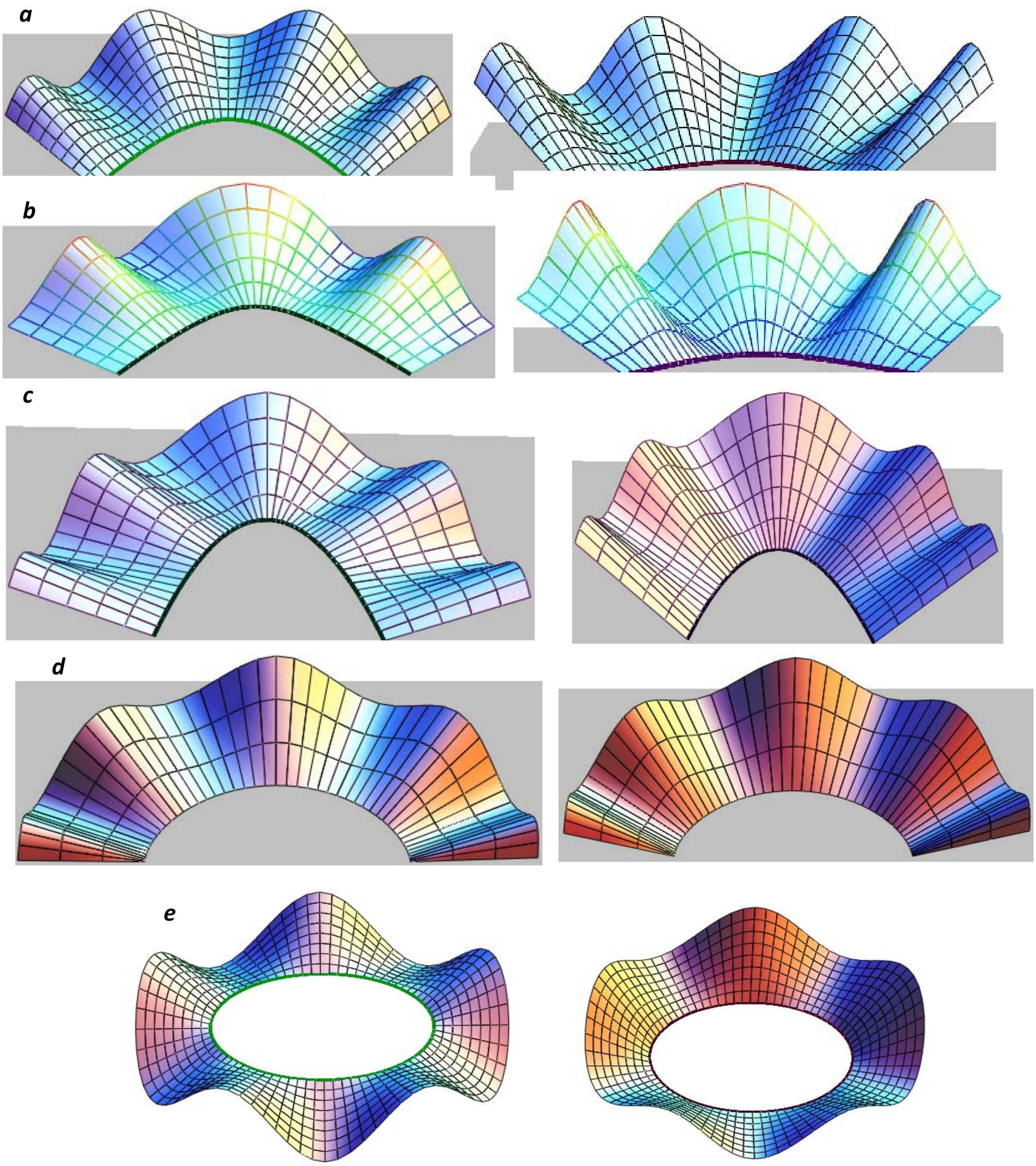

Fig. 5: Wavy linear surfaces with different directrix:
a – sine, *b* – hyperbola, *c* – parabola, *d* –cycloid, *e* - ellipse

The generating straight line at its moving along directrix made a wavy motion at the normal plane of the directrix: *a)* $z(u,v)=v(c+d\cos tu)$ or *б)* $z(u,v)=v(c+d\sin tu)$, $t=p\dfrac{\pi}{\Delta u}$, $\Delta u=u_k-u_n$ is a diapason of coordinate $v=(u_n \div u_k)$; p is a number of half waves of oscillation of the generating straight line; $c=\mathrm{tg}\theta$, θ is an angle of the generating line, around which it made the oscillations:

$$a)\quad z_u=-dtv\sin tu\,;\; z_{uu}=-dt^2v\cos tu\,;\quad z_v=c+d\cos tu\,;\;\; z_{uv}=-dt\sin tu\,;\;\; z_{vv}=0\,;$$

$$б)\quad z_u=dtv\cos tu\,;\; z_{uu}=-dt^2v\sin tu\,;\quad z_v=c+d\sin tu\,;\;\; z_{uv}=dt\cos tu\;\;\; z_{vv}=0\,. \qquad (15)$$

At the left row of the fig. 5 $\theta=0$, at the right row $\theta \neq 0$.

The coefficients of the fundamental forms and curvatures of these surfaces are determined on common formulas (4-9) with using formulas (19) and the fact that $N=0$, $k_v=0$.

If to take cycloid as the directrix (fig. 5,*d*) $X(u)=a(u\text{-}\sin u)$, $Y(u)=a(1\text{-}\cos u)$, $u=(0\div 2\pi)$, we'll receive:

$$s' = 2a\sin(u/2);\;\; s'' = a\cos(u/2);\;\; k = \frac{1}{4a\sin(u/2)};\;\; k_s = \frac{1}{2};\;\; k_s' = 0;$$

$$E = \left(2a\sin(u/2) + \frac{v}{2}\right)^2 + (dtv)^2\cos^2(tu);\quad G = 1 + (c + d\sin tu)^2;\; F = -dtv(c + d\sin tu)\sin tu;$$

$$\Sigma = \sqrt{\left[\left(2a\sin(u/2) + \frac{v}{2}\right)^2 + (dtv)^2\cos^2(tu)\right]\left[1 + (c + d\sin tu)^2\right] - \left[dtv(c + d\sin tu)\sin tu\right]^2}$$

$$L = \left[adtv\cos^2(u/2) + \left(2a\sin(u/2) + \frac{v}{2}\right)^2\frac{c + d\cos tu}{2} - \left(2a\sin(u/2) + \frac{v}{2}\right)dt^2 v\cos tu\right]\frac{1}{\Sigma};$$

$$M = \frac{dt}{2}\,\frac{v(\sin tu - \cos tu) - 4a\sin(u/2)\cos tu}{\Sigma};\quad N=0. \qquad (15)$$

II. Conclusion

The surfaces on the trapezium curved plans are formed by the moving of some generating curve at the normal plane of directrix curve. The generative curve may change its form when it is moving along the directrix, but has the constant wide of the plan. At the article there is received the vector equation of the surfaces on trapezium curved plans. On the base of the vector equation there are received the coefficients of the fundamental forms and the curvatures of the surfaces. If the function of vertical coordinates depends on coordinate parameter of the directrix (the form of generating curve changes at moving along the directrix), then the coordinate system of the surface isn't orthogonal and isn't conjugated. If along directrix there moving unchangeable curve then the coordinate lines of the surface are lines of principle curvatures and this type of surfaces is applied to the class of Monge's surfaces [5, 7-9]. On the base of common formulas there are received the formulas of geometric characteristics of the Monge's surfaces. If at the normal plane of the directrix there is moving a straight line with the constant slope to the directrix plane, then there will be received the torus surface of constant slope. Those type of surfaces belong to the class of Monge's surfaces as well.

On the base of common formulas of investigated class there are received the formulas of the surfaces and their geometric characteristics of the surfaces with concrete directrix and generating curves, as for surfaces of common type and so for Monge's and surfaces of constant slope. The using of common formulas made more simple the proses for receiving formulas for concrete surfaces. For every investigated surface there are given their figures.

Also there was investigated the type of wavy surfaces formed by the generating straight line which make oscillations at the normal plane of directrix. There are received the formulas of the geometric characteristics of this type of surfaces and given the figures of wavy line surfaces with some directrix lines.

The figures of the surfaces were made with using of vector equations of the surfaces in the "MathCad" system [5, 13]

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