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Strong Summability With Respect To A Sequence Of Orlicz Functions

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Abstract- **In this paper we define the notion of strong summability by a sequence of Orlicz functions and examine its relationship with A-statistical convergence.**

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I. INTRODUCTION

An Orlicz function is a function , which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ *for* $x > 0$ and $M(x) \to \infty$ *as* $x \to \infty$. If convexity of M is replaced by subadditivity, then this function is called a modulus function see Maddox [11].

Lindendstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let s be the space of all real or complex sequences x = (x_k) ,

$$\ell_M = \{x \in s : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0\}$$

which is called an Orlicz sequence space. ℓM is a Banach space with the norm

$$||x|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

Also, it was shown [9] that every space ℓ_M contains a subspace isomorphic to ℓ_p (p ≥ 1).The spaces of strongly summable sequences were discussed by Maddox [10]. Parashar and Choudhary [12] defined these spaces by using the idea of Orlicz function as follows:

Let p = (p_k) be a sequence of positive real numbers and s be the space of all real sequences. Then

$$W_0(M,p) = \{x \in s : \frac{1}{n}\sum_{k=1}^{n}(M\left(\frac{|x_k|}{\rho}\right)^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0\},$$

$$W(M,p) = \{x \in s : x - \ell e \in W_0(M,p),\ \ell > 0\},$$

$$W_\infty(M,p) = \{x \in s : \sup_n \frac{1}{n}\sum_{k=1}^{n}(M\left(\frac{|x_k|}{\rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0\}.$$

If M(x) = x, then the above spaces are deduced to [C, 1, p]0, [C, 1, p] and [C, 1, p]∞ respectively. For pk = p > 0 for each k, we denote these sequence spaces by Wp 0 (M), Wp(M) ,and Wp∞ (M) respectively.

Let X be a Banach space and s(X) denote the space of all sequences x = (x_k) in X. A scalar matrix A = (ank)∞n,k is called regular on s(X) if A maps c(X) into c(X) and limn An(x) = limk xk in X. It is known that a matrix A is regular on s(X) if and only if it is regular on s. The necessary and sufficient conditions for A to be regular [5] on s are

$(i)\ \sup_n \sum_k |a_{nk}| < \infty,\ (ii)\ \lim_n a_{nk} = 0$ $\lim_n$ ank = 0 for each k, and $(iii)\ \lim_n \sum_k a_{nk} = 1.$ These are well-known Silverman-Toeplitz conditions see [5]. A matrix A is said to be uniformly regular if it is regular, $a_{nk} \geq 0$ and $\lim_n \sum_{k \geq n} |a_{nk}| = 0$ uniformly on $n \in \mathbb{N}$

An Orlicz function *M* is said to satisfy *Δ2* - condition for all values u, if there exists a constant K > 0, such that

$$M(2u) \leq KM(u), \quad (u \geq 0)$$

We define the following Definition. An Orlicz function M is said to satisfy Δ λ - condition for all values u if there exists a constant K > 0 such that

$M(\lambda u) \leq K\lambda M(u)$ for all $u \geq 0$ and $\lambda > 1$.

We define the following sequence spaces Let A = (ank)∞ n,k be a non-negative regular matrix and m = (Mk) a sequence of Orlicz functions such that each Mk satisfies Δ λ − condition. Then for p > 0

$$W_0^p(m,A,X) = \left\{x \in s(X) : \lim_n \sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^p = 0, \text{ for some } \rho > 0\right\},$$

$$W^p(m,A,X) = \{x \in s(X) : \text{ there exists } x_0 \in X, (x_k - x_0) \in W_0^p(m,A,X)\}.$$

For $x \in W^p(m,A,X)$, we write $x_k \to x_0\ (W^p(m,A,X))$.

If $M_k(x)$ = x for each k, then these spaces are reduced to $W_0^p(A,X)$ and $W^p(A,X)$ respectively, where

$$W^p(A,X) = \left\{x \in s(X) : \text{there exists } x_0 \in X \text{ such that } \lim_n \sum_k a_{nk}\|x_k - x_0\|^p = 0\right\}$$

If (M_k) is replaced by (f_k) a sequence of modulus functions , then the above spaces are reduced to the spaces defined by Kolk [7].

II. INCLUSION RELATIONS

In this section, we prove the following results.

Theorem 2.1.-

$W_0^p(A,X) \subset W_0^p(m,A,X)$ if and only if

$$\lim_{t \to 0^+} \sup_k M_k(t) = 0 \quad (t > 0). \qquad (1)$$

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***Proof*-**Let $W_0^p(A,X) \subset W_0^p(m,A,X)$.
If we take A = I (unit matrix), then this inclusion is reduced to

$$c_0(X) \subset c_0(m,X)$$

where

$$c_0(m,X) = \left\{ x \in s(X) : \lim_n \sum_k \left(M_k \left(\frac{\|x_k\|}{\rho} \right) \right) = 0, \text{for some } \rho > 0 \right\}$$

Suppose that (2.1.1) fails to hold. Then there exists a number $\varrho_0 >$ 0 and an index sequence (k_i) such that

$$M_{k_i}(t_i) \geq \epsilon_0 \quad (i = 1, 2 \cdots) \tag{2}$$

for a positive sequence (t_i) ∈ c0. Define the sequence x = (x_k) by

$$x_k = \begin{cases} t_i y, & \text{for } k = k_i \text{ and for a fixed } y \in X \text{ with } \|y\| = 1; \\ 0, & \text{if } k \neq k_i. \end{cases}$$

Then x ∈ c0(X) since ti ∈ c0 , and hence x ∈ c0(m,X) . On the other hand, by (2.1.2) and ∆ λ −condition we have

$$M_{k_i}\left(\frac{\|x_{k_i}\|}{\rho}\right) = M_{k_i}\left(\frac{t_i}{\rho}\right) \geq \frac{\epsilon_0}{K\rho}, K > 0, \ \rho > 1, \quad (i = 1, 2 \cdots)$$

i.e., x /∈ c_0(m,X) , a contradiction. Therefore (2.1.1) must hold.
Conversely, suppose that (2.1.1) holds. Then for every ϱ > 0 there exists a number δ such that 0 < δ < 1 and

$$M_k(t) < \frac{\epsilon^{\frac{1}{p}}}{\|A\|} \quad , k = 1, 2 \cdots \quad for \ t \leq \delta \tag{3}$$

For a sequence x = $(x_k) \in W_0^p(A,X)$ let so that limn T_n = 0. Now

$$\sum_k a_{nk} \left[M_k \left(\frac{\|x_k\|}{\rho} \right) \right]^p = \Sigma_1 + \Sigma_2 \tag{4}$$

where Σ1 is the sum over k such that $\frac{\|x_k\|}{\rho} \leq \delta$; and Σ2 is the sum over k such that $\frac{\|x_k\|}{\rho} \leq \delta$; Since A is regular and by (2.1.3), we have

$$\Sigma_1 < \epsilon \tag{5}$$

By (2.1.1), we have

$$\sup_k M_k(\delta) = H < \infty \text{ for } \frac{\|x_k\|}{\rho} > \delta > 0 \tag{6}$$

Since each M_k is non-decreasing and convex, we have by (2.1.6) and ∆ λ −condition that for K > 0

$$M_k\left(\frac{\|x_k\|}{\rho}\right) = M_k\left(\delta\delta^{-1}\frac{\|x_k\|}{\rho}\right),$$
$$\leq K\delta^{-1}\frac{\|x_k\|}{\rho}M_k(\delta)$$
$$< K\delta^{-1}H\frac{\|x_k\|}{\rho},$$

i.e.

$$\Sigma_2 < (K\delta^{-1}H)^p T_n. \tag{7}$$

Hence $\Sigma_2 \to 0$ as $n \to \infty$. Therefore $x \in W_0^p(m,A,X)$.

This completes the proof of the theorem.

Theorem 2.2.(a) If $W_0^p(m,A,X) \subset W_0^p(A,X)$ and (M_k) is pointwise convergent for $t > 0$ (it is not necessary that every M_k satisfies the $\triangle_\lambda$ condition) then

$$\inf_k M_k(t) > 0, \quad t > 0, \tag{a}$$

(b) If

$$\inf_k M_k(t) > 0, \ t > 0, \tag{a}$$

and every Mk satisfies the ∆ λ condition then $W_0^p(m,A,X) \subset W_0^p(A,X)$.

Proof.-(a) Let $W_0^p(m,A,X) \subset W_0^p(A,X)$. Suppose that (2.2.1) does not hold. Then

$$\inf_k M_k(t) = 0 \ (t > 0) \tag{b}$$

and thus we can choose an index sequence (ki) such that

$$M_{k_i}(t_0) < \frac{1}{i} \text{ for certain } t_0 > 0 \ (i = 1, 2, \cdots) \tag{c}$$

Now, define a sequence x = (x_k) by

$$x_k = \begin{cases} t_0 y, & \text{for } k = k_i \text{ where } y \in X \text{ with } \|y\| = 1 \text{ and } t_0 > 0; \\ 0 & \text{,otherwise.} \end{cases}$$

Then $\|x_k\| = \|x_{k_i}\| = t_0$, and so by (2.2.2) and (2.2.3) we get

$$\lim_k M_k\left(\frac{\|x_k\|}{\rho}\right) = 0$$

and hence

$$\lim_k \left[M_k\left(\frac{\|x_k\|}{\rho}\right)\right]^p = 0$$

Further by regularity of A, we have

$$\lim_n \sum_k a_{nk} \left[M_k\left(\frac{\|x_k\|}{\rho}\right)\right]^p = 0$$

i.e. $x = (x_k) \in W_0^p(m,A,X)$. But on the other hand

$$\lim_n \sum_k a_{nk}\|x_k\|^p = t_0^p \lim_n \sum_k a_{nk} = t_0^p,$$

Hence $x \notin W_0^p(A,X)$. Which contradicts that $W_0^p(m,A,X) \subset W_0^p(A,X)$. Hence (2.2.1)must hold.

(b) Conversely, let (2.2.1) hold and $x \in W_0^p(m,A,X)$. Suppose that $x \notin W_0(A,X)$. Then for some number ϱ0 > 0 and index k0 we have $\|x_{k_i}\| > \epsilon_0 \quad (i \in \mathbb{N})$ for some subsequence of indices (ki), since A is regular. Thus

$$M_k\left(\frac{\epsilon_0}{\rho}\right) < M_k\left(\frac{\|x_{k_i}\|}{\rho}\right) \quad \text{for some } \rho > 0,$$

and further by regularity of A, we have $\lim_k M_k \left(\frac{\epsilon_0}{\rho}\right) = 0$ $x \in W_0^p(A, X)$.
which contradicts (2.2.1). Hence
This completes the proof of the theorem.

III. A-STATISTICAL CONVERGENCE

In this section we find relation of A-statistical convergence with strong A-summability defined by a sequence m = (M_k) of Orlicz functions.
Let K = {ki} be an index set, i.e. precisely the sequence (ki) of indices. Let ϕ k be the characteristic sequence of K, i.e. ϕ k = (ϕ kj), where

$$\phi_j^k = \begin{cases} 1: & \text{If } j = k, \quad j = 1, 2 \cdots \\ 0 & \text{, otherwise.} \end{cases}$$

If ϕ k is (C,1)-summable then the limit

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \phi_j^k$$

is called the asymptotic density of *K* and is denoted by δ (K). An index set K = {k_i} is said to have A-density if

$$\delta_A(K) = \lim_n A_n \phi^k$$
$$= \lim_n \sum_{k \in K} a_{nk}$$

exists, where $A = (a_{nk})_{n,k=1}^{\infty}$ is a non-negative regular matrix [cf,6].
The idea of statistical convergence was introduced by Fast [2] and studied by various authors, e.g. by ˘ Sal´at [13], Freedman and Sember [3], Fridy [4] Connor [1], and Kolk[6] .
sequence x = (x_k) ∈ s(X) is said to be A-statistically convergent to x0, [6] i.e. $x_k \to x_0(s_A(X))$ if for every ϱ > 0, $\delta_A(L_\varrho) = 0$, where $L_\varrho = \{k : \|x_k - x_0\| \geq \varrho\}$. We denote by SA(X) the set of all A-statistically convergent sequences in X. If *A* is C1−matrix, then A-statistical convergence is reduced to the statistical convergence.
Example. Define x_k if k is a square and xk = 0 otherwise. then $|\{k \leq n : x_k \neq 0\}| \leq (n)^{1/2}$, so $x = (x_k)$
Theoem 3.1.- Let A be uniformly regular matrix and the orlicz functions sequence m = (M_k) be a pointwise convergent. Then

$$x_k \to x_0(W_0^p(m, A, X)) \Rightarrow x_k \to x_0(S_A(X)),$$

if and only if

$$\lim_k M_k(t) > 0 \quad (t > 0). \tag{1}$$

Proof. Let ϱ > 0 . Then as in [8 , Theorem 3.8] we can find numbers s > 0 and r ∈ IN such that

$$\sum_{\substack{k \in L_\epsilon \\ k \geq r}} a_{nk} \leq s^{-p} \sum_{k \geq r} a_{nk} \left[M_k\left(\frac{\|x_k - x\|}{\rho}\right)\right]^p. \tag{2}$$

Where
$L_\epsilon = \{k : \|x_k - x_0\| \geq \epsilon\}$. Since $x_k \to x_0(W_0^p(m, A, X))$ implies that $\delta_A(L_\epsilon) = \lim_n \sum_{k \in L_\epsilon} a_{nk} = 0$. Therefore $x_k \to x_0(S_A(X)$
Conversely, suppose that $x_k \to x_0(W_0^p(m, A, X)) \Rightarrow x_k \to x_0(S_A(X))$. If (3.1.1.) is not true, we have

$$\lim_k M_k(t_0) = 0 \text{ for some } t_0 > 0.$$

Since *A* is uniformily regular, by Lemma 2.4. of Kolk [8], there exists an infinite index set K = (k_i) with $\delta_A(K) = 0$. Define a sequence y = (y_k) by

$$y_k = \begin{cases} 0, & k \in K, \\ t_0 z, & \text{otherwise;} \end{cases}$$

where z ∈ X with llzll = 1. Then

$$\lim_k \left[M_k\left(\frac{\|y_k\|}{\rho}\right)\right]^p = 0,$$

and by the regularity of A we have

$$y_k \to 0\,(W^p(m, AX))\,.$$

But for $0 < \epsilon \leq t_0$,

$$\delta_A(k : \|y_k\| \geq \epsilon) = \lim_n \sum_k a_{nk} - \delta_A(K) = 1 - 0 = 1.$$

Thus yk does not → 0(S_A(X)) , i.e. contradiction to the hypothesis. Hence (3.1.1) must hold. This completes the proof of the theorem.

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