A Subset Of The Space Of The Orlicz Space Of Gai Sequences

N.Subramanian¹, S.Krishnamoorthy², S. Balasubramanian³

Abstract
Let ~M denote the space of all Orlicz space gai sequences. Let M denote the space of all Orlicz space of analytic sequences. This paper is devoted to a study of the general properties of sectional space (~M)s of ~M.

Keywords: Sectional sequences, gai sequences, analytic sequences, Orlicz sequences.

2000 Mathematics subject classification: 40A05, 40C05, 40D05.

1 Introduction

A Complex sequence, whose kth term is xk is denoted by {xk} or simply x. Let φ be the set of all finite sequences. A sequence x = {xk} is said to be analytic if supk |xk|1/k < ∞. The vector space of all analytic sequence will be denoted by Λ . χ was discussed in Kamthan [13]. Matrix transformation involving ~ were characterized by Sridhar [21] and Sirajiudeen [22]. A sequence x is called entire sequence if limk!1 (k! xk)|1/k = 0. The vector space of all gai sequences will be denoted by χ. Kizmaz [19] defined the following difference sequence spaces

Z (Δ) = {x = (xk) : ∆x ∈ Z}

for Z = ℓ∞, c, c0, where ∆x = (Δx)k=1 = (xk − xk+1)k=1 and showed that these are Banach space with norm ∥x∥ = ∥x1∥ + ∥∆x∥∞. Later on Et and Colak [20] generalized the notion as follows : Let m ∈ ℜ

Z (Δm) = {x = (xk) : ∆mx ∈ Z} for Z = ℓ∞, c, c0 where m ∈ ℜ, Δ0x = (xk), Δx = (xk − xk+1), Δmx = (∆mx)k=1 = (∆m−1xk − ∆m−1xk+1)k=1.

The generalized difference has the following binomial representation:

Δmxk = ∑mγ=0 (−1)γ (mγ) xk+γ

They proved that these are Banach spaces with the norm

∥x∥Δ = ∑mγ=0 |xγ| + ∥Δmx∥∞

Orlicz [1] used the idea of Orlicz function to construct the space (L^M). Lin denstrauss and Tzafriri [2] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓM contains a subspace isomorphic to ℓp(1 ≤ p < ∞). Subsequently different classes of sequence spaces were defined by Parashar and Choudary[3], Mursaleen et al.[4], Bektas and Altin[5], Tripathy et al.[6], Rao and subramanian[7],and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [8]. An Orlicz function is a function M : [0, ∞) → [0, ∞) which is continuous, non-decreasing and convex with M(0) = 0, M(x) > 0, for x > 0 and M(x) → ∞ as x → ∞. If convexity of Orlicz function M is replaced by M(x + y) ≤ M(x) + M(y), then this function is called modulus function, introduced by Nakano[18] and further discussed by Ruckle[9] and Maddox[10], and many others.

Department of Mathematics, SASTRA University, Tanjore-613 402, India
nsmaths@yahoo.com

Department of Mathematics, Government Arts College(Autonomus), Kumbakonam-612 001, India.
drsk 01@yahoo.com

Department of Mathematics, Government Arts College(Autonomus), Kumbakonam-612 001, India
balssunjay@gmail.com
Orlicz [1] used the idea of Orlicz function to construct the space \((L^M)\). Lindenstrauss and Tzafriri [2] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space \(\ell_M\) contains a subspace isomorphic to \(\ell_p(1 \leq p < \infty)\). Subsequently different classes of sequence spaces were defined by Parashar and Choudhary[3], Mursaleen et al.[4], Bektas and Altın[5], Tripathy et al.[6], Rao and subramanian[7],and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [8].

An Orlicz function is a function 
\[
M : [0, \infty) \to [0, \infty) \quad \text{which is continuous, non-decreasing and convex with } M(0) = 0, M(x) > 0, \text{ for } x > 0 \text{ and } M(x) \to \infty \text{ as } x \to \infty.
\]
If convexity of Orlicz function \(M\) is replaced by \(M(x+y) \leq M(x) + M(y)\), then this function is called modulus function, introduced by Nakano[18] and further discussed by Ruckle[9] and Maddox[10], and many others.

An Orlicz function \(M\) is said to satisfy \(\Delta_2\) condition for all values of \(u\), if there exists a constant \(K > 0\), such that \(M(2u) \leq KM(u)(u \geq 0)\). The \(\Delta_2\) condition is equivalent to \(M(\ell u) \leq K\ell M(u)\), for all values of \(u\) and for \(\ell > 1\).

Lindenstrauss and Tzafriri[2] used the idea of Orlicz function to construct Orlicz sequence space

\[
\ell_M = \{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \}.
\]

where \(w = \{ \text{all complex sequences} \}\). The space \(\ell_M\) with the norm

\[
\|x\| = \inf \{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \}
\]

becomes a Banach space which is called an Orlicz sequence space. For \(M(t) = t^p, 1 \leq p < \infty\), the space \(\ell_M\) coincide with the classical sequence space \(\ell_p\).

Given a sequence \(x = \{x_k\}\) its \(n^{th}\) section is the sequence \(x^{(n)} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}\) \(\delta\) \((n) = (0, 0, \ldots, 1, 0, 0, \ldots)\), 1 in the \(n^{th}\) place and zero’s else where and \(s_k = \{0, 0, 0, \ldots, 1, -1, 0, 0, \ldots\}\), 1 in the \(n^{th}\) place and -1 in the \((n+1)^{th}\) place and zero’s else where.

If \(X\) is a sequence space, we define

(i) \(X'\) the continuous dual of \(X\).
(ii) \(X^\alpha = \{ a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X \}\);
(iii) \(X^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X \}\);
(iv) \(X^\gamma = \{ a = (a_k) : \sup_{n} |\sum_{k=1}^{n} a_k x_k| < \infty, \text{ for each } x \in X \}\);
(v) Let \(X\) be an FK-space \(\phi \phi\). Then \(X^f = \{ f(\delta^{(n)}) : f \in X' \}\). \(X^\alpha, X^\beta, X^\gamma\) are called the \(\alpha\)--(or Kō the-T öplitz) dual of \(X\), \(\beta\)--(or generalized Kō the-T öplitz) dual of \(X\), \(\gamma\)--dual of \(X\). Note that \(X^\alpha \subset X^\beta \subset X^\gamma\).

If \(X \subset Y\) then \(Y^\mu \subset X^\mu\), for \(\mu = \alpha, \beta, \text{ or } \gamma\).

An FK-space(Fréchet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals \(p_k(x) = x_k (k = 1, 2, \ldots)\) are continuous. We recall the following definitions[see [14]]. An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric space \((X, d)\) is said to have AK (or sectional convergence) if and only if \(d(x^{(n)} x) \to 0\) as \(n \to \infty\)[see 14].
The space is said to have AD or be an AD space if $\phi$ is dense in $X$. We note that AK implies AD by [11].

2 Definitions and Preliminaries

Throughout the paper $w, \chi_M$ and $\Lambda_M$ denote the spaces of all, Orlicz space of gai sequences and Orlicz space of bounded sequence respectively. Let $w$ denote the set of all complex sequences $x = (x_k)_{k=1}^\infty$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function. Let $t$ denote the sequence with $t_k = |x_k|^{1/k}$ for all $k \in \aleph$. Define the sets

$$\chi_M = \left\{ x \in w : \left( M \left( \frac{k!t_k}{\rho} \right) \right) \to 0 as k \to \infty for some \rho > 0 \right\}$$

$$\Lambda_M = \left\{ x \in w : \sup_k \left( M \left( \frac{t_k}{\rho} \right) \right) < \infty for some \rho > 0 \right\}$$

The space $\chi_M$ is a metric space with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{(k!|x_k - y_k|)^{1/k}}{\rho} \right) \right) \leq 1 \right\}$$

The space $\Lambda_M$ is a metric space with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left( M \left( \frac{|x_k - y_k|^{1/k}}{\rho} \right) \right) \leq 1 \right\}$$

Let $(\chi_M)_s = \{ x = x_k : \xi = \xi_k \in \chi_M \}$, where $\xi_k = 1!x_1 + 2!x_2 + \cdots + k!x_k$ for $k = 1, 2, 3, \cdots$ and $(\Lambda_M)_s = \{ y = y_k : \eta = \eta_k \in \Lambda_M \}$, where $\eta_k = y_1 + y_2 + \cdots + y_k$ for $k = 1, 2, 3, \cdots$. Then $(\chi_M)_s$ is a metric spaces with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left\{ M \left( \frac{|\xi_k - \eta_k|^{1/k}}{\rho} \right) : k = 1, 2, 3, \cdots \right\} \leq 1 \right\}$$

and $(\Lambda_M)_s$ is a metric spaces with the metric

$$d(x, y) = \inf \left\{ \rho > 0 : \sup_k \left\{ M \left( \frac{|\xi_k - \eta_k|^{1/k}}{\rho} \right) : k = 1, 2, 3, \cdots \right\} \leq 1 \right\}$$

Let $\sigma (\chi_M)$ denote the vector space of all sequences $x = (x_k)$ such that $\frac{\xi_k}{k}$ is an Orlicz space of gai sequence. We recall that $c_0$ denotes the vector space of all sequences $x = (x_k)$ such that $(\xi_k)$ is a null sequence.

3 Lemma

(see (14, Theorem 7.2.7)) Let $X$ be an FK-space $\supset \phi$. Then (i) $X^\gamma \subset X^f$. (ii) If $X$ has AK, $X^\beta = X^f$. (iii) If $X$ has AD, $X^\beta = X^\gamma$.

We note that $\chi_M^\beta = \chi^\beta_M = \chi_M^\gamma = \chi^\gamma_M = \Lambda$ for all $x = \{x_k\}$ and $y = \{y_k\}$ in $\chi_M$. 

Global Journal of Science Frontier Research  Vol. 10 Issue 6 (Ver 1.0), October 2010  Page | 55
4 Remark

\[ x = (x_k) \in \sigma(\chi_M) \iff \{\xi_k\} \in \chi_M \iff M\left(\frac{|\xi_{1/k}|}{\rho}\right) \to 0 \text{ as } k \to \infty. \iff M\left(\frac{|\xi_{1/k}|}{\rho}\right) \to 0 \text{ as } k \to \infty, \text{ because } k^{1/k} \to 1 \text{ as } k \to \infty \iff x = (x_k) \in (\chi_M)_s. \text{ Hence } (\chi_M)_s = \sigma(\chi_M), \text{ the cesàro space of order 1.} \]

5 Proposition

\((\chi_M)_s \subset \chi_M\)

Proof: Let \(x \in (\chi_M)_s\)

\(\Rightarrow \xi \in \chi_M\)

\(M\left(\frac{|\xi_{1/k}|}{\rho}\right) \to 0 \text{ as } k \to \infty\) \quad \quad (3)

But \(x_k = \xi_k - \xi_{k-1}\).

Hence \(M\left(\frac{|(k||x_k||)^{1/k}}{\rho}\right) \leq \left(M\left(\frac{|\xi_{1/k}}{\rho}\right)\right) + \left(M\left(\frac{|\xi_{1/k}}{\rho}\right)\right) \leq \left(M\left(\frac{|\xi_{1/k}}{\rho}\right)\right) + \left(M\left(\frac{|\xi_{1/k}}{\rho}\right)\right) \to 0 \text{ as } k \to \infty \text{ by using 3.} \)

Therefore \(M\left(\frac{|(k||x_k||)^{1/k}}{\rho}\right) \to 0 \text{ as } k \to \infty.\)

\(\Rightarrow x \in \chi_M. \text{ Hence } (\chi_M)_s \subset \chi_M.\)

Note: The above inclusions is strict. Take the sequence \(\delta^{(1)} \in \chi_M\). We have

\(M\left(\frac{|\xi_{1/k}}{\rho}\right) = 1\)

\(M\left(\frac{|\xi_{2/k}}{\rho}\right) = 1 + 0 = 1\)

\(M\left(\frac{|\xi_{3/k}}{\rho}\right) = 1 + 0 + 0 = 1\)

\(\vdots\)

\(M\left(\frac{|\xi_{k/k}}{\rho}\right) = 1 + 0 + 0 + \cdots + = 1\)

\(\to k - \text{terms} \to\)

and so on. Now \(M\left(\frac{|\xi_{1/k}}{\rho}\right) = 1\) for all \(k\). Hence \(\{M\left(\frac{|\xi_{1/k}}{\rho}\right)\}\) does not tend to zero as \(k \to \infty.\) So \(\delta^{(1)} \notin (\chi_M)_s.\) Thus the inclusion \((\chi_M)_s \subset \chi_M\) is strict. This completes the proof.

6 Proposition

\((\chi_M)_s\) has AK-property

Proof: Let \(x = (x_k) \in (\chi_M)_s\) and take \(x^{[n]} = (x_1, x_2, \ldots, x_n, 0, \ldots)\) for \(n = 1, 2, 3, \ldots.\)

Hence \(d(x, x^{[n]}) = \inf\left\{\rho > 0 : \sup_k \left\{M\left(\frac{|\xi_{k-\xi_{(n)}}_{1/k}}{\rho}\right)\right\} \leq 1\right\}\)

\(= \inf\left\{\rho > 0 : \sup_k \left\{M\left(\frac{|\xi_{n+1-\xi_{(n)}}_{1/n+1}}{\rho}\right), M\left(\frac{|\xi_{n+2-\xi_{(n)}}_{1/n+2}}{\rho}\right), \ldots \right\} \leq 1\right\}\)
Therefore $x[n] \to x$ as $n \to \infty$ in $(\chi_M)_s$. Thus $(\chi_M)_s$ has AK. This completes the proof.

7 Proposition

$(\chi_M)_s$ is a linear space over the field $C$ of complex numbers.

Proof: It is easy. Therefore omit the proof.

8 Proposition

$(\chi_M)_s$ is solid

Proof: Let $|x_k| \leq |y_k|$ with $y = (y_k) \in (\chi_M)_s$. So, $|\xi_k| \leq |\eta_k|$ with $\eta = (\eta_k) \in \chi_M$. But $\chi_M$ is solid. Hence $\xi = (\xi_k) \in \chi_M$. Therefore $x = (x_k) \in (\chi_M)_s$. Hence $(\chi_M)_s$ is solid. This completes the proof.

9 Proposition

$\Lambda \subset (\Gamma_M)_s \subset \Lambda_M (\Delta)$

Proof: Step 1. By Proposition 5., we have $(\chi_M)_s \subset \chi_M$. Hence $(\chi_M)_s^\beta \subset [(\chi_M)_s]^\beta$. But $(\chi_M)^\beta = \Lambda$. Therefore

$$\Lambda \subset [(\chi_M)_s]^\beta \quad (4)$$

Step 2: Let $y = (y_k) \in [(\chi_M)_s]^\beta$. Consider $f(x) = \sum_{k=1}^\infty x_k y_k$ with $x \in (\chi_M)_s$. Take $x = \delta^n - \delta^{n+1} = (0, 0, 0, \cdots, 1, -1, 0, 0, \cdots)$ in the $n^{th}$ place and zero's elsewhere. Then $f(\delta^n - \delta^{n+1}) = y_n - y_{n+1}$. Hence

$$M \left( \frac{|y_n - y_{n+1}|}{\rho} \right) = M \left( \frac{|f(\delta^n - \delta^{n+1})|}{\rho} \right) \leq \|f\| d((\delta^n - \delta^{n+1}), 0) \leq \|f\| \cdot 1.$$ 

So, $M \left\{ \left( \frac{y_n - y_{n+1}}{\rho} \right) \right\}$ is bounded. Consequently $M \left\{ \left( \frac{y_n - y_{n+1}}{\rho} \right) \right\} \in \Lambda_M$. That is $M \left( \frac{y_n}{\rho} \right) \in \Lambda_M (\Delta)$. But $y = (y_n)$ is originally in $[(\chi_M)_s]^\beta$. Therefore

$$[(\chi_M)_s]^\beta \subset \Lambda_M (\Delta) \quad (5)$$

From (4) and (5) we conclude that $\Lambda \subset [(\chi_M)_s]^\beta \subset \Lambda_M (\Delta)$. This completes the proof.
10 Proposition

The $\beta$– dual space of $(\chi_M)_s$ is $\Lambda$

**Proof:** **Step1.** Let $y = (y_k)$ be an arbitrary point in $[(\chi_M)_s]^\beta$. If $y$ is not in $\Lambda$, then for each natural number $n$, we can find an index $k(n)$ such that

$$M\left(\frac{|y_k(n)|^{1/k(n)}}{\rho}\right) > \frac{n!}{n^k}, \quad (n = 1, 2, 3, \ldots)$$

Define $x = (x_k)$ by

$$M\left(\frac{(k! x_k)}{\rho}\right) = 1/n^k, \text{ for } k = k(n); \text{ and}$$

$$M\left(\frac{(k! x_k)}{\rho}\right) = 0 \text{ otherwise}$$

Then $x$ is in $\chi_M$, but for infinitely many $k$,

$$M\left(\frac{(k! |y_k x_k|)}{\rho}\right) > 1 \quad (6)$$

Consider the sequence $z = \{z_k\}$, where $z_1 = 1!x_1 - s$ with $s = k! \sum x_k$; and $z_k = k! x_k \ (k = 2, 3, \ldots)$. Then $z$ is a point of $\chi_M$. Also $\sum M\left(\frac{z_k}{\rho}\right) = 0$. Hence $z$ is in $(\chi_M)_s$. But by the equation $(6)$ $\sum M\left(\frac{(k! z_k x_k)}{\rho}\right)$ does not converge. Thus the sequence $y$ would not to be in $[(\chi_M)_s]^\beta$. This contradiction proves that

$$[(\chi_M)_s]^\beta \subset \Lambda \quad (7)$$

**Step2.** By (4) of Proposition 9, we have

$$\Lambda \subset [(\chi_M)_s]^\beta \quad (8)$$

From (7) and (8) it follows that the $\beta$– dual space of $[(\chi_M)_s]^\beta$ is $\Lambda$. This completes the proof.

11 Proposition

$[(\chi_M)_s]^\mu = \Lambda$ for $\mu = \alpha, \beta, \gamma, f$

**Step1:** $(\chi_M)_s$ has AK by Proposition 6. Hence by Lemma 3(i) we get $[(\chi_M)_s]^\beta = [(\chi_M)_s]^f$. But $[(\chi_M)_s]^\beta = \Lambda$. Hence

$$[(\chi_M)_s]^f = \Lambda \quad (9)$$
Step 2: Since AK implies AD. Hence by Lemma 3(iii) we get \([\chi_M]^\beta = [\chi_M]^\gamma\). Therefore
\[ [\chi_M]^\gamma = \Lambda \] (10)

step3: \((\chi_M)_s\) is normal by Proposition 8. Hence, by [20, Proposition 2.7], we get
\[ [\chi_M]^\alpha = [\chi_M]^\beta = [\chi_M]^\gamma = [\chi_M]^f = \Lambda \] (11)

From (9), (10) and (11), we have
\[ [\chi_M]^\alpha = [\chi_M]^\beta = [\chi_M]^\gamma = [\chi_M]^f = \Lambda \]

12 Proposition

The dual space of \((\chi_M)_s\) is \(\Lambda\). In other words \([\Gamma_M]^* = \Lambda\)

Proof: We recall that \(s^k\) has 1 in the \(k\)th place, -1 in the \((k + 1)\)th place and zero’s else where with
\[ \xi = s^k, \{ M \left( \frac{|\xi|/k}{\rho} \right) \} = \left\{ \frac{M(1/1)}{\rho}, \frac{M(1/2)}{\rho}, \ldots, \frac{M(1/k)}{\rho}, \frac{M(1/k+1)}{\rho} \right\} = \left\{ 0, 0, \ldots, \frac{M(1/k)}{\rho}, 0, \ldots \right\} \]
which is exits. Hence \(s^{(k)} \in (\chi_M)_s\). \(f(x) = \sum_{k=1}^{\infty} \xi_k y_k\) with \(\xi \in (\chi_M)_s\) and \(f \in [(\chi_M)_s]^*\), where \([(\chi_M)_s]^*\) is the dual space of \((\chi_M)_s\). Take \(\xi = s^k \in (\chi_M)_s\). Then
\[ |y_k| \leq ||f|| d s^k, 0) < \infty \text{forall } k \] (12)

Thus \((y_k)\) is a bounded sequence and hence an analytic sequence. In other words, \(y \in \Lambda\). Therefore \([(\chi_M)_s]^* = \Lambda\). This completes the proof.

13 Lemma

[19, Theorem 8.6.1] \(Y \supset X \iff Y^f \subset X^f\) where \(X\) is an AD-space and \(Y\) an FK-space

14 Proposition

Let \(Y\) be any FK-space \(\supset \phi\). Then \(Y \supset (\chi_M)_s\) if and only if the sequence \(\delta^{(k)}\) is weakly analytic

Proof: The following implications establish the result.
\(Y \supset (\chi_M)_s \iff Y^f \subset [(\chi_M)_s]^f\), since \((\chi_M)_s\) has AD and by Lemma 13.
\(\iff y^f \subset \Lambda\), since \([(\chi_M)_s]^f = \Lambda\).
\(\iff \text{foreach } f \in Y', \text{the topological dual of } Y.f \delta^{(k)} \in \Lambda\).
\(\iff f \delta^{(k)}\) is analytic
\(\iff \delta^k\) is weakly analytic. This completes the proof.
15 Proposition

In \((\chi_M)_s\) weak convergence does not imply strong convergence. 

**Proof:** Assume that weak convergence implies strong convergence in \((\chi_M)_s\). Then we would have \([((\chi_M)_s)^{\beta\beta} = (\chi_M)_s\). [see 19]. But \([((\chi_M)_s)^{\beta\beta} = \Lambda^\beta = \Gamma\) by Proposition 5, \([((\chi_M)_s)] \) is a proper subspace of \(\chi_M\). Thus \([((\chi_M)_s)] \neq (\chi_M)_s\). Hence weak convergence does not imply strong convergence in \([((\chi_M)_s)]\). This completes the proof.

16 Definition

Fix \(k = 0, 1, 2, \cdots\). Given a sequence \((x_k)\), put 
\[ M\left(\frac{\xi_{k,p}}{\rho}\right) = M\left(\frac{(1+k)! x_{1+k}+(2+k)! x_{2+k}+\cdots+(p+k)! x_{p+k}}{\rho}\right) \text{ for } p = 1, 2, 3, \cdots. \]

Let \((\xi_{k,p} : p = 1, 2, 3, \cdots) \in \chi_M\) uniformly in \(k = 0, 1, 2, \cdots\). Then we call \((x_k)\) an “almost Orlicz space of gai sequence”. The set of all almost Orlicz space of gai sequences is denoted by \(\Delta_M\).

17 Proposition

\(\chi_M \bigcap \sigma^\alpha (\chi_M) = \Delta_M\), where \(\Delta_M\) is the set of all almost Orlicz space of gai sequences.

**Proof:** put \(k = 0\). Then \((\xi_{0,p}) \in \chi_M \iff \left(\frac{1! x_{1+2! x_2+\cdots+p! x_p}}{\rho p}\right) \in \chi_M\)

\[ \iff M\left(\frac{1! x_1 + 2! x_2 + \cdots + p! x_p}{\rho p}\right)^{1/p} \rightarrow 0 \text{ as } p \rightarrow \infty \]

\[ \iff (x_k) \in cs_0 \]

\[ \Delta_M \subset cs_0 \]

put \(k = 1\). Then \((\xi_{1,p}) \in \chi_M \iff \left(\frac{2! x_{2+3! x_3+\cdots+p! x_p}}{\rho p}\right) \in \chi_M\)

\[ \iff M\left(\frac{2! x_2 + 3! x_3 + \cdots + p! x_p}{\rho p}\right)^{1/p} \rightarrow 0 \text{ as } p \rightarrow \infty \]

\[ \iff M\left(\frac{2! x_2 + 3! x_3 + \cdots + p! x_p}{\rho p}\right)^{1/p} \rightarrow 0 \text{ as } p \rightarrow \infty \]

Similarly we get 

\[ M\left(\frac{3! x_3 + 4! x_4 + \cdots}{\rho}\right) = 0 \]
\[ M \left( \frac{4!x_4 + 5!x_5 + \cdots}{\rho} \right) = 0 \]  

and so on. From (13) and (14) it follows that
\[ M \left( \frac{1!x_1}{\rho} \right) = M \left( \frac{1!x_1 + 2!x_2 + \cdots}{\rho} \right) = M \left( \frac{2!x_2 + 3!x_3 + \cdots}{\rho} \right) = 0. \] Similary we obtain
\[ M \left( \frac{2!x_2}{\rho} \right) = 0, \quad M \left( \frac{3!x_3}{\rho} \right) = 0, \cdots \] and so on. Hence \( \Delta_M = \theta \) where \( \theta \) denotes the sequence \( (0, 0, \cdots) \). Thus we have proved that \( \chi_M \cap \sigma^\alpha (\chi_M) = \theta \) and \( \Delta_M = \theta \). In other words \( \chi_M \cap \sigma^\alpha (\chi_M) = \Delta_M \). This completes the proof.

18 Proposition

\( (\chi_M)_s = \chi_M \cap cs_0 \)

**Proof:** By Proposition 5. \( (\chi_M)_s \subset \chi_M \). Also, since every Orlicz space of gai sequence \( M (\xi_k/\rho) \) is exists, it follows that \( M (\xi_k/\rho) \) is exits. In other words \( M (\xi_k/\rho) \in cs_0 \). Thus \( (\chi_M)_s \subset cs_0 \). Consequently,

\[ (\chi_M)_s \subset \chi_M \cap cs_0 \quad (17) \]

On the other hand, if \( M \left( \frac{k!x_k}{\rho} \right) \in \chi_M \cap cs_0 \), then \( f(z) = \sum_{k=1}^{\infty} M \left( \frac{k!x_k}{\rho} \right) z^{k-1} \) is an Orlicz space of an gai function. But \( M \left( \frac{k!x_k}{\rho} \right) \in cs_0 \). So,

\[ f(1) = M \left( \frac{1!x_1 + 2!x_2 + \cdots}{\rho} \right) = 0. \] Hence \( \frac{f(z)}{1-z} = \sum_{k=1}^{\infty} M \left( \frac{\xi_k}{\rho} \right) z^{k-1} \) is also Orlicz space of an gai function. Hence \( M \left( \frac{\xi_k}{\rho} \right) \in \chi_M \). So \( x = (x_k) \in (\chi_M)_s \). But \( (x_k) \) is arbitrary in \( \chi_M \cap cs_0 \). Therefore

\[ \chi_M \cap cs_0 \subset (\chi_M)_s \quad (18) \]

From (17) and (18) we get \( (\chi_M)_s = \chi_M \cap cs_0 \). This completes the proof.

19 Definition

Let \( \alpha > 0 \) be not an integer. Write \( S^{(\alpha)}_n = \sum_{\gamma=1}^{n} A^{(\alpha-1)}_{n-\gamma} x_\gamma \), where \( A^{(\alpha)}_{\mu} \) denotes the binomial coefficient \( \binom{\mu+\alpha}{\mu} \). Then \( (x_n) \in \sigma^\alpha (\chi_M) \) means that \( \left\{ \frac{S^{(\alpha)}_n}{A^{(\alpha-1)}_{\mu}} \right\} \in \chi_M \).
20 Proposition

Let $\alpha > 0$ be a number which is not an integer. Then $\chi_M \bigcap \sigma^\alpha (\chi_M) = \theta$, where $\theta$ denotes the sequence $(0, 0, \cdots, 0)$

Proof: Since $(x_n) \in \sigma^\alpha (\chi_M)$ we have $\left\{ \frac{S_n^{\alpha}}{A_n^{\alpha - 1}} \right\} \in \chi_M$. This is equivalent to $(S_n^{(\alpha)}) \in \chi_M$. This, in turn, is equivalent to the assertion that $f_\alpha (z) = \sum_{n=1}^\infty s_n^{(\alpha)} z^{n-1}$ is Orlicz space of an gai function. Now $f_\alpha (z) = \frac{f(z)}{(1-z)^\alpha}$. Since $\alpha$ is not an integer, $f(z)$ and $f_\alpha (z)$ cannot both be integral functions, for if one is an integral function, the other has a branch at $z = 1$. Hence the assertion holds good. So, the sequence $0 = (0, 0, \cdots, 0)$ belongs to both $\chi_M$ and $\sigma^\alpha (\chi_M)$. But this is the only sequence common to both these spaces. Hence $\chi_M \bigcap \sigma^\alpha (\chi_M) = \theta$. This completes the proof.

References


