Convergence Of Some Doubly Sequences Iterations With Errors In Banach Spaces

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Abstract—In this paper, we obtain convergence theorems of G-iteration process with errors in the doubly sequences settings. Furthermore, we give some examples to support our results. Finally, we apply this G-iteration process to obtain a solution of a nonlinear equation. AMS: 47H10, 54H25

About a subset $E$ of a closed convex subset $X$ of a real Banach space $X$ and some real positive constant $c$. The modulus of smoothness of $X$ is defined by $\rho_X(t) = \sup \{ \|x+y\| - \|x\| - \|y\| : \|x\| = \|y\| = t, t > 0 \}$.

If $\rho_X(t) > 0$ for all $t > 0$ then $X$ is said to be smooth. If these exist a constant $c > 0$ and a real number $1 < p < \infty$ such that $\rho_X(t) \leq C t^p$, then $X$ is said to be $p$-uniformly smooth Banach space, then the following geometric inequality holds (see e.g., [4, 5]):

$$\|x + y\| \leq \|x\| + p\|y\| + C_p \|y\|^p, x, y \in X,$$

and some real positive constant $C_p \geq 1$. If $T$ is a self-mapping of a closed convex subset $E$ of $X$ and $I$ the identity of $X$: Then, $T$ is a nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\| \quad \text{for all} \ x, y \in E.$$

Krasnoselskii [11] proved that the sequence of iteration $\{T_n(x)\}$, starting from a given point $x_0 \in E$, does not converge necessarily to a fixed point $T$; whereas the sequence $\{T^\omega(x_0)\}$, where $T^\omega = (1 - \lambda)I + \lambda T$, $0 < \lambda \leq 1$, may converge to a fixed point of $T$, as shown by Krasnoselskii [11] which assumed $\lambda = \frac{1}{2} I$. The above scheme has been extended by means of so-called, Mann iterative process (see [14]), associated with $T$ and described in the following way:

Let $x_0 \in E$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1 - c_n)x_n + c_n T(x_n),$$

for $n = 0, 1, 2, \ldots$; where

i. $0 < c_n < 1, \ n \geq 0$,

ii. $\lim_{n \to \infty} c_n = 0$,

iii. $\sum_{n=1}^{\infty} c_n = \infty$.

The scheme (2) has been studied by many authors (see for example [4, 10, 12, 15, 16, 19, 23, 25, 26, 27, 29]) and others. In 1986, Pathak [20] introduced a generalization of the scheme (2) for two self-mappings $S, T$ on a closed subset $E$ of a Banach space $X$ which described in the following way:

Let $x_1 \in E$ and $\{Sx_n\}$ and $\{Sx_n\}$ be a sequence defined by

$$Sx_{n+1} = (1 - t)Tx_n + tT_{n-1}x_n, \quad n = 1, 2, 3, \ldots, \quad t \in (0, 1).$$

and he proved that under certain conditions if the sequence $\{Sx_n\}$ converges, then it converges to the unique common fixed point of $S$ and $T$: It $S = I$, the identity mapping, the iteration (3) is identical with the iteration (2).

The scheme (2) has been extended by means of the so-called, G-iteration process (see [21, 22]) associated with a single mapping $T$ and described in the following manner:

Let $x_0 \in E$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (\mu_n - c_n)x_n + c_n T(x_n),$$

for $n = 0, 1, 2, \ldots$; where

i. $\mu_n = \mu_0 = 1$,

ii. $0 < c_n < 1, \ 0 \leq \mu_n \leq 1$ such that $\mu_n \geq \lambda_n, \ n > 0$,

iii. $\lim_{n \to \infty} \lambda_n = h > 0$,

iv. $\lim_{n \to \infty} \mu_n = 1$.

We note that when $\mu_n = 1$, the $G$-iteration process reduces to Mann iteration (2). The idea of considering fixed point iteration procedures with errors comes from practical numerical computations. This topic of research play important role in the stability problem of fixed point iterations. In 1995, Liu [13] initiated a study of fixed point iterations with errors. Several authors have proved some fixed point theorems for Mann type iteration with errors using several classes of mappings (see [4, 6, 7],

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[17, 18, 25, 28] and others). On the other hand there are some attempts in the the doubly sequence setting see [1, 2, 2, 18].

To introduce the meaning of doubly sequence iterations, we introduce the following concepts.

**Definition 1.1** - (see e.g [18]) Let N denote the set of all natural numbers and let N be a normed linear space. By a double sequence in N is meant a function \( f : N \times N \rightarrow N \) defined by \( f(n,m) = x_{n,m} \in N \).

The double sequence \( \{x_{n,m}\}_{n,m} \) is said to converge strongly to \( x \) if for a given \( \epsilon > 0 \) there exist integers \( M,N \geq 0 \) such that
\[
\forall n,M, m,N \geq 0, |x_{n,m} - x| < \epsilon.
\]

If
\[
\forall n,M, m,N \geq 0, |x_{n,m} - x| < \epsilon.
\]

then the double sequence is said to be Cauchy. Furthermore, if for each fixed \( n, m \), \( x_{n,m} \to x \) as \( n,m \to \infty \) and then \( x_{n,m} \to x \) as \( n,m \to \infty \).

Let \( E_1 \) be a nonempty closed convex subset of the Banach space \( X \) and \( T : E_1 \to E_1 \). Then, the Mann double sequence \( \{x_{k,n}\}_{k,n=0}^{\infty} \) generated from an arbitrary \( x_{0,0} \in E_1 \) is defined by
\[
x_{k,n+1} = (1 - \alpha_n)x_{k,n} + \alpha_nT_{k,n}, \quad k,n \geq 0.
\]

where \( \alpha_n \in [0,1] \). Now, we define the double Ishikawa iteration process with errors as follows:
\[
y_{k,n+1} = (1 - \beta_n)x_{k,n} + \beta_nT_{k,n} + \epsilon_{k,n},
\]
\[
x_{k,n+1} = (1 - \alpha_n)x_{k,n} + \alpha_nT_{k,n} + \delta_{k,n}, \quad k,n \geq 0.
\]

In the next, we give the equivalence between Mann and Ishikawa iterates with errors in the doubly sequence setting.

**Theorem 1.1** - Let \( X \) be a real Banach space with uniformly convex dual and \( E_1 \) be a nonempty closed convex subset of \( X \). Let \( T : E_1 \to E_1 \) be a continuous and strongly contractive mapping. Then, for \( x_{0,0} \in E_1 \), the following assertions are equivalent:

i. Doubly Mann iterates with errors converges to the fixed point of \( T \);

ii. Doubly Ishikawa iterates with errors converges to the fixed point of \( T \).

**Proof** - The proof is very similar to the proof of Theorem 2.1 in [24] with some simple modifications, so we will omit it.

**II. FIXED POINTS AND DOUBLY G-ITERATION PROCESS**

In this section we consider the G-iteration process associated with two self-mappings \( S, T \) on a closed convex subset of a normed space \( (N, ||.||) \) as given below. For \( x_0 \in N \), set
\[
S_{x_{n+1,k+1}} = (\mu_n - \lambda_n)S_{x_{n,k}} + \lambda_nT_{x_{n,k}} + (1 - \mu_n)T_{x_{n-1,k+1}} + (1 - \mu_n)u_{n,k},
\]

For \( n,k \geq 0 \) where \( \{\mu_n\} \) and \( \{\lambda_n\} \) satisfy (i), (ii), (iii) and (iv).

If \( k = 0 \), \( \{\mu_n\} = 1 \) and \( S = I \); the identity mapping, the iteration (6) is identical with the Mann iteration (2).

In this section, it is proved that for two mappings \( S \) and \( T \) which satisfy conditions (6) and (7) below, if the sequences of G-iteration associated with \( S \); \( T \) as in (6), then it converges to a common fixed point of \( S \) and \( T \).

The contractive conditions to be used are the following:
For all \( x,y \in N \),
\[
||T_x - T_y|| \leq \alpha||S_x - S_y|| + \beta||S_x - T_x|| + \gamma||S_y - T_x|| + \delta \max(||S_x - T_y||, ||S_y - T_y||),
\]

where, \( \alpha, \beta, \gamma, \delta \geq 0 \) with \( 0 < \alpha + \beta + \gamma + \delta < 1 \).

The second contraction condition is the following:
\[
||T_x - T_y|| \leq \alpha||S_x - T_x|| + \beta|||S_x - T_x||^p + \gamma||S_y - T_x|| + \delta \max(||S_x - T_y||, ||S_y - T_x||^p),
\]

where \( p > 0 \) and \( 0 < q - (\alpha + \beta) < 1 \).

First of all we prove the following theorem:

**Theorem 2.1** - Let \( S, T \) be mappings satisfying condition (7) and the following condition:
\[
S = T = I; \quad \text{where I denotes the identity mapping.}
\]

The proof is very similar to the proof of Theorem 2.1 in [24] with some simple modifications, so we will omit it.

**Proof** - For each \( n \geq 0 \), we have
\[
||S_{n+1,k+1} - TS|| \leq \alpha||S_{n,k} - TS|| + \beta||S_{n,k} - TS|| + \gamma||S_{n,k} - TS|| + \delta \max(||S_{n,k} - TS||, ||S_{n,k} - TS||).
\]

From (10) and (11), we obtain
\[
||S_{n,k} - TS|| \leq \alpha||S_{n,k} - TS|| + \beta||S_{n,k} - TS|| + \gamma||S_{n,k} - TS|| + \delta \max(||S_{n,k} - TS||, ||S_{n,k} - TS||).
\]

From (6), one gets
\[
||S_{n,k} - TS|| \leq \frac{1}{\lambda_n} ||\mu_nS_{n,k} - S_{n+1,k+1}|| + \frac{1}{\lambda_n} ||\mu_nT_{n-1,k+1}|| + ||u_{n,k}||.
\]

Substituting (13) in (12) and letting \( n; k \to \infty \), we obtain
\[
||z - TS|| \leq (1 - h(1 - \theta))||z - TS||.
\]

Since \( 0 < |1 - h(1 - \theta)| < 1 \), then we have
\[
TSz = z.
\]

i.e., \( z \) is a fixed point of \( S \).

Now using (9), we have
\[
Tz = T^2Sz = Sz \quad \text{and hence,} \quad STz = STSz = z.
\]

i.e.,
\[
TSz = STz = z.
\]

Again using (7), (9), (14), (15) and (16), we have
\[
||z - Tz|| \leq \alpha||STz - S|| + \beta||STz - T^2z|| + \gamma||Sz - TSz|| + \delta \max(||STz - Tz||, ||STz - Tz||)
\]

which implies that,
\[
||z - Tz|| \leq \eta||z - Tz||
\]

a contradiction. Since \( 0 < \eta = \alpha + \gamma + \delta < 1 \), it follows that \( Tz = z \); i.e., \( z \) is a fixed point of \( T \); but \( Tz = Sz \). So, we have
i.e., \( z \) is a common fixed point of \( S \) and \( T \).

Now to prove the uniqueness of \( z \), let \( w ( w \neq z ) \) be another common fixed point of \( S \) and \( T \). Then, we have

\[
|z - w| = |T(z) - T(w)| \leq \alpha |Stz - STw| + \beta |STz - T^2w| + \gamma |STw - T^2z| + \delta \max \{|STw - T^2w|, |STz - T^2w|\}
\]

\[
\leq \eta |z - w|,
\]

(18)

a contradiction, since \( 0 < \eta < 1 \), then \( z = w \). This completes the proof of the theorem.

Theorem 2.2 - Let \( K \) be a closed convex subset of a normed space \( N \). Let \( S, T : K \rightarrow K \) be mappings satisfying the conditions (8) and (9) as given before. Let \( \{Sx_n,t\} \) be the sequence of G-iteration process as given by (6). If \( \{Sx_n,t\} \) converges to \( z \) in \( K \) and if

\[
\max \{c, 2q\} < 1 + \alpha_1 + \alpha_2,
\]

then \( z \) is the unique common fixed point of \( S \) and \( T \).

Proof - We consider two cases.

Case (I) - When \( p \) is a positive integer. Consider,

\[
|Sx_{n+1,k} - TSz|^p \leq \max \{0, |Sx_n - Sx_{n-1,k} + TSz|, |Sx_n - TSz| \} \leq \max \{0, |Sx_n - Sx_{n-1,k}|, |Sx_n - TSz| \}
\]

(20)

Since \( S \) and \( T \) satisfy (8), then by using (9), we have

\[
|T(x) - TSz|^p \leq \max \{|Sx_n - Sx_{n-1,k}|, |Sx_n - TSz| \},
\]

(21)

Letting \( n, k \rightarrow \infty \) in (21), we obtain

\[
\lim_{n,k \rightarrow \infty} |T(x) - TSz|^p \leq \max \{|Sx_n - Sx_{n-1,k}|, |Sx_n - TSz| \},
\]

(22)

which implies that,

\[
\lim_{n,k \rightarrow \infty} ||T(x) - TSz||^p \leq \max \{|Sx_n - Sx_{n-1,k}|, |Sx_n - TSz| \},
\]

(23)

From (21), we obtain

\[
|Sx_{n+1,k} - TSz|^p \leq \Phi^p + \lambda_n \Phi^p |Sx_n - TSz|^p + \lambda_k \Phi^p |Sx_{n-1,k} - TSz|^p + \ldots + \lambda_k \Phi^p |Sx_1 - TSz|^p
\]

(24)

where,

\[
\Phi^p = \left( \gamma (q - a_1 + a_2) \right)^p |Sx_n - TSz|^p
\]

(25)

Hence, as \( n, k \rightarrow \infty \) (25) and using (24), we obtain

\[
|z - TSz|^p \leq \lim_{n,k \rightarrow \infty} \Phi^p + \lambda_n \Phi^p + \lambda_k \Phi^p |Sx_n - TSz|^p + \ldots + \lambda_k \Phi^p |Sx_1 - TSz|^p
\]

(26)

Now, we shall compute

\[
\lim_{n,k \rightarrow \infty} \Phi^p , \lim_{n,k \rightarrow \infty} \Phi^p \leq 1.
\]

From (25) if we letting \( n, k \rightarrow \infty \) we obtain

\[
\lim_{n,k \rightarrow \infty} \Phi^p = \left( 1 - h \right)^p |z - TSz|^p.
\]

(27)

Substituting from (27) in (26), we thus obtain

\[
|z - TSz|^p \leq \left( 1 - h \right)^p |z - TSz|^p + \alpha \left( 1 - h \right)^p |z - TSz|^p + \ldots + \alpha \left( 1 - h \right)^p |z - TSz|^p
\]

(28)

This implies that

\[
|z - TSz|^p \leq \lambda |z - TSz|^p,
\]

then \( TSz = z \). Now, let \( Tz \neq z \), then using (7), we have

\[
|z - Tz|^p \leq \max \left( 1 + \alpha_1 + \alpha_2 \right) |z - TSz|^p.
\]

(29)

Similarly, we can prove that \( STz = z \). Now, let \( Tz \neq z \), then using (7), we have

\[
|z - Tz|^p \leq \max \left( 1 + \alpha_1 + \alpha_2 \right) |z - TSz|^p.
\]

(30)

Therefore, as \( n, k \rightarrow \infty \) in (28), we obtain

\[
\lim_{n,k \rightarrow \infty} |z - TSz|^p \leq (1 - h)^p |z - TSz|^p,
\]

(31)

which implies that,

\[
|z - TSz|^p \leq \left( 1 - h \right)^p |z - TSz|^p
\]

(32)

this implies that,

\[
|z - TSz|^p \leq \lambda |z - TSz|^p,
\]

then \( TSz = z \), i.e., \( z \) is a fixed point of \( TS \), where

\[
0 < \lambda = \left( 1 - h \right)^p |z - TSz|^p < 1.
\]

Similarly, as in Case (I), we can prove that

\[
TSz = z,
\]

i.e., \( z \) is a common fixed point of \( S \) and \( T \). Now, let \( w ( w \neq z ) \) be another fixed point of \( S \) and \( T \):

Then using (8) and (9), we obtain

\[
|z - w|^p = |T(z) - T(w)|^p \leq \max \{c, |STz - STw|^p, |STz - T^2z|^p, |STz - T^2w|^p, |STz - T^3z|^p + |STz - T^3w|^p + |STz - T^4z|^p + |STz - T^4w|^p + |STz - T^5z|^p + |STz - T^5w|^p\}
\]

(33)

Hence,
\[ ||z - w||^p \leq \frac{q}{1 + a_1 + a_2} \max\{c||z - w||^p, 0, 2||z - w||^p\} \]
\[ = ||z - w||^p \frac{1}{1 + a_1 + a_2} \max\{c, 2q\}, \]
a contradiction. Hence \( z = w \): This completes the proof of the theorem.

III. EXAMPLES

In this section, we give some examples to discuss the validity of the hypothesis and degree of generality of some of our theorems.

Example 3.1 Let \( N = R^2 \); the set of all 2-tuples i.e., \( x = (x_1, x_2) \) of real numbers and the norm \( ||x|| \) is defined by
\[ ||x|| = \left( \sum_{i=1}^{2} |x_i|^2 \right)^{\frac{1}{2}}, \quad x \in R^2. \]

Further, let \( K = \{ x : ||x - O|| \leq 1, O, x \in R^2 \} \) and define the mappings \( S,T : K \to K \) such that for arbitrary \( x = (x_1, x_2) \in K \),
\[ Sx = (x_2, x_1), \]
and
\[ Tx = (-x_2, -x_1). \]

Suppose \( \{Sx_{n,k}\} \) be a sequence of elements of \( K \) satisfying condition (6) with \( a_{n,k} = 0 \) where
\[ \lambda_n = 1 - \frac{n}{2n + 1} \quad \text{and} \quad \mu_n = \frac{n + 3}{2n + 3} \quad \text{for} \quad n \geq 0. \]

Consider, \( x_{1,1} = (0.5; 0) \in K \), then it is easy to see that \( x_{2,1} = (0.166667; 0) \) and \( x_{3,1} = (0.21903; 0) \) etc:

Now it is easy to see that all conditions of Theorem 2.1 are satisfied, for instance taking \( x = x_{2,1} \) and \( y = x_{3,1} \), then we have 0.4713996 which is true, since
\[ 0 < \lambda_n < 1, \quad 0 \leq \mu_n \leq 1 \quad \text{such that} \quad \mu_n \geq \lambda_n, \quad n > 0, \]
\[ \lim_{n \to \infty} \lambda_n = 0 > 0, \]
\[ \lim_{n \to \infty} \mu_n = 1. \]

remains in \( B \) and converges strongly to \( x^* \) with convergence being at least as fast as geometric progression.

Proof: Since \( T \) is locally Lipschitz, there is an \( r > 0 \) such that \( T \) is Lipschitz on
\[ B = B(x_0) = \{ x \in X : ||x - x^*|| \leq r \} \subset D(T) \]

Let \( k \in (0, 1) \) and \( L > 1 \) denote the strong acrrettivity and Lipschitz constant of \( A \) respectively. Observe that \( f = Tx^* \).

\[ h_k = \left( \frac{k}{L-D(T)} \right)^{\frac{1}{r-1}} \]

and generate the sequence \( x_{n,k} \in B \) as in (6). We now prove that \( x_{n,k} \in B \), \( n, k > 0 \). Suppose that \( x_{n,k} \in B \), then
\[ ||x_{n+1,k+1} - x^*|| = \left| ||x_n - x_k|| + \lambda_n(x_k - x_{n,k}) + (1 - \mu_n)(x_{n,k} - x^*) \right| \]
\[ = \left| ||x_n - x_k|| + \lambda_n(x_k - x_{n,k}) + (1 - \mu_n)(x_{n,k} - x^*) \right| \]
\[ = ||x_n - x_k|| + \lambda_n(x_k - x_{n,k}) + (1 - \mu_n)(x_{n,k} - x^*) \]
\[ = x_{n,k} - x^* - (\lambda_n + (1 - \mu_n)||x_{n,k} - x^*||) \]

Now using (1), we obtain
\[ ||x_{n+1,k+1} - x^*||^p \]
\[ \leq ||x_{n,k} - x^* + (1 - \mu_n)x_{n,k}||^p + p(||x_n - x_k|| + \lambda_n(x_k - x_{n,k}) + (1 - \mu_n)(x_{n,k} - x^*))^p \]
\[ + C_p(\lambda_n + (1 - \mu_n)||x_{n,k} - x^*||)^p \]
\[ \leq (1 - \mu_n(||x_{n,k} - x^*||^p + (1 - \mu_n)||x_{n,k} - x^*||^p) + (1 - \mu_n)(x_{n,k} - x^*)) \]
\[ \leq (1 - \mu_n(||x_{n,k} - x^*||^p + (1 - \mu_n)||x_{n,k} - x^*||^p) \]
\[ \leq \left( 1 - \mu_n(||x_{n,k} - x^*||^p + (1 - \mu_n)||x_{n,k} - x^*||^p) \right) \]
\[ \leq \left( 1 - \mu_n(||x_{n,k} - x^*||^p + (1 - \mu_n)||x_{n,k} - x^*||^p) \right) \]

Where \( K(x_{n,k}, p) \) is a function depends on \( u_{n,k} \) and \( p \). Now, since \( x_{n,k} \in B \), by choice of the initial guess, it follows by the inductive hypothesis that the sequence \( \{x_{n,k}\} \) remains in \( B \).

Set
\[ \delta' = \left( 1 - (p - 1)||x_{n,k} - x^*||^p \right)^{\frac{1}{p}} \]
and observe that
\[ \delta' \in (0, 1), \quad \text{since} \quad \frac{1}{p} \leq \frac{1}{4} \quad \forall \quad 1 < p < \infty. \]

Hence, we obtain
\[ ||x_{n,k} - x^*||^p \leq \left( \delta' \right)^p \leq ||x_{n,k} - x^*||^p + (1 - \mu_n)K(u_{n,k}, p), \]
since \( \delta' \to 0 \) as \( n \to \infty \) the assertions of the theorem follows and the proof is complete.
V. REFERENCES


12) Kim, Tae-Hwa and Xu, Hong-kun.: Strong convergence of Modified Mann iterations for asymptotically nonexpansive mappings and semigroups, Nonlinear Analysis. Theory Methods Appl. 64 No. 5(A), 1140-1152 (2006).


