Global Journal of Science Frontier Research

dIscoverIng thoughts and InventIng future

Volume 10 Issue 5 Version 1.0

ISSN: 0975-5896

September 2010

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Global Journal of Science Frontier Research

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Volume 10 Issue 5 (Ver. 1.0)

Global Academy of Research and Development

Incorporation No.: 0423089
License No.: 42125/022010/1186
Registration No.: 430374
Import-Export Code: 1109007027
Employer Identification Number (EIN):
USA Tax ID: 98-0673427

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From the Chief Author's Desk

We see a drastic momentum everywhere in all fields now a day. Which in turns, say a lot to everyone to excel with all possible way. The need of the hour is to pick the right key at the right time with all extras. Citing the computer versions, any automobile models, infrastructures, etc. It is not the result of any preplanning but the implementations of planning.

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Intentions are very clear to do best in all possible way with all care.

Dr. R. K. Dixit
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Global Environmental Change And Human Health: A Thought On The African Inner-City

RAHEEM Usman Adebimpe

GJSFR Classification – H (FOR) 050204,050101,040604

***Abstract*- The need to keep the teaming urban dwellers healthy is of paramount importance to governments particularly of the third world. The reason for this assertion is related to the unprecedented movement of the people to live in cities. Urban future therefore challenges scholars of urban health to producing a comprehensive understanding of urban development strategy that enables governments to act to secure the slippery good health of urban dwellers. Within this scenario, studies have emphasized the gap between health in rural and urban environments while variations in health outcomes that may occur within areas of the same city are taken for granted. The difference is clearer when the inner-city is compared with the frontier. The inner-city depicts a clearer case of dilapidation and a greater vulnerability on the part of its population. Incidentally, the effects of global environmental change are not selective of space rather it occurs over all manners of spaces and locations. What differentiates populations is the variability of response to the shocks produced, often externally, by the circumstances of transformation. Vulnerability of the population is therefore related to the structural characteristics of the population per unit of time. The present paper isolates the health effect of global environmental change and examines its implications on the life chances of one of the most vulnerable groups in the third world- the inner-city dwellers. The paper argues that, given their life circumstances this group are least prepared to cope with the health effects of environmental changes that may sometimes be adverse and irreversible. The contribution from this effort includes strengthening the understanding of the variable nature of the space of vulnerability in developing country cities. The implications drawn by the paper includes that GEC may be triggers of poverty and deprivation when inner-city dwellers are exposed to its effects that require additional fund to cope with. The paper proffers a number of possible institutional and household level shock ameliorating mechanisms.**

I. Introduction

The fact that man and elements of his environment are on collision course that increasingly threatens the harmonious balance between them is now well appreciated. Part of the appreciation includes the contemplation of such phrases as global change, global warming, ozone hole, earthquakes and volcanoes, overpopulation, urbanization and migration. These are natural and human phenomena that are daily altering the planet on which man live. The magnitude of the alteration -enroute technology and ever-increasing population – is at a pace unknown to natural history. Globalization is the generic term from which the understanding of 'Global change' or 'Global Environmental Change (GEC)' emanates. This term refers "primarily to the global effects, notoriously unintended and unanticipated, rather than to global initiatives and undertakings. It is not about what we all, or at least the most resourceful and enterprising among us, wish or hope to do. It is what is happening to us all" (Shaw, 1999).

Ipso facto, GEC had attracted several meanings and perspectives, often genuinely value laden. Some of these illuminate while some others obfuscate its understanding. The wisdom in this paper agrees with Lee and others that GEC is the "expression of concerns surrounding economic globalization (which) extend to the environmental domain where the unprecedented and unsustainable use of resources, degradation of ecosystem and loss of biodiversity threaten the very basis of life on earth"(Lee, etal, 2002 see also Vitousek, et al 1997). Although GEC is resulting from both natural and anthropogenic phases of lived existence, today's rapid changes in global climate systems, the concentration of stratospheric ozone, biodiversity loss and the dwindling vitality of food- producing ecosystem, etc are largely due to the impacts of the human angle in the rigorous use and abuse of the planet earth from time geologic (See McMichael, 2001). Hence the concerns we are expressing today can be appropriately titled 'regrets' for the actions of forefathers; and the worry that our children may not be as luck, and may have greater cause to regret our own actions or inactions, given the level of the alterations that are being effected on the earth surface through technology, industrialization, and urbanization.

It is therefore narrow to view global change merely in terms of time-space compression that homogenizes the human conditions but it is good wisdom to evaluate the spatial dimensions of the technological annulment of temporal and spatial distances that serve rather to polarize the world. Within this scenario, human health and environmental health are acknowledged to be intertwined. To this extent, to enjoy basic rights to life, food, housing, livelihood and culture, the existence of clean water, stable climate, thriving wildlife and well-managed natural resources are in the center stage (see Gopalan, 2003). The aim of this paper is to identify the impacts of environmental change on human health generally, and to underscore the peculiar nature of the inner-city dwellers' vulnerability using, in most cases, the African example.

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II. Africa in the Global Setting: Some Critical Indicators

Within the global setting, there are three 'worlds'. The first world comprises of some countries that border less with development in its rudimentary connotations because they are already 'developed'. The second group of nations ranks next in the hierarchy of wealth and international power structure; while the third world comprises of humans and nations that denies neither their poverty nor their deprivation but try endlessly to explain them. Today, the first two 'worlds' operates in a cybernetic isolation, while the third world is cut off and forced to pay cultural, psychological and political prices of their new isolation. This is because the global social and economic media are not truly interactive.

African situation is precarious among the third world countries. It is the world's poorest continent. The first ten countries in the world with the highest percentage of their population living below US1D per day are African countries. In these countries, between 74.5 and 92.4 percent live below US2D per day. (WorldBank, 2005). Out of the 18.8 million HIV/AIDS deaths registered globally from the beginning of the epidemic through the end of 1999, 14.8 million or about 80% occurred in Sub-Saharan Africa (Joint United Nations Programme on HIV/AIDS, 2000). Due to this and other conflicts, 43 million children were already orphaned at the beginning of the 21st century; out of which up to 10% have lost both parents. In some 11 African countries, 15% of their children are orphans (WorldBank, 2005). Life expectancy at birth is thus slim and evidences are slender that they will improve. It ranges between 44years in Nigeria- its most populous nation- to 53 in South Africa, 57years in Zambia, 39 in Zimbabwe and 40 years in Mozambique. Malawi and Botswana's population are expected to live for only 37 years (WorldBank, 2005). High levels of inequality still characterize most of the countries in the continent with difficult development challenges. Thus in Africa, technology and industrial development have met their hardest soils while diseases and infections navigate its massive landmass with unparalled ease. These have combined to result in a widespread deterioration in living standards. The mortality rate had worsened from 107 in 1970 and now stands at a toll of 165 per 100,000 and AIDS is threatening to reduce this by 20 years in the affected countries (UNCHS, 2001).

These and similar evidences reluctantly agree with globalization as signifying the end of territorialism. If globalization is a positive compression of space, to the extent that it captures "the notion that the economic and information features .are penetrating even the remotest corners of the earth and that each locality is now forced to participate in the new globalized world…" (UNCHS, 2001); then a fascinating debate is required to determine the 'quantity' of the benefit of this phenomenon to Africa. The objectives of this study include

1) to examine the characteristic 'city divides' in Africa, and
2) identify the impact of GEC on African city dwellers' health with particular reference to the inner cities and where the inner-city problems are present.

The world over, city divide is a growing feature of the contemporary disparity. The growth of disparities between the affluent and the income-deprived is exemplified in the residential and human geography of a typical African city. In most parts of the continent, the affluent neighborhood and slums coexist, albeit; in clearly defined but invisible boundaries (see Plate 1). This is to the extent that " it is entirely possible …to spend months in any of today's world cities, as well as in the capital of a developing country, without ever coming into visual contact with a slum or a derelict neighbourhood" (UNCHS, 2001). African inner cities are typical because "histories of marked social inequality and poverty (as well as) extremes of wealth and poverty are found in close juxtaposition" (Lloyd- Evans, 1998).

Plate 1: A typical City Divide Scenario In the Third World. The plate Shows Contrasts in Sanitation practices
(Source: Mobile Nig. Unlimited, 2004)

III. Defining and Delimiting the Inner-City

The most problematic area in inner-city study is the unambiguous delimitation of the inner-city. Jones (1979) notes that it is much easier to recognize an inner-city problem than to define where the inner-city itself is. In delimiting the inner-city, scholars have adopted two popular approaches. These are the spatial contiguity and the pathological schools. The former idea is implied from the Burgess' concentric zone model which asserts that differentiation into natural economic and cultural groupings gives form to the city and segregation offers the grouping (and hence individuals therein) a place and a role in the total organization of the city life. (See Stewart, 1972). Based on this general framework, the inner-city refers to an ill defined area close to the city centre with characteristic old dilapidated housing having few basic amenities (Eyles, 1986). Burgess' conception excludes the city centre but included areas that are close to it. Considering a third world city structure, the city centre may be included since the existence of dilapidation, multiple occupation and few amenities possess a graph with a positive linear relationship beginning from the city centre. Spatial contiguity school therefore believes that it is possible to demarcate spatial units of the city based on age and/or dominant activity with the innermost part being the inner-city. This pre-colonial ring surrounding most-large cities and conurbation in Africa has aroused special interest because of the discovery that poverty and deprivation were disproportionately localized in and around this ring (see Carter and Jones, 1998).

The pathological school on the other hand, associates the inner-city pathology to a general vulnerability of a group of people to structural economic changes such as those basic characteristics that are dominant in the native areas of African cities. This approach recognizes that it is possible to identify areas of the city where the problems of unemployment, employment loss poor public and private housing, accessibility to and availability of basic amenities and general decay are jointly or singly confined or at least predominate. This approach suggests that the inner-city can be defined as a series of overlapping zones or areas delimited on the basis of the social problem(s) in them. These areas may neither be coincidence or contiguous because several reasons have, over time, shaped the city structure and its population to produce the city mosaic as are found in most parts of the pre-colonial African cities today. In the present paper, while we agree with the fact that inner-city problems are to be found in the innermost parts of the towns or cities, they are not confined to these areas. The paper therefore refrain from a rigid geographical delimitation of the inner-city; rather it believes that it is more important to map the forces which fashion urban deprivation in the city centre which are also replicable in the periphery. The paper adopts the 'inner-city' as an umbrella concept for the examination of the interaction of urban economy decline, physical decay and the concentration of the socially disadvantaged, particularly the poor people who may be located in any part of the city where these pathology may be present.

This approach had also found empirical and theoretical credence in the earlier works of the Lamberth inner Area Study, (Great Britain, 1977), Karn, (1979) and Sim (1984). In Africa, the inner-city concept as used here is an essential characteristic of traditional pre-colonial cities. It is used to express areas of rundown structures that have been overtaken by modern high rises which are dominant in the city centres but can also be found in some other parts of the city existing in close juxtaposition with the modern structures.

Within a pre-colonial African city, it is possible to identify areal pathology of the characteristics of disadvantages and a marked inequality in facility distribution. This is such that a typical pre-colonial African city is a mirror image of the political economy. At the structural level, it is possible to observe such indicators as the social character of access to facilities like health, education, sanitation, etc as well as the general level and severity of poverty and deprivations. This phenomenon often depicts a regular gradient from the city centre to the frontier. Jones (1979) described the major attributes of the inner-city space as that of dereliction, high concentration of unemployment, drab and vandalized council estates, low educational achievement by school children, factory closure, decaying neighbourhoods and population decline. The African inner-city dwellers thus represent the most disadvantaged segment of a typical African urban setting. They include those families that have experienced long term spell of poverty and welfare dependency because their members are unemployed and/or '*unemployable*' for lack of basic training or skill. They also include those that have dropped out of labour force altogether due to age or for other structural reasons. The inner-city is therefore less of physical or geographical distance between different areas of a city but related more to the social distance between its people even when they appear close in Euclidean relations.

The inhabitants of this '*space*' are particularly vulnerable to structural changes such as those occasioned by macroeconomic and institutional arrangements involving shifts from goods producing to service producing industries, the polarization of the labour market into low and high wage sectors; as well as large scale and micro-level environmental and climatic changes. It is therefore reasonable to characterize inner-cities broadly in terms of a general vulnerability of a group of people to such changes. Global change affects individuals and population groups in ratios determined by levels of vulnerability, which is in turn related to several other factors that determine the impact and severity of the change on their health. These factors may include socioeconomic status, gender, level of education and geographical location within the urban space. To the extent that these impacts occur differently, vulnerability therefore is bound to vary according to the relative positions of individuals in relation to the parameters identified above. In seeking to explain this variation we have deliberately ignored the possible differences between several compositional and contextual factors. In other words, the variation between vulnerability brought about by the social and material circumstances of people living in an area and the social, material and environmental characteristics of the area itself are held constant so that we could explain vulnerability to the impact of GEC as due largely to the characteristics of the particular locations containing a complex web of social and cultural milieu (see Macintyre et al., 2002 and Gatrell et al., 2004). The idea of vulnerability therefore becomes a guiding concept for this study.

IV. Vulnerability and Global Environmental Change

The concept of vulnerability (as used here) does not connote the sense of resistance as may exist in ecology texts (see Begon et al., 1990), but the sense of resilience; suggesting ability of adaptation, ability to return to an original resting state or achieve a new equilibrium after a shock (see Watson et al., 1996). When the concept is applied to climate or environmental change, it is defined as a "state of increased probability of adverse outcomes for a given environmental exposure" (Woodward et al., 1998). Hence vulnerability is at once relative and undesirable. The impacts of 'change' in environmental parameters function through the crevices of the non-linear vulnerability to the change by individuals and community who are the victims. It is therefore this concept of vulnerability that act to differentiate individuals and is thus the most important concept as far as the debate on the human dimensions of climate change is concerned.

The need to critically examine levels of vulnerability in this case therefore stems from three related reasons:

i. Human species have grown to appreciate the fact about the debate on the variations and/or alterations in the basic attributes of climate; it is the variabililities of the vulnerability that is still poorly understood; whereas this is the root cause of health disparity resulting from GEC.

ii. It is the events that take place locally that have the greatest impact and similar events may (and do) produce differential impacts on human health; whereas the spatial scale for impact forecasting is often regional. One of the dangers of such impact forecasting at larger resolution is the optimism in the late 1940s that the battle between humans and parasitic/infectious diseases had been decided in a victory for the former to the extent that it was time to close book on infectious diseases (see Fisher, 1994), only to be ravaged by formidable health problems from the *'relics'* of ancient scourges such as malaria (Pearce, 1995), tuberculosis (Bloom and Murray, 1990, Brown, 1992, etc) cholera (Glass et al., 1992) and HIV/ AIDS(Ainsworth and Over, 1994, Barrett and Rudalema, 2001). Today, local happenings are shaped in a complex way by events occurring many miles away because in space, "everything is connected to everything else to the extent that "most events and phenomena are connected, caused by, and interacting with a huge number of other pieces of a complex universal puzzle. We have come to see that we live in a small world….." (Barabas, 2002).

iii. There is also the tragic reality that the most vulnerable groups are also those that receive the most devastating effects of GEC whereas these same groups are the least likely to return to equilibrium when they are devastated. Thus, GEC happens to us all, certainly, not equally and any intervention that takes for granted the pattern of vulnerability is at once frustrated.

This new inequality is well known but least acknowledged and poorly appreciated.

The connection between GEC and Vulnerability on one hand as well as the inner-city and human health on the other is complex. One way to view this connection is to note that environmental, social, economic and political decisions taken by major players on the world stage can (and do) indeed have consequences for the health of those in the most peripheral locations and that even the most privileged can be affected by diseases with their origins in the most materially deprived parts of the world (see Gatrell, 2005). The implication of this is that a major property of the global climate impact is that small changes in one component of the elements of the earth system do not lead to correspondingly small changes in others due to varying levels of vulnerability in the social context.

V. Global Environmental Change in Africa: Some Evidences

There are various symptoms of environmental changes that are happening at dimensions that are threatening "…the homeostasis of the marvelous planet we call home"(Rosenblatt, 2005, see also McMichael, 1998). These changes are also threatening the very existence of the human species that inhabit 'the home' in non-linear ways. The impacts of some of these changes are general and affect humanity at different levels. Some of the changes or impacts that are peculiar to the African sub-region are also identified.

A significant portion of Africa is located in the tropics where bush burning and shifting cultivation are major agricultural practices. This alone, according to Paul Crutzen, as quoted by Pearce (1996), releases between 1.8-4.7 billion tones of carbon a year. The main by-products of these tropical fires are carbon monoxide, methane, nitrogen, oxides and carbon dioxide. Ozone is also a by-product of these emissions (Adelekan and Gbadegesin, 2005). The amount of emission from traffic on the African country roads is also increasing daily and may sometimes account for 90-95% of lead and carbon monoxide emission. This emission rate is exacerbated by the *'off- the- road'* conditions of most of these vehicles in terms of their unacceptable combustion levels. The rush to industrialize in most parts of Africa also contributes to the observed rate of emission of GHGs and to Global Environmental Change. In Africa, the trio of Egypt, Nigeria and Algeria combine to account for up to 35.5% of total fossil fuels from the continent (Marland et al., 2003). Indeed, Nigeria is the largest emitter of carbon dioxide and related GHGs in the West African sub-Region. This is partly because only a small part of the country's natural gas is utilized while as much as 76% is flared (WorldBank, 1995). The composition of the flare includes 90% methane, 1.5-2.0% carbon dioxide, 1.4-2.4% heavier hydrocarbons and 3.9-5.3% ethane among other compounds (Jones, et al., 1998). This composition is such a major impetus to the local GHGs and to say the least, to global climate change. The World Health confirms this assertion that:

"Human–induced increase in the atmospheric concentration of GHGs is amplifying the greenhouse effects. In recent times,the great increase in fossil fuel burning, agricultural activity and several other economic activities has greatly augmented greenhouse gas emissions. The atmosphere concentration of carbon dioxide has increased by one-third since the inception of the industrial Revolution" (WHO, 2003).

Table 1 provides examples of some of these GHGs that may amplify the climate change rate and variations in their concentration and rate of change over the period 1990-1999 as well as their atmospheric lifetime.

Table1: Changes In GHGs' Concentration (1990-1999)

Time Period/GHG	**Co_2**	**CH_4**	**N_2o**	**$CFC\text{-}_{11}$**	**$HFC\text{-}_{23}$**	**CF_4**
Pre-Industrial Concentration	-280ppm	-700ppb	-270ppb	Zero	Zero	40ppt
Concentration in 1998	365ppm	1745ppb	314ppb	268ppt	14ppt	80ppt
Concentration Change rate (1990-1999)	1.5ppm/yr	7.0ppb/yr	0.8ppb/yr	-1.4ppt/yr	0.55/yr	1ppt/yr
Atmospheric Lifetime	5-200yrs	12yrs	114yrs	45yrs	260yrs	750,000yrs

$C0_2$ =Carbon dioxide, CH_4 =Methane, N_2o= Nitrous Oxide, $CFC\text{-}_{11}$ = Chlorofluorocarbon, $HFC\text{-}_{23}$ =Hydroflurocarbon$_{23}$, CF_4 =Perfluromethane

(Source: WHO, 2003).

From the above, it is evident that since the industrial Revolution, C_{02} have risen more than one third while other GHGs have similarly risen and estimates ahead are not less grievous. More over, the Energy White Paper (DTI, 2003) highlights some fundamental ways by which rising temperature is affecting the planet earth:

i. Retreating of ice caps from many mountain peaks
ii. Global mean sea level rose by an average of 1-2mm per year during the last century.
iii. Summer and autumn arctic sea ice thinned by 40% in recent decades.
iv. El-Nino events have become more frequent and intense during the last two to three decades;

Weather related economic losses to communities and businesses have increased ten folds over the last 40years.Although, not the most powerful GHG, $C0_2$ is the prime target for mitigation action against climate change because of its sheer abundance that accounts for 80% of total emission and more importantly, its very long atmospheric life span also means that its concentration will continue to rise and be a cause of problems for years and decades to come (CURE, 2003).

.There is a stronger evidence to support the hypothesis that human activities are also responsible for most of the alterations that are identified above. For all these alterations, human systems or his livelihood sectors are vulnerable both separately and in a complex web of causation. Some of the vulnerable human sectors include water and sanitation, agriculture and food security, the coastal and marine ecosystem and human health which are affected by each of these sectors and by their joint impact. The impact on health is thus left out here for separate discussion in a dedicated section. Importantly, the precarious situation of African cities suggests that most of these impacts possess implications for urban health and urban life in general.

A. Water and Sanitation

One third of the world's population lives in countries where more than 20% of the renewable water supply has been used. These countries are otherwise called 'water- stressed' countries. By 2025, the IPCC (2001) predicts that an estimated 5billion people will be affected by the stress from the current number of 1.5. One out of every five world people also lack access to safe drinking water while as much as 50% does not have access to sanitation (see Kasperson and Kasperson, 2001). Hence up to 3-4 million people die annually of diseases that are water borne or water related because about 10% of global annual water consumption is sourced from depleting groundwater resources undergoing contamination (Cosgrove and Rijsberman, 2000). African situation is worrisome, where approximately two- third of its rural population and one-quarter of the urban population lack safe drinking water; the percentage with poor sanitation is higher (Zinyowera, et al., 1998). It is estimated that by the year 2025, the number of water -stressed African countries would have reached 18 from 8 in 1990 and this may affect up to 4billion people (Kasperson and Kasperson, 2001, Cosgrove and Rijsberman, 2000, World Bank, 1996). Though the effects are yet uncertain, the continent has experienced devastating droughts in 1965-66, 1972-74, 1981-84, 1986-87, 1991-92 and 1994-95 affecting different parts. Runoffs may also increase in regions with increased precipitation. Episodes of flood are therefore not also uncommon as in the case of the Ibadan Flood Disaster in 1980 and 1982 (see Akintola, 1989). According to URT (2003), reduced runoff into two out of three major rivers in Tanzania will have far-reaching socio- economic impacts because River Pangani's annual flow could decrease by between 6-9% and Ruvu by 10% due to increased temperature. Such decrease in precipitation in Sudan is likely to be more devastating because more than 50 and 25 percents of its landmass already lie in the desert or semi-desert and arid savannah respectively. In contrast, the potential for heavy flood damage may increase during the long rainy season as a result of increased rainfall due to low temperature in humid areas whose river flow may increase to about 20% (MLWE, 2002, Orindi and Murray, 2005). This may affect urban infrastructure like major hydropower stations, communication infrastructure, farms and human settlements. For example, the El-Nino of 1997/98 resulted in the death of over 1000 people and caused major disruptions in communication and service provision in Uganda (MLWE, 2002, Ropeleski, 1999).

B. Food and Agricultural Production

The impact of climate variability and change depends upon the rate, magnitude, and geographic pattern of the change. Studies suggest that adverse effects of climate change on agriculture and food security will be concentrated in developing countries out of which Africa is a major stakeholder. The IPCC (2001) hints that these impacts will also interact with other environmental and socio-economic vulnerabilities to exacerbate hunger and undermine food security. Income of vulnerable groups will decline as the risk from hunger increase. Over all, because of the geographic and temporal shifts in agriculture, coupled with worsening social and economic situations, food security in many parts of Africa may worsen. For instance, in Egypt where the rich agricultural system is predicated mainly on favourable temperature, fertile soil and abundant irrigation water from the Nile, agricultural activities had been expanded into desert lands adjoining the Nile Basin and reclaimed long used areas that have become Stalinized or water logged. Thus Egypt's 63million people remain totally dependent upon Nile fed agriculture which poses severe challenge to its economy even without climate change. With any change in climate, serious threats from such phenomenon as sea level rise are imminent because this would jeopardize areas of the Nile Delta lying below one metre in elevation. Hence up to 15% of the existing agricultural land in the Delta area of the Nile could be lost (see Nicholls and Leatherman, 1994). This will also accelerate the intrusion of saline water into surface water sources and the coastal aquiver (Sestini, 1992)

The case of Egypt illustrates how far- reaching the impacts of climate change may be for some African countries. In essence, through changes in temperature and precipitation, changes in soil moisture and fertility, changes in the length of growing season sand and an increased probability of extreme climate conditions, Global Climate Models (GCMs) predict that climate change may lead to significant reduction in agricultural productivity in developing countries like Africa.

C. Marine Ecosystem And Coastal Population

Climate change will change oceans in many important ways; from decreased sea level cover in the polar region to sea-surface temperature and mean sea level rise in different parts of the world. When such changes interact with growing population densities and peculiar African developmental pressures particularly in coastal areas, a major ecological and human impact of climate change will be imposed on Africa. These may include

i. loss of land and displacement of coastal population;
ii. flooding of low lying coastal areas,
iii. loss of agricultural yields and employment;
iv. negative impact on coastal and
v. urban agriculture ;
vi. erosion of sandy beaches and associated losses in tourism.

It is anticipated that in this regard, the most serious impact will befall fisheries and marine ecosystems-coral reefs, atolls, salt mashes and mangrove forests of Africa, depending on the rate of sea level rise. An estimated 1billion people are said to be dependent on fish and fishing as either their primary source of protein or occupation (see French, 2000); out of which a significant proportion are African city dwellers.

Certainly, it is more difficult to monitor the marine the marine environment in the same way as crop yields can be investigated, but scientific assessments suggest changes in oceans' make-up in the face of atmospheric temperature. The IPCC, 2001 for instance predicts changes in the abundance, distribution and specie composition of some fish populations as well as collapse of some fisheries. Currently, with a 2^0C increase in the mean temperature of the world' ocean per century (Souler, Obura and Linden, 2000) combined with the strongest El-Nino of 1998, coral, throughout the world's tropical Islands suffer extensive bleaching and mortality. along the East African Coasts, 90-100 percent of corals exposed to water temperatures higher than 32^0C died; and with no sign of recovery even after about the first 18months(Souler, Obura and Linden,2000, see also Kasperson and Kasperson,2001). More over, estimates suggest also that land loss due to sea-level rise of one metre would total 30,000sq. Km in Bangladesh, 6,000 sq. km in India, 34,000sq.km in Indonesia and 7,000sq.km in Malaysia (Nicholls, Mimura and Topping, 1995). Similar estimates for Africa are not available to the author, but indications do not suggest that they will be less tremendous.

Many societies, particularly urban coastal populations depend heavily upon fish as a significant part of food supply. As much as one- third of protein supply in densely populated Nigeria comes from fish (Kasperson and Kasperson, 2001, see also Raheem, 2005).

According to Carpenter and others, (1992), climate may affect fisheries through.

i. changes in water temperature
ii. the timing and duration of temperature extremes
iii. the magnitude and patterns of annual stream flows
iv. surface water elevations; and
v. the shorelines of lakes, reservoirs, and near shore marine environments.

Thus arising from the above factors in varying combinations and severity, OECD, (2001) indicates that more than one forth of the world's marine fisheries are "already either exhausted, over- fished, or recovery from over fishing". Vulnerable areas from these impacts will face severe stresses due to

i. a shift in centres of fish production and the composition of species as ecosystems change in their duration and internal structure;
ii. falling economic return until long stability in the fisheries is achieved; and
iii. disproportionate suffering among subsistence and other small scale fishermen in Africa (Zinyowera,et al., 1998).

"The situation in Africa suggests that it s the most vulnerable people and places that will bear most of the costs" (Kasperson and Kasperson, 2001).

VI. Health impacts of GEC and the inner-city vulnerabililty: an example.

The concern for human health resulting from the impact of climate and environmental changes have been greatly amplified by the recognition of the inherent danger in the McMichael's (1993) *'planetary overload'*-the term he used to capture the fact that "humankind in aggregate is …exceeding capacities of the biosphere to absorb our wastes and to regenerate natural condition"(McMichael and Kjellstorm,2002).This development arouses both policy and research orientation at paying more than ordinary attention to the impact they may posses for not only human health but also for the survival of the highly vulnerable population.

The impact of GEC on health could manifest either directly or indirectly. The impact identified herein occur both as the impact on health of climate change on one hand and the relapsing joint impact of other influences of climate change may posses for human health. Patz et al's (2000) framework for assessing the potential health effects of climate change is a realistic one that captures major perspectives of both direct and indirect effect of climate change on human health. His framework (fig 1) does not however include the impact of 'other impacts' of climate change on human health.

The direct impact include changes in temperature and humidity leading to heat-stress morbidity and mortality, flood, expansion in the ecology of tropical vector borne diseases and possible decreases in cold related illnesses. Kasperson and Kasperson (2001) also predict that climate change will also affect the proportion of the world population living in the cities, availability of sanitation and portable water supplies, human migration and living condition. The World Health Organization concludes that climate change is also likely to alter the incidence and range of major tropical vector borne diseases. This is shown in table 2.The table, among other indications, shows that the geographical range of two infectious diseases-Malaria and Dengue will increase. If temperature should rise above the IPCC projected upper bound of 3-5^0 C by 2100,the proportion of the world population affected by Malaria may increase by 15percent.This will lead to the addition of close to 80million annually(Watson et al,1998). Kovats, et al (2000) are of the opinion that this change might already be in place because according to them, Malaria is already one of the most serious and complex global diseases and it is the disease most likely to be affected by climate change .There are at least 270million cases at any given time, out of which 1-2million people die annually .Majority of the deaths are coming from Africa.(see McGranahan, et al 1999).

Table 2: Climate Change and Major Tropical Vector –Borne Diseases.

Disease	Likelihood of with Climate Change	Vector	Present Distribution	People at Risk (Millions)
Malaria	+++	Mosquito	Tropical/Sub-Tropics	*2020*
Schistosomiasis	++	Water Snail	Tropical/Sub-Tropics	*600*
Leisnmaniasis	++	Phlatomine Sandfly	Asia/Southern Europe/Africa/Americas	*350*
American Trypanosomiasis *Chagas Disease*	+	Triatomine bug	Central and South America	*100*
African Trypanosomiasis *Sleeping Sickness*	+	Tsetse fly	Tropical Africa	*55*
Lymphatic Flariasis	+	Mosquito	Tropics/Sub-Tropics	*1100*
Dengue	++	Mosquito	All Tropical Countries	*2500-3000*
Onchocerciasis	+	Blackfly	Africa/ Latin America	*120*

River Blindness				
Yellow Fever	+	Mosquito	Tropical South America and Africa	-
Dracunculiasis *Guinea worm*	?	Crustacean *Copepod*	South Asia/Arabian Peninsular/Central – West Africa	*100*

+++Highly Likely, ++Very Likely, +Likely, ?Unknown

Source: (Kovats et al., 2000, 20 cited in Kasperson and Kasperson,2001)

This is made possible because climate change will affect the underlying ecological conditions for Malaria breeding, transmission and multiplication. These include higher temperatures, increased precipitation and relative humidity, mosquito habitats, and access to washing and drinking water. Thus areas adjoining current Malarial Zones and high altitudes within the zones such as the mountainous East Africa, where low temperatures hitherto curtailed malarial transmission (McGranahan, et,al., 1999) may experience malaria infections. Apart from malaria, schistosomiasis would spread beyond its current distribution limits. Dengue will expand in the direction similar to that of malaria; while trypanosomiasis (sleeping sickness) is also expected to undergo redistributions that are climate induced.

Finally flooding and heavy storm events will increase and could produce heavy consequences for areas of cities with old and weak structures as are found in the inner cities of Africa. This vulnerability also increases in Island states due to their physical circumstances. for instance, most of such states' critical infrastructure and economic activities are located close to the coast where just a 50cm to 1metre rise in sea level may convert significant part of it into sandbars with great economic loss and health consequences. The consequences are complicated by the high population of Island cities. Island populations grow at more than 3 percent yearly (Pernetta, 1992, Kasperson and Kasperson, 2001).

VII. Inner-City Vulnerability: Some Peculiarities

Poorer African nations are disproportionately vulnerable to disasters and to the effects of climate change. The capacity of peoples to be 'wounded' from the stress, produced from abrupt changes in environmental parameters whether environmental or socioeconomic, vary widely in response to many structural and institutional factors of the group or institutions. These factors may include wealth, technology, knowledge, infrastructure, institutional capabilities, and preparedness, and access to resources. This is because human endowments in such assets also depict wide variability particularly in a world of mounting and widening inequalities. The Kaspersons hence note with emphasis that

"Developing countries and particularly the least developed countries, are clearly the most vulnerable regions to climate change. They will experience the greatest loss of life, the most negative effects on economy and development, and the largest diversion of resources from other pressing needs" (2001).

This is because developing countries often lack the resources to adapt and/or cope with weather hazards and thus are ill-prepared in terms of general preparedness such as coastal protection, early warning and disaster response systems, as well as victim relief and recovery assistance (see IPCC, 1995 and GEF, 2001).

In Nigeria, inland flood and structure collapse are also among the effects of climate change that are becoming more regular rather than episodic. In the last few years, several instances of structure collapse have been recorded with professional builders and engineers proposing a more regular occurrence. The example of rainstorm and flood disaster in Kwara state, Nigeria depicts one of the micro-level impacts of GEC in African cities. Kwara is one of the 36 states to which Nigeria is subdivided for administrative and political purposes. The state is predominately rural with Ilorin, its administrative headquarter, Offa and a few other centres being urban. Ilorin is a city of some one million people in 2006. It is a city with distinct structural and morphological semblance of a traditional city since its history predates colonialism in Nigeria. Rainstorms and flood disasters are regular events during the early and late rains in the city. Statistics from the Kwara Chapter of the National Emergency Management Agency (NEMA) indicate that between 2003 and 2006, a total of 769 houses were affected by rainstorm or flood or both in Ilorin metropolis. The breakdown of the figure shows further that (666) or more than 86 percent of the houses affected were located in wards classified as the 'interior' while the remaining (103) or 14 percent occurred in wards classified as 'other areas' of the city.

Period/ Area	**2003-2004**	**2005-2006**	**Total**
Interior Ward	120	546	666
Other Wards	28	75	103
Total	148	621	769

Source: Compiled from Kwara NEMA, 2006.

The Ilorin examples above shows that the inner most parts of African cities where the poorest urban population also reside are more vulnerable to the impact of GEC. There are two dimensions to the vulnerability. The first is the vulnerability of the houses in inner most parts of traditional cities to the impact because of their age, as well as materials used for the roofs and walls. Secondly, the vulnerability of the inhabitants of these areas of the city produced by a combination of extremes of poverty, deprivation, unemployment as well as the general lack of skills to take up well paid jobs. This inability had combined to suppress the capability of this group to recover from the shock produced by the effects of flood and rainstorm. For this reasons, majority of households were unable to, on their own, bear the cost of recovering in terms of roof replacement and other immediate expenses produced by the disaster. In another study, Raheem, (2006) found that this event is ecologically restricted to the old traditional areas of the city where houses were old and built in the bye-gone age with inferior materials both for the walls and roofs. Often, government responses were inadequate and a substantial part of the victims were reduced to local .environmental refugees. This can be devastating since a typical African inner-city dweller is already operating on critical and unstable socio economic conditions.

The direct health implication of this may include population displacement, unintended injuries that can, sometimes, be fatal and life claiming as shown in plate 2

.

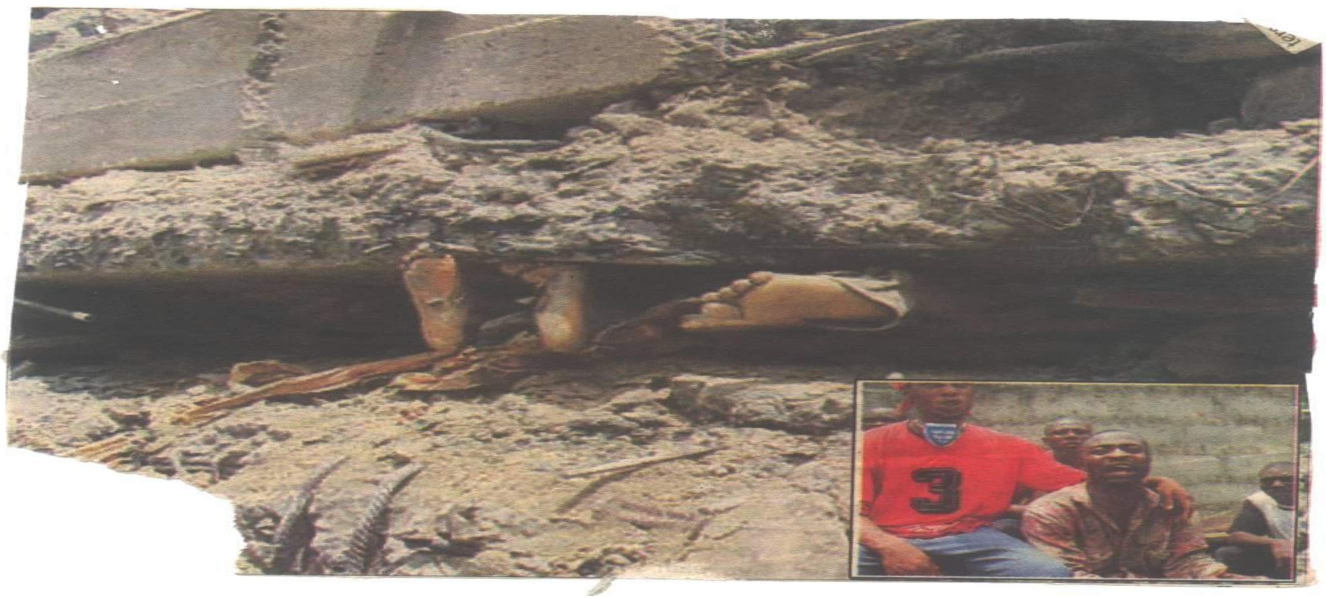

Plate 2: Collapsed building in Port Harcourt, Nigeria (Punch, August, 16, .2005)

Since health is the ecological characteristics of population reflecting the wider condition of the social and natural environment-a reflection of the level of biological functioning achievable within the prevailing environmental conditions(see McMichael and Kjellstorm 2002), the events described above also lead to a series of indirect health consequences. One of the indirect consequences of extreme weather conditions produced by GEC is the alteration in the patterns and dynamics of common infectious diseases. For instance Malaria is the most disabling vector borne disease. It ranks first in accounting for morbidity, mortality and loss of productive time in Sub Saharan Africa. Although, 40 percent of the world population carry the risk of contact, 75 percent of all active cases occur in Africa leading to the death of 3,000 children daily (Snow, et al, 2005, WHO, 2003). Studies have implicated warming and weather extremes in the prevalence of this and other tropical vector-borne diseases. This is because flood provides conditions for large outbreaks while high temperature reduces the incubation periods for their vectors. For instance while McArthur (1972) notes that the incubation period for *plasmodium falciparium* which takes 26 days can be reduced by half with a 5^0 C increase in temperature, Ebi, and others (2005) reports “a four-to-five spike in malaria cases” following a devastating series of three topical cyclones and heavy rains in Mozambique as shown below

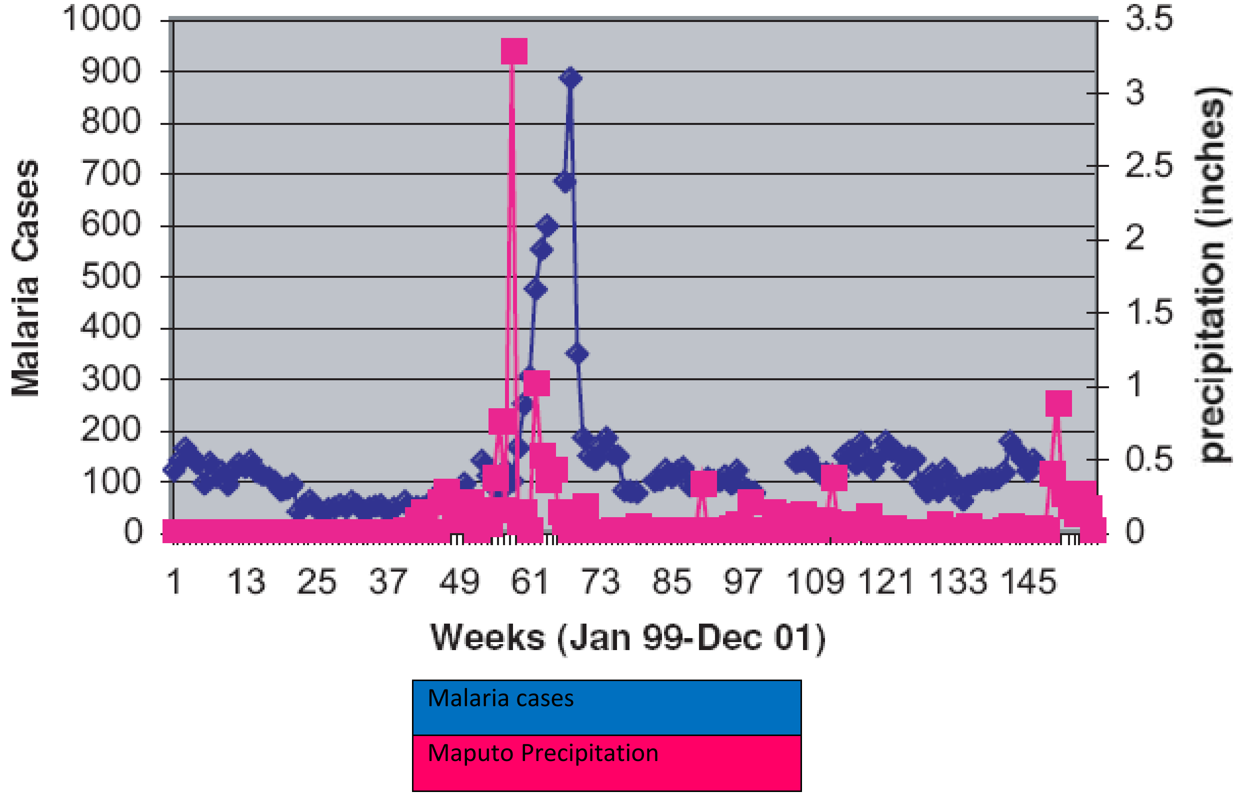

Fig. 2: Malaria and Floods in Mozambique (Adapted from Centre for Health and Global Environment, 2005).

VIII. SUMMARY AND CONCLUSION

In this paper we have acted the Farmer, et al (2000)'s Prophet: *Predicting that the poor will do poorer even in boom times*. The African inner-city exemplifies poor area and this possesses greater vulnerability to the impacts of climate change. Scientific research and many practical projects are showing the tendencies that "global climate change can be partially mitigated if the world big cities substantially reduce their environmental impact" (Hunt, 2004). The big poser: how far can cities of Africa assist themselves to substantially reduce their environmental impact? To date, the impetus for urban growth in Africa is higher than the capability to cope with the growth and the changes there from. Hence, cities dwellers are excessively prone to the consequences of environmental and socioeconomic changes arising from such urban growth.

Global environmental Change is affecting the incidence of key infectious diseases and the dynamics of their prevalence. Food insecurity, poor water and sanitation coverage are also shown to be part of the responses of the life support systems to the rapid changes in the global environment. The vulnerability of the population to these impacts is also affected by the existing levels of poverty, deprivation and empowerment. Although some of these form important policy areas for many African governments, evidences on ground is to the effect that a great lag exists between fundamental livelihood structure of the African population and the attention given to such issues as deprivation, poverty, inequality and empowerment. As outlined earlier, the impact of climate change will serve to aggravate this prevailing situations with great consequences for human health particularly those of the inner-city dwellers.

Africa's inability to cope with Global Environmental Change borders greatly on, not only development but also on sustainability. The progress made by African cities in the area of poverty and hunger reduction, maternal and child mortality as well as life expectancy significantly demonstrates that Africa cities have been at least lagging and at most stranded in most of the global transitions-demographic, epidemiological and risk. Hence, the fourth transition – sustainability transition – is an enormous challenge to the continent (see McMichael and Kjellstorm (2002).

To meet up with the rest of the world, Africa must be assisted first by ensuring that African cities are given particular priorities in most global initiatives like the Kyoto Protocol, the world summit on Sustainable Development of Johannesburg in 2002, Agenda 21 of the United Nations in 1992 and the Millennium Development Goals. Such attention may make clearer the nature and level of intra-urban inequality within a typical African city so that the inner-city dwellers may receive greater than ordinary attention they often receive by their governments.

Finally, the struggle to rescue our biological system from imminent collapse is a role to be assumed by all facets of the research community. The consolation in this regard is the increasing 'liberalization' of health research. The current effort at harnessing the transdisciplinary initiative and ideas about health, medicine and disease holds a lot of promise for a sustainable global health

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X. Acknowledgement

The author acknowledges the Grant received from the University of Ilorin to attend the 6th Open meeting of the International human Dimensions Programme which took place in Bonn, Germany, where the first draft of this paper was presented. I also wish to thank Delali Dovie of the University of Witwatersrand, South Africa, for painstakingly reading through the manuscript and for the many useful suggestions on the earlier drafts of the paper.

Application Of Special Functions And SIRP In Wireless Communication Fading Statistics

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GJSFR Classification – F (FOR) 010204,100502,100510

***Abstract*- In the present paper, we obtain the distribution of quotient of two independent I-function random variables. We show that I-function and spherically- invariant random processes (SIRP) can be used to provide a unified theory on wireless communication fading statistics. Further, we show that various performance measures of fading communication system, such as error probability and amount of fading can be evaluated using the SIRP statistical characterization.**

Keywords-Probability density function, distribution function, spherically-invariant random process, I-function.

I. Introduction

The distribution of quotient of I-function random variables is of interest in many areas of physics and engineering. Wireless communication systems are the major modern communications that impact the human lifestyle in the last several years. Now due to the convergence of technology, many diverted technologies such as radio, camcorder, digital camera or even television are combined with wireless equipment. The frequency and data management are decisively concerned.Therefore, the quality and capacity of the channels are the most significantly predicament. Most coded and uncoded digital communication systems are analyzed and designed under the ideal free-space propagation Gaussian noise assumption. For wireless communication, however, multipath fading, scattering and shadowing can significantly reduce the performance of the system compared to the ideal Gaussian assumption. Under various outdoor/indoor narrowband flat fading scenarios, a variety of statistical model of envelope distributions, such as Rayleigh, Rician, exponential, Nakagami -m, Weibull, lognormal K_v etc. have been proposed by various authors. However, almost all these distributions have been proposed purely from empirical fitting of measured data to a statistical distribution with neither analytical nor physical justification. Yao et al. (2004) gave a systematic approach to show that spherically-invariant random process can be used to provide a unified theory to model fading channel statistics. In recent papers Chaurasia and Kumar (2010) and Chaurasia and Singh (2010) have investigated the distributions of random variables pertaining to special functions. In this paper, we obtain the distribution of quotient of two independent I-function random variables. Further, we show that the distribution of I-function random variables can be used in wireless communication fading statistics based on spherically-invariant random processes.

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II. Spherically-invariant random processes

The nth order pdf of a spherically-invariant random process $\{X(t), -\infty < t < \infty\}$ is given in the form

$$p_X(x) = D_n\, h_n(Q_n(x, \mu, \rho)), \quad x \in R^n,$$

where $D_n = (2\pi)^{-n/2}|\rho|^{-1/2}$ is a normalization constant, μ is the mean vector, ρ is the positive definite covariance matrix, quadratic form is given by $Q_n(x, \mu, \rho) = (x-\mu)^T \rho^{-1}(x-\mu)$ and $h_n(.)$ belongs to a classes of positive scalar-valued functions. Clearly, the SIRP includes the Gaussian process as a special case, and in fact many of the analytically tractable properties of the Gaussian process are still valid for SIRP. Yao (1973) provide a Representation Theorem that yields a necessary and sufficient condition for the explicit form of the nth order pdf of the SIRP, which is stated as

SIRP Representation Theorem 2.1

A necessary and sufficient condition for the nth order pdf of a SIRP is that

$$p_X(x) = (2\pi)^{-1/2}|\rho|^{-1/2} h_n(Q_n(x, \mu, \rho)), \quad x \in R^n, \quad (1)$$

where $h_n(r)$ is given by

$$h_n(r) = \int_0^\infty v^{-n} e^{-r^2/(2v^2)} f_V(v)\, dv, \quad 0 < r < \infty \quad (2)$$

and $f_V(v)$ is any univariate pdf defined on $(0,\infty)$.

Now, using (2) in (1), we get

$$p_X(x) = \int_0^\infty (2\pi)^{-n/2}|v^2\rho|^{-1/2} e^{-(1/2)(x-\mu)^T (v^2\rho)^{-1}(x-\mu)} f_V(v)\, dv, \quad (3)$$

which shows that the nth order pdf of a SIRP is the statistical average of the nth order pdf of a Gaussian process with an arbitrary positive valued univariate random variable V having the above mentioned pdf $f_V(v)$. This means that every SIRP process X has the simple interpretation of being equivalent to $\{X(t) = V\,Y(t), -\infty < t < \infty\}$, where $\{Y(t), -\infty < t < \infty\}$ is a Gaussian process independent of V.

A narrowband Gaussian process can be expressed as $Y(t) = Y_I(t)\cos(2\pi ft) - Y_Q(t)\sin(2\pi ft)$, where $Y_I(t)$ and $Y_Q(t)$ are two zero-mean independent low pass Gaussian processes with its envelope denoted by $R_Y(t) = (Y_I(t)^2 + Y_Q(t)^2)^{1/2}$, which has a Rayleigh pdf. Recently Yao (2003), Abdi et al. (2000) and Aalo and

Zhang (2000) have proposed to use a SIRP to model fading on the narrowband Gaussian processes as given by $\{X(t)= V Y(t), -\infty < t < \infty\}$. Then, X(t) is also a zero-mean narrowband process and its envelope can be expressed as $R_X(t)=(X_I(t)^2+X_Q(t)^2)^{1/2}=(VY_I(t)^2+V Y_Q(t)^2)^{1/2}=V R_Y(t).$

To simplify our notations, if we denote the original envelope of the Gaussian process by R, and the fading SIRP envelope by X. Then, we have

$$X = V.\,R \tag{4}$$

Incidentally, the use of a random mixture model consisting of the product of positive valued random variables against a Rayleigh random process has been proposed by Ward (1981) for modeling radar clutter envelopes and Loo (1985) for modeling Land Mobile Satellite fading channel statistics. If the SIRP model is to be useful to characterize the fading channel statistic, then we must be able to show that for any of the known $f_X(.)$ envelope pdfs (e.g., Weibull, Nakagami-m, etc.) and $f_R(.)$ being a Rayleigh pdf, there must be $f_V(.)$ pdfs satisfying (4). But from elementary probability theory proposed by Dwass (1970), if V and R in (4) are two independent positive valued univariate random variables, then their pdfs must satisfy

$$f_X(x)=\int_0^\infty f_V(v) f_R\left(\frac{x}{v}\right)\left(\frac{1}{v}\right)dv,\quad 0<x<\infty \tag{5}$$

The Mellin transform (Erdélyi (1954)) of any $f_X(x)$ univariate defined on $(0,\infty)$ is given by

$$M\{f_X(x)\} = \int_0^\infty f_X(x)\, x^{s-1}\,dx = E\,(X^{s-1}) \tag{6}$$

Taking (s – 1)th power on both sides of (4) and taking expectations, we get

$$E\,(X^{s-1}) = E\,(V^{s-1})\,E\,(R^{s-1}) \tag{7}$$

Now, making use of (6) and after a little simplification, we get

$$f_V(v) = M^{-1}\left\{\frac{M\{f_X(x)\}}{M\{f_R(r)\}}\right\} \tag{8}$$

The Mellin transform $M\{f_R(r)\}$ of the Rayleigh pdf and $M\{f_X(x)\}$ of most fading envelope pdfs (e.g., Weibull, Nakagami-m, etc.) can easily be evaluated. However, the inverse Mellin transform of the ratio $M\{f_X(x)\}/M\{f_R(r)\}$ usually involves the ratio of various Gamma functions and thus is not readily available of $M\{f_X(x)\}$ functions of interest in Erdélyi (1954). Therefore, the explicit expression for $f_V(v)$ in (8) can not be analytically evaluated from the known Mellin transform table (Erdélyi (1954)) and symbolic mathematical programs.

III. I-FUNCTION

The I-function introduced by Saxena (1982), is defined as

$$I(z)=I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m}:(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}z^s\,ds \tag{9}$$

where $p_i\,(i=1,\ldots,r), q_i\,(i=1,\ldots,r)$, m, n are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i$ $(i = 1,2,\ldots,r)$; r is finite, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}$ are real and positive and a_j,b_j,a_{ji},b_{ji} are complex numbers such that $a_j(b_h+\nu)\neq\beta_h(a_j-1-k)$, for $\nu,k=0,1,2,\ldots,\ m;\ i=1,2,\ldots,\ r.$

The contour extends from $-i\infty$ to $i\infty$ such that all the poles of $\Gamma(b_j-\beta_j s)$ for j = 1,…,m and those for $\Gamma(1-a_j+\alpha_j s)$ for j = 1,…,n are separated by this contour.

The I-function can also be expressed in terms of the inverse Mellin transform as

$$I(z)=I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\right\}}z^{-s}\,ds$$

$$=M^{-1}\left\{\frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)\right\}}\right\} \tag{10}$$

Clearly, the I-function includes the H-function as a special case when r = 1 and the G-function when r, α_j and β_j are set equal to unity.

Consider the generalized gamma pdf defined by

$$f(x) = \frac{\beta\, a^{\alpha/\beta}\, x^{a-1}\exp\,(-ax^{\beta})}{\Gamma(\alpha/\beta)}$$

$$=\frac{a^{1/\beta}}{\Gamma(\alpha/\beta)}I^{1,0}_{0,1:1}\left[a^{1/\beta}x\left|\begin{matrix}----\ ;\ ----\\\left(\frac{\alpha-1}{\beta},\frac{1}{\beta}\right);----\end{matrix}\right.\right] \tag{11}$$

Various positive-valued univariate pdfs are special cases of the generalized gamma pdf and can be expressed in terms of the I-function such as the gamma pdf, Rayleigh pdf, Weibull pdf, Nakagami-m pdf, Chi-Squared pdf, half-Gaussian pdf, One-sided exponential pdf, half-Cauchy pdf, half-Cauchy like pdf, beta pdf, lognormal pdf, Rician pdf, K_v pdf, etc.

IV. THE DISTRIBUTION OF QUOTIENT OF I-FUNCTION RANDOM VARIABLES

In this section, we obtain the distribution of quotient of two independent random variables X_1 and X_2 with pdfs $f_k(x_k)$ for k = 1, 2 given by

$$f_k(x_k)=\begin{cases} C_k\, I^{m_k,n_k}_{p_{i_k},q_{i_k}:r_k}\left[\mu_k x_k \left| \begin{matrix} (a_j^{(k)},\alpha_j^{(k)})_{1,n_k};(a_{ji_k}^{(k)},\alpha_{ji_k}^{(k)})_{1,p_{i_k}} \\ (b_j^{(k)},\beta_j^{(k)})_{1,m_k};(b_{ji_k}^{(k)},\beta_{ji_k}^{(k)})_{1,q_{i_k}} \end{matrix}\right.\right], & x_k>0, \\ 0\ , & \text{otherwise} \end{cases} \tag{12}$$

where C_k and μ_k are normalizing constants.

Theorem 4.1. If X_1 and X_2 are two independent positive valued univariate I-function random variables with pdfs $f_k(x_k)$ for k = 1,2 given by (12), then the pdf of the random variable $V = X_1/X_2$ is given by

$$f_V(v)=\begin{cases} \frac{C_1C_2}{\mu_2^2}\, I^{m_1+n_2,n_1+m_2}_{p_{i_1}+q_{i_2},q_{i_1}+p_{i_2}:r_1r_2}\left[\left(\frac{\mu_1}{\mu_2}v\right)\left| \begin{matrix} (a_j^{(1)},\alpha_j^{(1)})_{1,n_1},(1-b_j^{(2)}-2\beta_j^{(2)},\beta_j^{(2)})_{1,m_2}; \\ (b_j^{(1)},\beta_j^{(1)})_{1,m_1},(1-a_j^{(2)}-2\alpha_j^{(2)},\alpha_j^{(2)})_{1,n_2}; \end{matrix}\right.\right. \\ \qquad \left.\begin{matrix} (a_{ji}^{(1)},\alpha_{ji}^{(1)})_{1,p_{i_1}},(1-b_{ji_2}^{(2)}-2\beta_{ji_2}^{(2)},\beta_{ji_2}^{(2)})_{1,q_{i_2}}; \\ (b_{ji_1}^{(1)},\beta_{ji_1}^{(1)})_{1,q_{i_1}},(1-a_{ji_2}^{(2)}-2\alpha_{ji_2}^{(2)},\alpha_{ji_2}^{(2)})_{1,p_{i_2}} \end{matrix}\right], \quad v>0 \\ 0\ , \qquad \text{otherwise} \end{cases} \tag{13}$$

Proof- To obtain the pdf of $V = X_1/X_2$, we use the method of Mellin transform and its inverse. The pdf $f_V(v)$ of the random variable $V = X_1/X_2$ is given by

$$f_V(v)=\begin{cases} M^{-1}[M_s\{f_1(x_1)\}M_{s-2}\{f_2(x_2)\}],\ v>0 \\ 0\ , \qquad \text{otherwise}. \end{cases}$$

Now, substituting the values of $f_1(x_1)$ and $f_2(x_2)$ from (12), using equation (10), rearranging and expressing the result thus obtained in terms of I-function, we get the value of $f_V(v)$. The result obtained above is quite general in nature and a large number of corresponding results involving simpler pdfs can easily be obtained on merely specializing the various parameters involved therein.

If we set X_2 = R to have a Rayleigh pdf, then $f_V(v)$ in the equation(13) is valid for any X_1 given by any positive valued univariate I-function pdf independent of X_2 having the form (12). In particular, if X_1 = X is a generalized Gamma pdf of the form (11), then it includes the various pdfs such as the gamma pdf, Rayleigh pdf, Weibull pdf, Nakagami-m pdf, Chi-Squared pdf, half-Gaussian pdf, One-sided exponential pdf, etc.

Denoting

$$m_1=q_{i_1}=m_2=q_{i_2}=1,\ n_1=p_{i_1}=n_2=q_{i_2}=0,\ r_1=r_2=1,$$

$$m_1+n_2=1,\ n_1+m_2=1,\ p_{i_1}+q_{i_2}=1,\ q_{i_1}+p_{i_2}=1,\ r_1\, r_2=1,$$

$$\mu_1=a_1^{1/\beta_1},\ \mu_2=a_2^{1/2},\ C_1=\frac{a_1^{1/\beta_1}}{\Gamma(\alpha_1/\beta_1)},\ C_2=a_2^{1/2},$$

$$\frac{\mu_1}{\mu_2}=a_1^{1/\beta_1}a_2^{-1/2},\ b_1^{(1)}=\frac{\alpha_1-1}{\beta_1},\beta_1^{(1)}=\frac{1}{\beta_1},\ b_1^{(2)}=1/2,\ \beta_1^{(2)}=\frac{1}{2},$$

then $f_V(v)$ in (13) becomes

$$f_V(v) = \frac{C_1 C_2}{\mu_2^2} I_{1,1;1}^{1,1}\left[\frac{\mu_1 v}{\mu_2}\left|\begin{matrix}\left(-\frac{1}{2},\frac{1}{2}\right)\\ \left(\frac{\alpha_1-1}{\beta_1},\frac{1}{\beta_1}\right)\end{matrix}\right.\right]$$

$$= \frac{a_1^{1/\beta_1} a_2^{-1/2}}{\Gamma(\alpha_1/\beta_1)} I_{1,1;1}^{1,1}\left[a_1^{1/\beta_1} a_2^{-1/2} v\left|\begin{matrix}\left(-\frac{1}{2},\frac{1}{2}\right)\\ \left(\frac{\alpha_1-1}{\beta_1},\frac{1}{\beta_1}\right)\end{matrix}\right.\right], v > 0. \quad (15)$$

Thus, (15) provides the solution of (8). In addition, the I-function technique can provide explicit evaluation of the moment generating function and moments of the random variable V with the pdf of (15).

V. PERFORMANCE MEASURES OF FADING COMMUNICATION SYSTEMS

In digital cellular communication systems, the probability of error is an important tool for measuring the system performance. Consider coherent detection of a binary phase shift keying (BPSK) system over a frequency flat slow SIRP fading channel. The average bit error probability (BEP) for such a communication system is given by

$$P_b(E) = \int_0^\infty Q\left(\sqrt{2r^2 E_b/N_0}\right) p_R(r)\, dr, \quad (16)$$

Where

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta,\ x \geq 0 \quad (17)$$

and E_b/N_0 the bit energy-to-noise spectral density ratio. Substituting the value of Q(x) from (17) in (16), we get

$$P_b(E) = \int_0^{\pi/2} d\theta \int_0^\infty \exp\left(-\frac{r^2 E_b/N_0}{\sin^2\theta}\right) p_R(r)\, dr \quad (18)$$

Now, using (5) in (17), we get

$$P_b(E) = \int_0^\infty v^{-2} f_V(v)\, dv \frac{1}{\pi}\int_0^{\pi/2} d\theta \int_0^\infty \frac{r}{\sigma^2} \exp\left(-r^2\left(\frac{E_b/N_0}{\sin^2\theta} + \frac{1}{2\sigma^2 v^2}\right)\right) dr$$

$$= \int_0^\infty f_V(v)\, dv \frac{1}{\pi}\int_0^{\pi/2} \frac{\sin^2\theta}{\sin^2\theta + v^2\bar{\gamma}}\, d\theta$$

$$= \int_0^\infty f_V(v) \frac{1}{2}\left[1 - \sqrt{\frac{v^2\bar{\gamma}}{1 - v^2\bar{\gamma}}}\right] dv$$

$$= \int_0^\infty f_V(v) P_{bRay}(E; v^2\bar{\gamma})\, dv \quad (19)$$

where $\bar{\gamma} = 2\sigma^2 E_b/N_0 = \Omega E_b/N_0$ is the average fading signal-to-noise ratio (SNR), and $P_{bRay}(E;\bar{\gamma}) = \frac{1}{2}\left[1 - \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}\right]$ is the average BEP for a Rayleigh fading process.

As an example, consider the case of a half-Cauchy-like fading pdf $f_X(x) = 2ax/(1+2ax^2)^{3/2}, x > 0, a > 0,$ for which $f_V(v)$ is given by

$$f_V(v) = \sqrt{\frac{2}{\pi}}\, v^{-2} \exp\left(-\frac{1}{2v^2}\right). \quad (20)$$

Clearly, the half-Cauchy-like pdf has the same asymptotic behavior as the half-Cauchy pdf $f_X(x) = (2a/\pi)/(a^2 + x^2), x > 0, a > 0.$ Both of these pdf asymptotically behave as $O\left(\frac{1}{x^2}\right)$ as x approaches infinitely. Then, for coherent detection of BPSK, we have from (20)

$$P_b(E) = \sqrt{\frac{1}{2\pi}}\int_0^\infty v^{-2} \exp\left(-\frac{1}{2v^2}\right)\left[1 - \sqrt{\frac{v^2\bar{\gamma}}{1+v^2\bar{\gamma}}}\right] dv \quad (21)$$

Putting $\frac{1}{2v^2} = x,$ we get

$$P_b(E) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{x}} \exp(-x)\left[1 - \sqrt{\frac{\bar{\gamma}}{2x + \bar{\gamma}}}\right] dx \quad (22)$$

It is easy to show that the limits of the integrand at x = 0 and x = ∞ are both equal to zero. Thus, since the integrand is always positive in the domain of the integral and vanishes at its end points, it has a maximum somewhere in between. A plot of $P_b(E)$ versus $\bar{\gamma}$ determined from (22) comparison with the comparable result for a Rayleigh fading distribution reveals that the two are quite similar with the half-Cauchy-like SIRP having slightly better average BEP behavior at small $\bar{\gamma}$ .Thus, even though the half-Cauchy-like SIRP appears to have an infinite amount of fading, it does result in a finite and well behaved average BEP performance. Other results based on the explicit use of the Theorem 4.1 for error probabilities can also be obtained.

The "amount of fading" (AF) is another performance measure, which is introduced by Charash (1974, p.29, eq.(6)) as a measure of the severity of the fading channel by itself. It is suggested (Simon and Alouini (2000), Chap.2, Sec.2.2)) that the AF measure is often appropriate in the more general content of describing the behavior of systems with arbitrary combining techniques and channel statistics and thus can be used as an alternative performance criterion whenever convenient. If γ_t denote the total instantaneous SNR at combiner output, the AF is defined by

$$AF = \frac{Var\{\gamma_t\}}{(E[\gamma_t])^2} = \frac{E[\gamma_t^2]-(E[\gamma_t])^2}{(E[\gamma_t])^2} \tag{23}$$

which can be expressed in terms of the MGF of γ_t by

$$AF = \frac{\left.\frac{d^2M_{\gamma_t}(s)}{ds^2}\right|_{s=0} - \left(\left.\frac{dM_{\gamma_t}(s)}{ds}\right|_{s=0}\right)^2}{\left(\left.\frac{dM_{\gamma_t}(s)}{ds}\right|_{s=0}\right)^2} \tag{24}$$

Consider the case when R is Rayleigh distributed. Defining $Z = R^2$ and using (5), the pdf of Z is, for $0 \le z \le \infty$

$$p_z(z) = \frac{p_R(\sqrt{z})}{2\sqrt{z}} = \frac{1}{2\sigma^2}\int_0^\infty v^{-2}\exp\left(-\frac{z}{2\sigma^2 v^2}\right) f_V(v)\,dv, \tag{25}$$

with first and second moments given by

$$\overline{Z} = 2\sigma^2\,\overline{V}^2 \tag{26}$$

and

$$\overline{Z}^2 = 8\,\sigma^2\,\overline{V}^4 \tag{27}$$

Now, letting $\gamma = (Z/\overline{Z})\overline{\gamma}$ and using (25) the amount of the SIRP process in the no diversity case is given by

$$AF = \frac{Var\{Z\}}{(E[Z])^2} = \frac{\overline{Z}^2-(\overline{Z})^2}{(\overline{Z})^2} = \frac{2\overline{V}^4}{(\overline{V}^2)^2} - 1 \tag{28}$$

As an example, consider again the case of the half-Cauchy-like SIRP for which the pdf of V is given in (20). To evaluate the AF for this process, first we compute

$$\frac{\overline{V}^4}{(\overline{V}^2)^2} = \sqrt{\frac{\pi}{2}}\lim_{\omega\to\infty}\frac{\int_0^\infty v^2\exp\left(-\frac{1}{2v^2}\right)dv}{\left[\int_0^\infty \exp\left(-\frac{1}{2v^2}\right)dv\right]^2} \tag{29}$$

Putting $\frac{1}{2v^2} = x$, we get

$$\frac{\overline{V}^4}{(\overline{V}^2)^2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\lim_{\omega\to\infty}\frac{\int_{1/2\omega^2}^\infty x^{-5/2}\exp(-x)\,dx}{\left[\int_{1/2\omega^2}^\infty x^{-3/2}\exp(-x)\,dx\right]^2} \tag{30}$$

Using known result (Gradshteyn and Ryzhik (1994), eq. (3.381.6)), we get

$$\frac{\overline{V}^4}{(\overline{V}^2)^2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\lim_{u\to\infty} u^{1/4} e^{u/2}\frac{W_{-5/4,-3/4}(u)}{[W_{-5/4,-1/4}(u)]^2} \tag{31}$$

Other results on the average SNR at the output of a dual diversity selection combiner and its outage probability can also be evaluated in terms of the I-function.

VI. References

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Algorithm For Finding The Proper Continuous Distribution Function To A Unimodal Empirical Distribution Function

GJSFR Classification – F (FOR) 010106,010406,010404

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***Abstract*- We present an algorithm containing the first step towards solving one of the Fundamental Problems of Non-parametric Mathematical Statistics: determining the distribution of an unknown unimodal continuous population from which we have a large random sample of discrete observations. We will present the "algorithm of non-fitting" with the help of which, by using the so-called relative increment functions as auxiliary functions, one can eliminate a large class of classical continuous unimodal distributions which our population does not belong to. In the remaining class of unimodal and smooth distributions one can approximate the distribution of the population in question. The algorithm is illustrated in three numerical examples.**

Keywords- unimodal smooth distribution function,relative increment function, large sample, curve fitting.

I. Introduction

By using only the information obtained from a large random sample of an unknown unimodal continuous population, we can, in some cases, determine the distribution of this population.

Assume UCP denotes an unknown Unimodal Continuous Population with the unknown probability density function f whose open support is the open real interval $I \coloneqq (r,s) \subseteq (-\infty,+\infty)$. Assume that $f \in C^{\infty}(r,s)$.

Suppose that, from UCP, we have a large random sample of size N spanning the closed real interval J $(\subseteq closure(I))$. Assume we have no other information on our particular UCP. We form an equidistant partition of the sample interval J into n subintervals $I_k = [x_{k-1}, x_k)$ of equal length $x_k - x_{k-1} \equiv a(>0)$, for $k = 1,2,...,n$.

The frequency distribution of the sample will be ν_k, $k = 1,2,...,n$ (ν_k is the number of sample values belonging to the interval I_k ,for all k).Assume $\nu_n \geq 1$.

One can form both the relative frequencies $r_k (= \frac{\nu_k}{N})$ and the cumulative relative frequencies $y_k (= \sum_{j \leq k} r_j)$ which induce the empirical cumulative distribution F_{emp} whose points of discontinuity are at equidistant places x_k , and we have $\boldsymbol{F_{emp}(x_k) = y_k}$ ***for*** $\boldsymbol{k} = 0,1,...,\boldsymbol{n}$ .In addition, we have $F_{emp}(x_k) = y_k < 1$ (for $\boldsymbol{k} = 0,1,2,...,\boldsymbol{n}-1$), since $\nu_n \geq 1$.

Our aim is to determine the unimodal and infinitely smooth (in I) distribution of our continuous population.

By an infinitely smooth function, we mean a real function which is infinitely many times differentiable in its open support $I \coloneqq (r,s) \subseteq (-\infty,+\infty)$.

By a unimodal distribution function, we mean a distribution function whose probability density function is unimodal.

II. Algorithm

Step 1-.[containing "the method of non-fitting" which will allow us to eliminate a relatively large class of unimodal and infinitely smooth distributions which our continuous population does not belong to.]

We consider an arbitrarily chosen "classical" distribution function $F(x) = \int_{-\infty}^{x} f(t)dt$ from the class of Unimodal and infinitely Smooth Distribution Functions (USDF) with at most one point of inflection and $\boldsymbol{F(x)} < 1$ for $x \in I$.

We assign our auxiliary functions, the so-called Relative Increment Functions (RIF) to both $\boldsymbol{F_{emp}}$ and $\boldsymbol{F}$, and then we compare the monotonic properties of the RIFs. If they do not match, then we drop $\boldsymbol{F}$, otherwise we keep it (and put it in S).

More precisely, we form the empirical relative *Increment Function-*

$$h_{emp}(x_k) \quad := \quad \frac{y_{k+1} - y_k}{1 - y_k} \quad \text{for} \quad \boldsymbol{k} = 0,1,2,...,\boldsymbol{n}-1,$$

and then we check the monotonic behavior of $\boldsymbol{h_{emp}}$.

We keep (and put it in S) the continuous cumulative distribution function $\boldsymbol{F}$ whose relative increment function

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(abbreviated as RIF) $h_F(x) := \dfrac{F(x+a)-F(x)}{1-F(x)}$ (for $a = x_{k+1} - x_k > 0$) has the same monotonic behavior as h_{emp}
(like, e.g., both relative increment functions h_{emp} and h_F increase,
or both decrease,
or both have a $\bigcap$ -shape: first increase and then decrease,
or both have a $\bigcup$ -shape: first decrease and then increase, etc.),
otherwise we drop F .

This way we can eliminate a large part of all known types of unimodal and infinitely smooth classical distribution functions, and we will keep and use a relatively small part S of the USDF functions.

Step 2- We will work with the remaining family S of unimodal and smooth cumulative distribution functions and, by using e.g. the Least Square's Method, we select the unimodal and infinitely smooth distribution function $F \ (\in S)$ providing the best fit to the cumulative relative frequencies y_k : $\sum_{k=1}^{n}\left[F(x_k)-y_k\right]^2 \to \min!$

[or, if F does not have an explicit form, then we choose F whose density function f provides the best fit to the relative frequencies r_k : $\sum_{k=1}^{n}\left[f(x_k)-r_k\right]^2 \to \min!$].

If there are more best-fit candidates for F , then Chi-Squer Test (or Kolmogorov's Test) applies for testing goodness of fit ; furthermore, we will compare h_{emp} and h_F , and choose the "best F " in the sense of $\min_F \sum_{k=1}^{n}\left[h_F(x_k)-h_{emp}(x_k)\right]^2$.

Remark 1- Our method is completely different from Density Estimation Methods like Average Shifted Histograms, Smoothing Splines etc.(Venables and Ripley 1994), because the result of our algorithm is an infinitely smooth (in the support I) distribution function with at most one point of inflection, one of the "classical" continuous distribution functions mentioned by Feller (1971), Johnson and Kotz (1970), Stuart and Ord (1987).

Remark 2-Dental researchers have discovered the importance of relative increment of decay or RID Index long time ago. RID Index was first defined by Porter and Dudman (1960), and then used intensively by the author of this paper and co-authors (Sobkowiak, Szabo, Radtke and Adler 1973; Adler and Szabo 1974a,b, 1975, 1976, 1979a,b, 1984; Szabo 1974, 1976, 1989, 1994, 1996, 1999, 2004, 2006; Szondy and Szabo 1976).

Remark 3-. The hazard rate of a continuous distribution function F , being defined to be $\dfrac{f}{1-F} = \lim_{a \to 0} \dfrac{h_F(x)}{a}$ (Barlow and Proschan 1967) , is closely related to the RIF of F . Clear that the hazard rate and the relative increment function of F have the same monotonic behavior.

Remark 4.-This algorithm has a disadvantage. It can handle only samples whose empirical RIF has a relatively simple shape that is either increasing, or decreasing, or it has only two monotonic phases
[increases first and, having culminated, it decreases; or decreases first and, having reached its minimum, it increases].

The main research problem was to find sufficient conditions with the help of which one can easily check the monotonic behavior of the relative increment function h_F of a distribution function F [without using the inconvenient term $1-F \quad (= \int_x^{\infty} f(t)dt)$ appearing in h_F]. These sufficient conditions have been worked out by Adler and Szabo (1974a), and Szabo (1976, 1989, 1994, 1996, 1999, 2006). These conditions contain relatively simple assumptions under which the RIF h_F of the unimodal and infinitely smooth distribution function F increases or decreases, or has two monotonic phases. These assumptions deal with the simple expression f / f' .

These results have been applied to logistic, extreme value, z-, Pareto of the third kind, Weibull, trigonometric, normal, gamma, beta of the first kind, Pareto of the second kind, chi-square and many other distributions [See the Propositions below.]

Monotonic behavior of the relative increment functions of some unimodal and infinitely smooth distribution functions from Feller, W. (1971), Gradshteyn and Ryzhik (1980), Johnson and Kotz (1970), Stuart and Ord (1987) have been checked, and some of the results in Szabo (1976, 1996, 1999, 2006) are summed up as follows.

Examples of Unimodal and infinitely Smooth Distribution Functions (USDF), and monotonic properties of the corresponding Relative Increment Functions (RIF) are given in the following five Propositions.

Prop.1. The following USDFs have increasing RIFs

1. $F(x) = 1 - x^2, \quad I = (-1,0)$;
2. $F(x) = 1-(-x)^k, k > 1$ integer, $I = (-1,0)$;
3. $F(x) = \sin x$; $\quad I = (0, \pi/2)$;
4. $F(x) = 1 + \tan x, I = (-\pi/4, 0)$;
5. $F(x) = 1 + sh(x), I = (\ln(\sqrt{2}-1), 0)$;
6. $F(x) = 2 - ch(x); \quad I = (\ln(2-3^{1/2}), 0)$;
7. $F(x) = \ln x, \quad I = (1, e)$;

8. Uniform distribution;
9. Any convex (below) USDF.
10. $F(x) = (1 - \exp(-\lambda x))^k, \lambda > 0, k > 1, I = (0, \infty).$
11. $F(x) = 1 - \exp(-\lambda \cdot e^x), \lambda > 0, \ I = (-\infty, \infty).$
12. $F(x) = (1 + e^{-x})^{-k}, k > 0, \ I = (-\infty, \infty).$
13. $F(x) = 2^{-k} \cdot (1 + th(x))^k, k > 0, \ I = (-\infty, \infty).$
14. Logistic Distribution $F(x) = (1 + e^{-\lambda x})^{-1}, \lambda > 0, I = (-\infty, \infty).$
15. Fisher's z-distribution $F(x) = C \cdot \int_{-\infty}^{x} e^{nt} \cdot (1 + k \cdot e^{2t})^{-\alpha} dt,$ $k = \frac{n}{n'}$ $\alpha = \frac{n + n'}{2} (> 0)$, $(0 <) C = 2 \cdot k^{n/2} \cdot \Gamma(\alpha) \cdot \left[\Gamma\left(\frac{n}{2}\right) \cdot \Gamma\left(\frac{n'}{2}\right) \right]^{-1}$ and n, n' are positive integers, $I = (-\infty, \infty).$
16. Weibull distribution when $\alpha > 1, F(x) = 1 - \exp\left(-\lambda \cdot x^{\alpha}\right), \lambda > 0, \quad I = (0, \infty).$
17. Extreme value distribution $F(x) = \int_{-\infty}^{x} \exp(-t - e^{-t}) dt, \quad I = (-\infty, \infty).$
18. $F(x) = 1 - 2[c \cdot (1 + e^x)^k - c + 2]^{-1}, c > 0, k = 1,2; I = (-\infty, \infty).$
19. Normal distribution

$$F(x) = K \cdot \int_{-\infty}^{x} \exp(-1/2 \cdot \sigma^{-2} \cdot (t - m)^2) dt, K = \sigma^{-1} \cdot (2\pi)^{-1/2}, \sigma > 0, I = (-\infty, \infty).$$

20. (Special) Gamma distribution $F(x) = K \cdot \int_{0}^{x} t^{\alpha - 1} \cdot \exp(-\lambda t) dt, K = \lambda^{\alpha} / \Gamma(\alpha), \lambda > 0, \alpha > 1, I = (0, \infty).$
21. Beta distribution of the first kind $F(x) = C \cdot \int_{0}^{x} t^{\alpha} \cdot (1 - t)^{\beta} dt, C = \Gamma(\alpha + \beta + 2) / [\Gamma(\alpha + 1) \cdot \Gamma(\beta + 1)], \alpha, \beta > -1, I = (0,1).$
22. $F(x) = C \cdot \int_{-s}^{x} (1 - t^2 / s^2)^n dt, C = \left[s \cdot \mathrm{B}(1/2, n + 1) \right]^{-1}, s > 0,$ n is a positive integer, $I = (-s, s)$.

Prop.2. - The following USDFs have DECREASING RIFs:

23. $F(x) = 1 - x^{-\lambda}, \lambda > 0, I = (1, \infty)$;
24. $F(x) = 1 - (\ln x)^{-\lambda}, \lambda > 0, I = (e, \infty)$;
25. $F(x) = 1 - (\ln \ln x)^{-\lambda}, \lambda > 0, I = (e^e, \infty)$;
26. Weibull distribution when $0 < \alpha < 1$, $F(x) = 1 - \exp(-\lambda \cdot x^{\alpha}), \lambda > 0, \quad I = (0, \infty)$;
27. $F(x) = 1 - a \cdot \exp(-bx) - c \cdot \exp(-dx), a, b, c, d > 0, a + c = 1, I = (0, \infty)$;
28. $F(x) = 1 - \sum_{j=1}^{N} a_j \cdot \exp(-b_j x), \quad a_j, b_j > 0, \quad \sum_{j=1}^{N} a_j = 1, I = (0, \infty)$;
29. Pareto distribution of the third kind $F(x) = 1 - k \cdot \exp(-bx) \cdot x^{-a}, a, b, k > 0, I = (k, \infty)$;
30. Chi-Square distribution $F(x) = K \cdot \int_{0}^{x} t^{n/2 - 1} \cdot \exp(-t/2) dt, K = 2^{-n/2} / \Gamma(n/2),$ n is a positive integer, $I = (0, \infty)$;
31. Pareto distribution of the second kind $F(x) = 1 - x^{-k}, k > 0, I = (1, \infty).$
32. (Special) Gamma distribution $F(x) = K \cdot \int_{0}^{x} t^{\alpha - 1} \cdot \exp(-\lambda t) dt, K = \lambda^{\alpha} / \Gamma(\alpha), \lambda > 0, \alpha < 1, I = (0, \infty).$

Prop.3. - The Exponential Distribution Function $F(x) = 1 - \exp[-\lambda(x - a)], \lambda > 0, \quad I = (a, \infty)$ has a *CONSTANT* Relative Increment Function, and this property of RIF is characteristic for the Exponential Distribution.

Prop.4- The following USDFs have RIFs that INCREASE first and, having culminated, they DECREASE:

34. Cauchy distribution $F(x) = \frac{1}{2} + \frac{1}{\pi} \cdot \mathrm{arctg}\, x,$ $I = (-\infty, \infty)$.
35. Inverse Gaussian distribution

$$F(x) = \int_{0}^{x} \left(2\pi t^3 / \lambda\right)^{-1/2} \cdot \exp(-\lambda \cdot (t - m)^2 / (2m^2 \cdot t)) dt, \lambda, m > 0, I = (0, \infty).$$

36. Lognormal distribution

$$F(x) = K \cdot \int_{0}^{x} \exp(-1/2 \cdot \sigma^{-2} \cdot (\ln t)^2) / t dt, K = \sigma^{-1} \cdot (2\pi)^{-1/2}, K = 1/\left[\sigma \cdot \sqrt{2\pi}\right], \sigma > 0, I = (0, \infty).$$

Prop.5.- The following USDFs have RIFs that DECREASE first and, having reached their minima, they INCREASE:

37. $F(x) = 1 + \frac{2}{\pi} \cdot \arcsin x, \quad I = (-1, 0)$;

38. $F(x) = \sqrt{x}, \ I = (0,1)$;

39. $F(x) = (1 - x^2)^{1/2}, \ I = (-1,0)$.

Remark 4- By using the transformation $x' = Ax + B, \ (for \quad A > 0)$, the interval I can be transformed, but the monotonic behavior of RIF over the transformed interval I' will NOT change.

Numerical Example 1.

Time points x_k	Relative frequencies r_k
1	0.7059
2	0.1612
3	0.0621
4	0.0324
5	0.0154
6	0.0092
7	0.0054

Table 1. Data

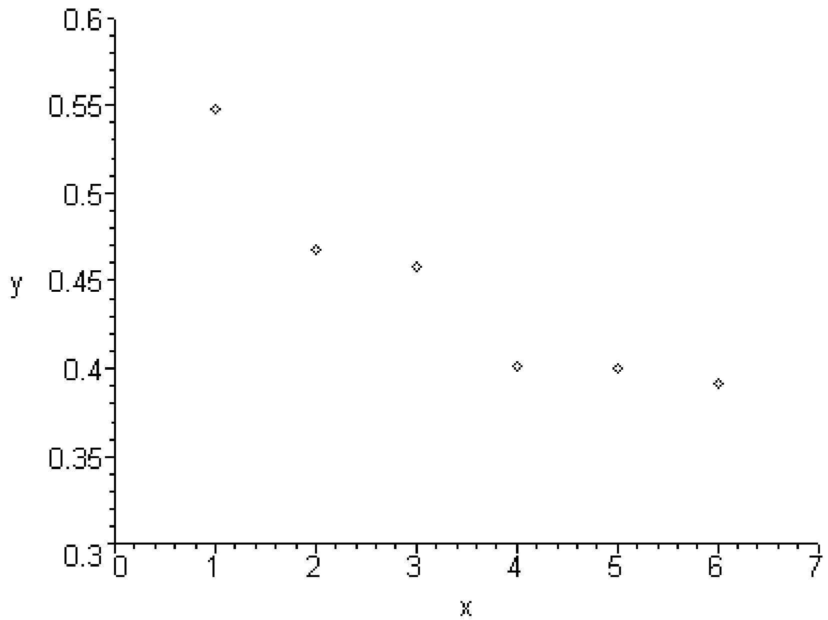

Figure 1. Graph of the corresponding finite sequence of the empirical relative increments $\{(x_k, h_{emp}(x_k))\}$ has a decreasing shape.

This monotonic behavior of the RIF h_F can be found only for USDFs No. 23-32 in Proposition 2. Among these functions, the Weibull distribution function $F(x) = 1 - \exp[-1.2141(x + 0.0771)^{0.6667}]$ provided the best least squares fit $\sum_{k=1}^{7} [F(x_k) - y_k]^2 \approx 0.00037$.

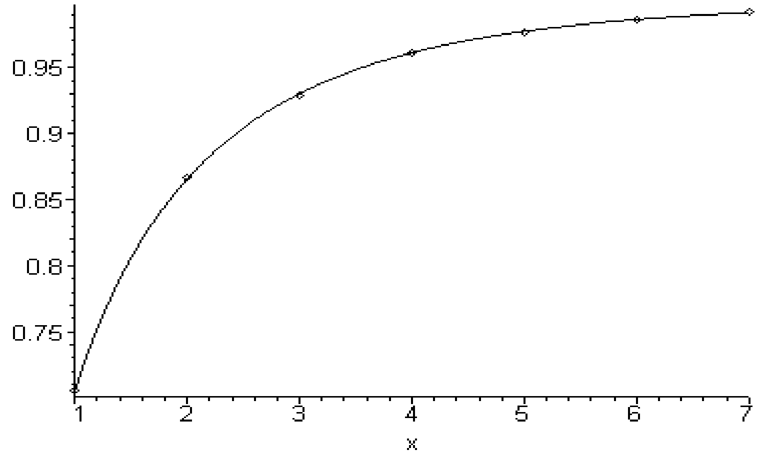

Figure 2. The graphs of F and $\{(x_k, y_k)\}$.

Numerical Example 2. (From Dental Science).

Age x_k	Cumulative caries prevalence value y_k
7	0.050
12	0.116
17	0.195
22	0.282
27	0.372
32	0.460
37	0.543
42	0.618
47	0.684
52	0.732
57	0.761
62	0.781

Table 2. Longitudinal data $\{x_k, y_k\}$ collected by Prof.P.Adler, Late Director of Dental Clinics of Debrecen University, Debrecen, Hungary (European Union).

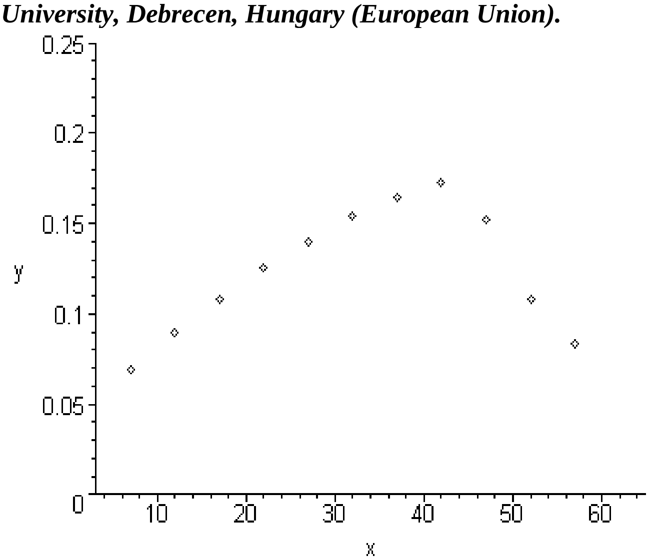

Figure 3. Graph of the corresponding empirical relative increment sequence $\{(x_k, h_{emp}(x_k))\}$.

According to this graph, and the overall clinical experience in Dental Science (Porter and Dudman 1960; Sobkowiak, Szabo, Radtke and Adler 1973; Adler and Szabo 1974a,b, 1975, 1976, 1979a,b, 1984), the relative increments of decay [i.e. the RIF of cumulative caries prevalence curve] increases up to a certain age, and decreases thereafter. This monotonic behavior of the RIF h_F can be found only for USDFs No. 34-36 in Proposition 4. Among these three functions, the truncated Cauchy distribution function $F(x) = \max\{1 + 1.48939\left(-0.5 + \frac{1}{\pi}\arctan\left(0.042887x - 0.84988\right)\right), 0\}$ provided the best least squares fit $\sum_{k=1}^{12}[F(x_k) - y_k]^2 \approx 0.0053478$

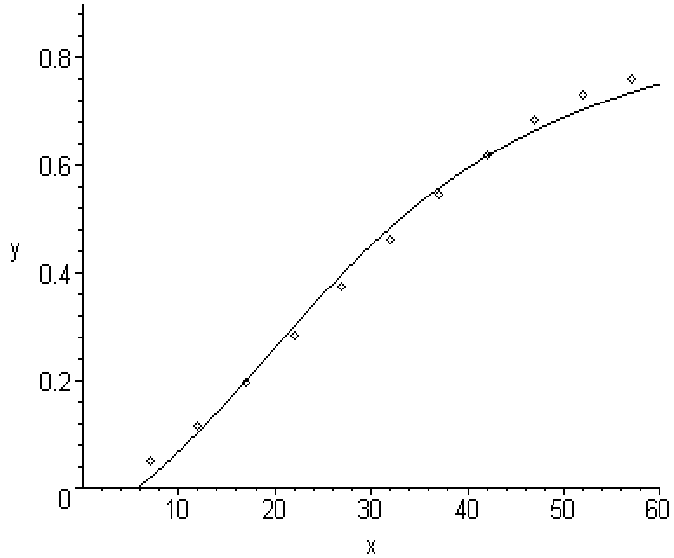

Figure 4. The graphs of F and $\{(x_k, y_k)\}$.

Our method can be used for modeling bounded growth processes as well:

Assume we have the following information on a bounded growth process

$$\boldsymbol{C = \{(x_k, g_k)} \quad \textbf{for} \quad \boldsymbol{k = 1, 2, \ldots, n\}}:$$

1.A finite and increasing sequence of values of observations $g_k (>0)$ measured at some equidistant points x_k [$x_{k+1} - x_k = a(>0),$ for all k] . Suppose our measurements are free from error.

A reasonable a'priori strict upper bound $B(>0)$ given for the process C beforehand: $g_k < B$ for all values of k ,

(B is greater than any possible value of g_k),

and no other information is given on C .

We want to model the growth process C by using a properly chosen, bounded, strictly increasing and infinitely smooth real function G whose values lie between 0 and B : $0 \le G(x) < B$ (for all x) , $G(-\infty) = 0$ and $G(\infty) = B$. We use strict mathematical principles, some auxiliary functions and the above mentioned information only ,and no specific information on the growth process C will be used.

Our modeling procedure is as follows.

First, we consider our transformed (normed) data $y_k = \frac{g_k}{B} \quad (<1)$ to be the values of an empirical cumulative distribution function F_{em}, at the points x_k :

$$F_{emp}(x_k) = y_k \quad for \quad k = 0,1,...,n \quad .$$

Then we use our method described above and obtain the USDF F .Eventually, the increasing function $G(x) = B \cdot F(x)$ will serve as a good smooth model for the growth process C in question (Szabo 2004)

Let me illustrate this method by showing

Numerical Example 3:

Modeling and forecasting the average temperature of changing climate.

	x_k	y_k
1926-35	1	13.470
1936-45	2	14.290
1946-55	3	14.883
1956-65	4	15.310
1966-75	5	15.624
1976-85	6	15.927
1986-95	7	16.273
1996-2005	8	16.758

Table 3. For the growth process of average temperatures at a given geographic place of the Earth, the data of ten-year average temperatures (in Celcius) in the past 80 years.

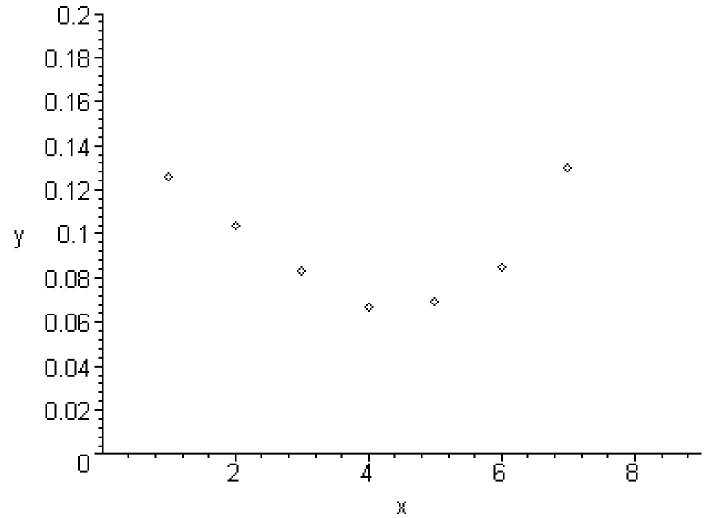

Figure 5. Graph of the corresponding finite sequence of the empirical relative increments $\{(x_k, h_{emp}(x_k))\}$.

According to this graph, the empirical relative increments decrease up to a certain time, and increase thereafter. This monotonic behavior of the RIF h_F can be found only for USDFs No. 37-39 in Proposition 5. Among these three types of functions, the transformed function

$$G(x) = B \cdot F(x) = B \cdot \sqrt{1-[a(x-b)]^2} \quad \text{for}$$

$$x \in \left(b - \frac{1}{a}, b\right) \approx (-8.226, 23.621) \,,$$

$G(x) = 0$ for $x \le b - \frac{1}{a} \approx -8.226$, and $G(x) = B$ for $x \ge b \approx 23.621$

provided the best least squares fit $\sum_{k=1}^{12} [G(x_k) - y_k]^2 \approx 0.13187935$ with $a = 0.0314, b = 23.621, B = 19.295$

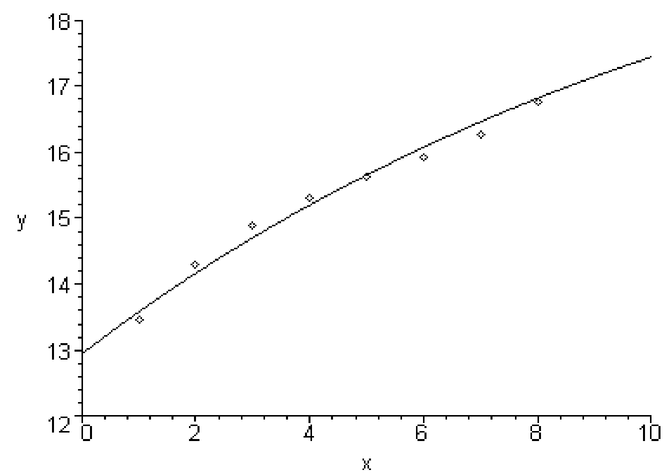

Figure 6. The graphs of G and the data $\{(x_k, y_k)\}$.

According to the value of the limit at infinity

$\lim_{x \to \infty} G(x) = B \approx 19.295$, the expected average temperature after a long period of time can be predicted to be $19.295^o C$ provided the climatic conditions at that very place will not change dramatically and significantly.

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Some Geometric And Physical Properties Of Generalized Peres Space-Time

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GJSFR Classification – F (FOR) 010102,010204,010504

***Abstract*-In this paper we have investigated some geometric and physical properties of generalized Peres space-time P* in the sense of Takeno (1961) and made some discussions.**

***Keywords* - General relativity,plane fronted waves, Ricci tensor, curvature tensor, electromagnetic energy momentum tensor, co-ordinate condition,phase velocity, energy momentum pseudo tensors,parallel vector field vi etc.**

I. Introduction

The pioneer work of Einstein (1916, 18) and Rosen (1937) forms the corner stone of the investigations of plane gravitational waves in general relativity.Plane and plane fronted gravitational waves in general relativity have been studied by number of authors, Peres (1959), Takeno H (1961), Zakhanov (1973) etc.According to Takeno[4], a plane wave ij g is a non flat solution of $R_{ij}=0$ and in some co-ordinate system all its components are functions of A single variable $Z = Z(x^i)$. In formulating this definition he assumed a cartesian like co-ordinate system and obtained a plane wave metric for both $Z = (z - t)$ and $Z = (t\ z)$ type plane gravitational waves in $_4V$ represented by

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - (C - E)dz^2 - 2Edzdt + (C + E)dt^2, \quad \text{and} \qquad (1)$$

$$ds^2 = -Adx^2 - 2Ddxdy - Bdy^2 - Z\ 2\ (C - E)dz^2 - 2ZEdzdt + (C + E)dt^2 \qquad (2)$$

Where *A,B,C, D* and E are functions of *Z.*

A.Peres [2] has found an exact plane-wave like solutions of field equations by considering the line element of the space-time P ,

$$ds2 = - dx2 - dy2 - dz2 + dt\ 2 - 2\ f(x, y,Z)\ (dz - dt)2, \qquad (3)$$

where f = f (x, y,Z) satisfying $\partial_{11} f + \partial_{22} f = 0.$ (4)

He also showed that the curvature tensor of the space-time belongs to thesecond class of petrov's classifications .Furthermore Takeno[5] call the space-time P defined in (1.3) in which $f = f(x, y, Z)$ is an arbitrary function of its arguments and accordingly does not satisfy (1.4) in general, a space-time P and the coordinate system in which the metric takes the form (1.3), the (0)-system alongwith he investigated physical, geometrical and differential properties of space time (1.1), (1.2) and (1.3) respectively with respect to, phase velocity,the coordinate condition, the energy momentum pseudo tensor, parallel null vector field vi , canonical form of curvature tensor, etc.

Takeno [3] considered the generalized Peres space-time represented by the metric

$$ds^2 = -Adx^2 - Bdy^2 - (1 - E)dz^2 - {}^2Edzdt + (1 + E)dt^2 \qquad (5)$$

and we denote it by P^* space-time,

where $A = A(x, Z)$, $B = B(y, Z)$ and $E = E(x, y, Z)$; $Z = Z - t$, and satisfies Einsteins field equations for empty region. E is a function of x, y and Z satisfying,

$$\epsilon = 0 \text{ where } \epsilon = \frac{\alpha}{A} + \frac{\beta}{B}, \qquad (6)$$

$$\alpha = \frac{1}{2}\left[\overline{\overline{A}} - E_{xx} - \frac{1}{2}\left(\frac{A^2 - A_x E_x}{A}\right)\right], \beta = \frac{1}{2}\left[\overline{\overline{B}} - E_{yy} - \frac{1}{2}\left(\frac{B^2 - B_y E_y}{B}\right)\right].$$

The space-time P^* have been studied by number of authors, Pandey [8] ,Lal and Pandey [7], Patel and Vaidya [6] etc. and obtained the plane wave solutions of the field equations. Lal and Pradhan [9] obtained the plane wave- like solution of weakend field equations. Recently Khapekar and Deshmukh [11] obtained the generalized Peres plane wave-like solution of Einstein Maxwell field equation in thepresence of null fluid and null currents for space- time (1.5).Summarizing above investigations, in this paper we have obtained and discussed some geometric and physical properties of space- time P^* in the sense of Takeno (1961).

II. The Curvature Tensor and Ricci Tensor

We confine to the space-time (1.5) for which it follows that

$$(g_{ij}) = \begin{bmatrix} -A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & -(1-E) & -E \\ 0 & 0 & -E & 1+E \end{bmatrix} \qquad (1)$$

and

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$$\left(g^{ij}\right)=\begin{bmatrix}-1/A & 0 & 0 & 0\\ 0 & -1/B & 0 & 0\\ 0 & 0 & -(1+E) & -E\\ 0 & 0 & -E & 1-E\end{bmatrix} \quad (2)$$

The non-vanishing components of christoffel symbol, curvature tensor R_{ijkl} and Ricci Tensor R_{ij} are respectively,

$$\left.\begin{aligned}
&\begin{Bmatrix}1\\11\end{Bmatrix}=\frac{A_x}{2A},\ \begin{Bmatrix}1\\13\end{Bmatrix}=-\begin{Bmatrix}1\\14\end{Bmatrix}=\frac{\overline{A}}{2A},\ \begin{Bmatrix}1\\33\end{Bmatrix}=-\begin{Bmatrix}1\\34\end{Bmatrix}=\begin{Bmatrix}1\\44\end{Bmatrix}=\frac{E_x}{2A},\\
&\begin{Bmatrix}2\\22\end{Bmatrix}=\frac{B_y}{2B},\ \begin{Bmatrix}2\\23\end{Bmatrix}=-\begin{Bmatrix}2\\24\end{Bmatrix}=\frac{\overline{B}}{2B},\ \begin{Bmatrix}2\\33\end{Bmatrix}=-\begin{Bmatrix}2\\34\end{Bmatrix}=\begin{Bmatrix}2\\44\end{Bmatrix}=\frac{E_y}{2B},\\
&\begin{Bmatrix}3\\11\end{Bmatrix}=\begin{Bmatrix}4\\11\end{Bmatrix}=-\frac{\overline{A}}{2},\ \begin{Bmatrix}3\\22\end{Bmatrix}=\begin{Bmatrix}4\\22\end{Bmatrix}=-\frac{\overline{B}}{2},\ \begin{Bmatrix}a\\13\end{Bmatrix}=-\begin{Bmatrix}a\\14\end{Bmatrix}=-\frac{E_x}{2}\\
&\begin{Bmatrix}a\\23\end{Bmatrix}=-\begin{Bmatrix}a\\24\end{Bmatrix}=-\frac{E_y}{2},\ \begin{Bmatrix}a\\33\end{Bmatrix}=-\begin{Bmatrix}a\\34\end{Bmatrix}=\begin{Bmatrix}a\\44\end{Bmatrix}=-\frac{\overline{E}}{2},\text{where a =3,4.}
\end{aligned}\right\} \quad (3)$$

Here bar **(-)** over a letter denotes the derivative with respect to Z.The non-vanishing components of the curvature tensor R_{ijkl} and Ricci tensor R_{ij} are

$$\left.\begin{aligned}
&R_{1313}=-R_{1314}=R_{1414}=\alpha,\\
&R_{1323}=-R_{1324}=-R_{1423}=R_{1424}=\frac{1}{2}E_{xy},
\end{aligned}\right\} \quad (4)$$

$$R_{2323}=-R_{2324}=R_{2424}=\beta.$$

$$R_{33}=-R_{34}=R_{44}=\epsilon \quad . \quad (5)$$

From (2.1) and (2.5)

$$R=g^{ij}R_{ij}=0.$$

Hence from (2.3) we have,

Theorem - A condition that a space-time (1.5) be the Minkoswkian is

$$\alpha=\beta=E_{xy}=0, \quad (6)$$

in the *(0)*- system

III. Exact Solution of Field Equations in the Case When an Electromagnetic Field Is Present

We shall consider the following field equations for a system in which an electromagnetic field F_{ij} co-exists with gravitational field.

$$R_{ij}=-8\pi\ E_{ij},\quad (i,j,k=1,\ldots,4), \quad (1)$$

$$F_{ij;k}+F_{jk;i}+F_{ki;j}=0, \quad (2)$$

$$F^{ij}_{;j}=0. \quad (3)$$

Where E_{ij} is the electromagnetic energy tensor. We consider in (1.5)
a transverse electromagnetic field whose components in the (0)-system are,

$$F_{31}=F_{14}=\sigma,\quad -F_{23}=F_{24}=-\rho \qquad F_{12}=F_{34}=0, \quad (4)$$

where r and s are functions of x, y and Z .
The non-vanishing components of E_{ij} are,

$$E_{33}=-E_{34}=E_{44}=\frac{\sigma^2}{A}+\frac{\rho^2}{B}, \quad (5)$$

and the field equations (8.3.1) becomes,

$$\epsilon=-8\pi\left(\frac{\sigma^2}{A}+\frac{\rho^2}{B}\right). \quad (6)$$

The generalized Maxwell equations (3.2) and (3.3) are equivalent to the following Cauchy-Riemann type equations:

$$\partial_1\rho=-\partial_2\sigma \quad , \quad \partial_2\rho=\partial_1\sigma. \quad (7)$$

Hence we have
Theorem : A necessary and sufficient conditions that g_{ij} given by (2.1) and F_{ij} given by (3.4) where E, r, s are functions of x, y and Z satisfy the field equations (3.1), (3.2) and (3.3) is that E, r, s satisfy (3.6) and (3.7).

IV. Some Properties of P* and F [ij]

Some tensor properties of P^* and F_{ij} can be obtained as direct consequences of formulas in section §2, which we shall enumerate in terms of simple theorems

Theorem 1-g^{ij} given by (1.5) cannot satisfy the co-ordinate condition

$$\frac{\partial}{\partial x^j}\left(\sqrt{-g}\,g^{ij}\right)=0.$$

Theorem 2 -The phase velocity of the waves is not the fundamental velocity general in the sense that the equiphase hypersurface E = constant are not null in general. The velocity is the fundamental only when the (1.5) is Minkowskian.

Proof -We can consider that g_{ij} given by (1.5) is of the form $g_{ij}=g_{ij}(E)$,where the phase function E is the $E=E(x, y, Z)$ i.e the phase velocity propogating in the negative direction of z-axis.
Consider the relation

$$g^{ij}(\partial_i E)(\partial_j E)=0,$$

$$g^{11}(\partial_1 E)(\partial_1 E)+g^{22}(\partial_2 E)(\partial_2 E)+g^{33}(\partial_3 E)^2+2g^{34}(\partial_3 E)(\partial_4 E)+g^{44}(\partial_4 E)^2=0,$$

$$\frac{-E_x^2}{A}-\frac{E_y^2}{B}-\overline{E}^2-E\overline{E}^2+2E\overline{E}^2+\overline{E}^2-E\overline{E}^2=0,$$

$$\frac{E_x^2}{A}+\frac{E_y^2}{B}=0,$$

$$\frac{E_x^2}{A}=0 \quad \text{or} \quad \frac{E_y^2}{B}=0,$$

$$E_x=0 \ \text{or}\ E_y=0 \quad , \quad \text{since A, B} \neq 0.$$

$$\therefore \partial_1 E=\partial_2 E=0.$$

Hence E is independent of x and y or E = constant.
Hence the result.

Theorem 3-The g $_{ij}$ of (1.5) is null in the sense of Ehlers and Sachs,

$$i.e \quad R_{ijkl}R^{kl}_{pq} = 0.$$

Proof-

$$R_{ijkl}R^{kl}_{pq} = R_{ij13}R^{13}_{pq} + R_{ij14}R^{14}_{pq} + R_{ij23}R^{23}_{pq} + R_{ij24}R^{24}_{pq}$$

$$= (R_{1313} + R_{1413} + R_{2313} + R_{2413})(g^{11}g^{33}R_{1313} + g^{11}g^{33}R_{1413} + g^{11}g^{33}R_{2313} + g^{11}g^{33}R_{2413} + g^{11}g^{34}R_{1314} + g^{11}g^{34}R_{1414} + g^{11}g^{34}R_{2314} + g^{11}g^{34}R_{2414})$$

$$+ (R_{1314} + R_{1414} + R_{2314} + R_{2414})(g^{11}g^{43}R_{1313} + g^{11}g^{43}R_{1413} + g^{11}g^{43}R_{2313} + g^{11}g^{43}R_{2413} + g^{11}g^{44}R_{1314} + g^{11}g^{44}R_{1414} + g^{11}g^{44}R_{2314} + g^{11}g^{44}R_{2414})$$

$$+ (R_{1323} + R_{1423} + R_{2323} + R_{2423})(g^{22}g^{33}R_{1323} + g^{22}g^{33}R_{1423} + g^{22}g^{33}R_{2323} + g^{22}g^{33}R_{2423} + g^{22}g^{33}R_{1324} + g^{22}g^{33}R_{1424} + g^{22}g^{33}R_{2324} + g^{22}g^{33}R_{2424})$$

$$+ (R_{1324} + R_{1424} + R_{2324} + R_{2424})(g^{22}g^{43}R_{1323} + g^{22}g^{43}R_{1423} + g^{22}g^{43}R_{2323} + g^{22}g^{43}R_{2423} + g^{22}g^{44}R_{1324} + g^{22}g^{44}R_{1424} + g^{22}g^{44}R_{2324} + g^{22}g^{44}R_{2424})$$

$$= \left(\alpha - \alpha + \frac{1}{2}E_{xy} - \frac{1}{2}E_{xy}\right)(\kappa_1) + \left(-\alpha + \alpha - \frac{1}{2}E_{xy} + \frac{1}{2}E_{xy}\right)(\kappa_2) + \left(\frac{1}{2}E_{xy} - \frac{1}{2}E_{xy} + \beta - \beta\right)(\kappa_3) + \left(-\frac{1}{2}E_{xy} + \frac{1}{2}E_{xy} - \beta + \beta\right)(\kappa_4)$$

$= 0,$ [From(2.5)].

Where

$$\kappa_1 = (g^{11}g^{33}R_{1313} + g^{11}g^{33}R_{1413} + g^{11}g^{33}R_{2313} + g^{11}g^{33}R_{2413} + g^{11}g^{34}R_{1314} + g^{11}g^{34}R_{1414} + g^{11}g^{34}R_{2314} + g^{11}g^{34}R_{2414}),$$

$$\kappa_2 = (g^{11}g^{43}R_{1313} + g^{11}g^{43}R_{1413} + g^{11}g^{43}R_{2313} + g^{11}g^{43}R_{2413} + g^{11}g^{44}R_{1314} + g^{11}g^{44}R_{1414} + g^{11}g^{44}R_{2314} + g^{11}g^{44}R_{2414}),$$

$$\kappa_3 = (g^{22}g^{33}R_{1323} + g^{22}g^{33}R_{1423} + g^{22}g^{33}R_{2323} + g^{22}g^{33}R_{2423} + g^{22}g^{33}R_{1324} + g^{22}g^{33}R_{1424} + g^{22}g^{33}R_{2324} + g^{22}g^{33}R_{2424}),$$

$$\kappa_4 = (g^{22}g^{43}R_{1323} + g^{22}g^{43}R_{1423} + g^{22}g^{43}R_{2323} + g^{22}g^{43}R_{2423} + g^{22}g^{44}R_{1324} + g^{22}g^{44}R_{1424} + g^{22}g^{44}R_{2324} + g^{22}g^{44}R_{2424}).$$

Theorem 4-Four eigen values of Ricci tensor are all zero.

Proof -Consider, $\det(R_{ij} - \lambda g_{ij}) = 0$,

$$\begin{vmatrix} \lambda A & 0 & 0 & 0 \\ 0 & \lambda B & 0 & 0 \\ 0 & 0 & \epsilon + \lambda(1-E) & -\epsilon + \lambda E \\ 0 & 0 & -\epsilon + \lambda E & \epsilon - \lambda(1+E) \end{vmatrix} = 0,$$

$$\lambda^4 = 0.$$

Theorem 5:-Four eigen values of Fij are all zero.

Proof -Consider $\det(F_{ij} - \lambda g_{ij}) = 0$

$$\begin{vmatrix} \lambda A & 0 & -\sigma & \sigma \\ 0 & \lambda B & \rho & -\rho \\ \sigma & -\rho & \lambda(1-E) & \lambda E \\ -\sigma & \rho & \lambda E & -\lambda(1+E) \end{vmatrix} = 0,$$

$$\lambda^4 = 0.$$

Theorem 6- $R_{ij}R^{jl} = 0$

Proof-

$$R_{ij}R^{jl} = g^{jr}g^{ls}R_{ij}R_{rs}$$

$$= g^{jr}g^{ls}R_{1j}R_{rs} + g^{jr}g^{ls}R_{2j}R_{rs} + g^{jr}g^{ls}R_{3j}R_{rs} + g^{jr}g^{ls}R_{4j}R_{rs}$$

$$= g^{jr}(R_{1j} + R_{2j} + R_{3j} + R_{4j})R_{rs}g^{ls}$$

$$= [g^{1r}(R_{11} + R_{21} + R_{31} + R_{41}) + g^{2r}(R_{12} + R_{22} + R_{32} + R_{42}) + g^{3r}(R_{13} + R_{23} + R_{33} + R_{43}) + g^{4r}(R_{14} + R_{24} + R_{34} + R_{44})]R_{rs}g^{ls}$$

$$= (R_{33} + R_{34})(g^{3r} + g^{4r})R_{rs}g^{ls}$$

$= 0,$ [using(2.5)].

Theorem 7- $F_{ij}F^{jk}F_{kl} = 0$

Proof-

$$F_{ij}F^{jk}F_{kl} = F_{ij}\{F^{j1}F_{1l} + F^{j2}F_{2l} + F^{j3}F_{3l} + F^{j4}F_{4l}\}$$

$$= F_{1l}(F_{ij}F^{j1}) + F_{2l}(F_{ij}F^{j2}) + F_{3l}(F_{ij}F^{j3}) + F_{4l}(F_{ij}F^{j4})$$

$$= F_{1l}(F_{i3}F^{31} + F_{i4}F^{41}) + F_{2l}(F_{i3}F^{32} + F_{i4}F^{42}) + F_{3l}(F_{i1}F^{13} + F_{i2}F^{23}) + F_{4l}(F_{i1}F^{14} + F_{i2}F^{24})$$

$$= F_{1l}(F_{i3} + F_{i4})F^{41} + F_{2l}(F_{i3} + F_{i4})F^{23} + (F_{3l} + F_{4l})(F_{i1}F^{13} + F_{i2}F^{24})$$

$= 0,$ where $(F^{i3} = F^{i4}, F_{i3} = -F_{i4})$, [using(3.4)].

Theorem *8*- $R_{ijkl}F^{kl} = 0.$

Proof-

$$R_{ijkl}F^{kl} = R_{ij1l}F^{1l} + R_{ij2l}F^{2l} + R_{ij3l}F^{3l} + R_{ij4l}F^{4l}$$

$$= R_{ij11}F^{11} + R_{ij12}F^{12} + R_{ij13}F^{13} + R_{ij14}F^{14} + R_{ij21}F^{21}$$
$$+ R_{ij22}F^{22} + R_{ij23}F^{23} + R_{ij24}F^{24} + R_{ij31}F^{31} + R_{ij32}F^{32}$$
$$+ R_{ij33}F^{33} + R_{ij34}F^{34} + R_{ij41}F^{41} + R_{ij42}F^{42} + R_{ij43}F^{43}$$
$$+ R_{ij44}F^{44}$$
$$= 2\left[R_{ij13}F^{13} + R_{ij14}F^{14} + R_{ij23}F^{23} + R_{ij24}F^{24}\right]$$
$$= 2\left[\left(R_{ij13} + R_{ij14}\right)F^{14} + \left(R_{ij23} + R_{ij24}\right)F^{24}\right]$$
$$= 0. \qquad [\text{using}(2.4)]$$

Theorem*9*- $R_{ij}R^{jl} = 0$

Proof-

$$R_{ij}R^{jl} = g^{jr}g^{ls}R_{ij}R_{rs}$$
$$= g^{jr}\left(g^{l3}R_{r3} + g^{l4}R_{r4}\right)R_{ij}$$
$$= R_{ij}\left[g^{j3}g^{l3}R_{33} + g^{j3}g^{l4}R_{34} + g^{j4}g^{l3}R_{43} + g^{j4}g^{l4}R_{44}\right]$$
$$= \epsilon\ R_{ij}\left(g^{l3} - g^{l4}\right)\left(g^{j3} - g^{j4}\right)$$
$$= \epsilon\ R_{ij}\left(g^{l3} - g^{l4}\right)\left[R_{i3}\left(g^{33} - g^{34}\right) + R_{i4}\left(g^{43} - g^{44}\right)\right]$$
$$= -\epsilon\ \left(g^{l3} - g^{l4}\right)\left[R_{i3} + R_{i4}\right],$$
$$= 0, \qquad \left(\because R_{i3} + R_{i4} = 0\right), \qquad [\text{using}(2.5)].$$

Theorem *10*-The curvature tensor satisfies the identity

$$R_{ijkl}R^{jk} = 0.$$

Proof-

$$R_{ijkl}R^{jk} = R_{ijkl}g^{jr}g^{ks}R_{rs}$$
$$= R_{ijkl}\left[g^{j3}g^{k3}R_{33} + g^{j3}g^{k4}R_{34} + g^{j4}g^{k3}R_{43} + g^{j4}g^{k4}R_{44}\right]$$
$$= R_{ijkl}\ \epsilon\ \left(g^{j3} - g^{j4}\right)\left(g^{k3} - g^{k4}\right)$$
$$= \epsilon\ \left[\left(g^{33} - g^{34}\right)R_{i33l}\left(g^{33} - g^{34}\right) + \left(g^{33} - g^{34}\right)\left(g^{43} - g^{44}\right)R_{i43l}\right.$$
$$\left. + \left(g^{43} - g^{44}\right)\left(g^{33} - g^{34}\right)R_{i34l} + \left(g^{43} - g^{44}\right)\left(g^{43} - g^{44}\right)R_{i44l}\right]$$
$$= \epsilon\ \left[R_{i33l} + R_{i43l} + R_{i34l} + R_{i44l}\right]$$
$$= 0, \qquad [\text{using}(2.4)\text{and } (2.5)].$$

Theorem 11- The ij F is null in the sense of Synge,

i.e $F_{ij}F^{ij} = 0.$

Proof-

$$F_{ij}F^{ij} = F_{1j}F^{1j} + F_{2j}F^{2j} + F_{3j}F^{3j} + F_{4j}F^{4j}$$
$$= 2\left[F_{13}F^{13} + F_{14}F^{14} + F_{23}F^{23} + F_{24}F^{24}\right]$$
$$= 2\left[F_{13}F^{13} - F_{13}F^{13} + F_{23}F^{23} - F_{23}F^{23}\right]$$
$$= 0, \quad \text{where} \quad \left(F_{i3} = -F_{i4}, \quad F^{i3} = F^{i4}\right), \ [\text{using}(3.4)].$$

Theorem *12*- $F^*_{ij}F^{ij} = 0$, *where* F^*_{ij} *is the dual of* F^{ij}.

$$F^*_{ij} = \frac{1}{2}\eta_{ijkl}F^{kl}$$

Proof -We have,

Where η_{ijkl} is a tensor whose components changes the sign under interchange of any pair of indices and the components are zero in which pair of indices are identical. The only non-vanishing components are those for which all four indices are different.

$$F^*_{ij}F^{ij} = \frac{1}{2}\eta_{ijkl}F^{kl}F^{ij}$$
$$= F^{kl}\left[\eta_{13kl}F^{13} + \eta_{14kl}F^{14} + \eta_{23kl}F^{23} + \eta_{24kl}F^{24}\right]$$
$$= F^{13}\left[\eta_{1313}F^{13} + \eta_{1413}F^{14} + \eta_{2313}F^{23} + \eta_{2413}F^{24}\right]$$
$$+ F^{14}\left[\eta_{1314}F^{13} + \eta_{1414}F^{14} + \eta_{2314}F^{23} + \eta_{2414}F^{24}\right]$$
$$+ F^{23}\left[\eta_{1323}F^{13} + \eta_{1423}F^{14} + \eta_{2323}F^{23} + \eta_{2423}F^{24}\right]$$
$$+ F^{24}\left[\eta_{1324}F^{13} + \eta_{1424}F^{14} + \eta_{2324}F^{23} + \eta_{2424}F^{24}\right],$$
$$= F^{13}F^{14}\left[\eta_{1413} + \eta_{1314}\right] + F^{23}F^{13}\left[\eta_{1323} + \eta_{2313}\right] + F^{24}F^{13}\left[\eta_{1324} + \eta_{2413}\right]$$
$$+ F^{14}F^{23}\left[\eta_{1423} + \eta_{2314}\right] + F^{24}F^{14}\left[\eta_{1424} + \eta_{2414}\right] + F^{23}F^{24}\left[\eta_{2324} + \eta_{2423}\right],$$
$$= F^{13}F^{14}\left[\eta_{1413} - \eta_{1413}\right] + F^{23}F^{13}\left[\eta_{1323} - \eta_{1323}\right] + F^{24}F^{13}\left[\eta_{1324} - \eta_{1324}\right]$$
$$+ F^{14}F^{23}\left[\eta_{1423} - \eta_{1423}\right] + F^{24}F^{14}\left[\eta_{1424} - \eta_{1424}\right] + F^{23}F^{24}\left[\eta_{2324} - \eta_{2324}\right],$$
$$= 0, \quad \text{where} \quad \left(F^{i3} = F^{i4}\right), \qquad [\text{using}(3.4)].$$

V. ENERGY- MOMENTUM PSEUDO-TENSORS IN P*

With a view to some physical applications, in this section we shall first gives energy-momentum peudo-tensors of Einstein and of Landau-Lifshitz

Theorem-In P* , both the peudo-tensors of Einstein j i t and that of Landau-Lifshitz t*ij does not vanish in general in an O - system.

Proof*-PSEUDO-TENSOR OF EINSTEIN*

The formula which gives the energy-momentum pseudo-tensor introduced by Einstein as

$$16\pi\sqrt{-g}\,t_i^{\,j} = \begin{Bmatrix} j \\ mn \end{Bmatrix}\left(\sqrt{-g}\,g^{mn}\right)_{,i} - \left(\log\sqrt{-g}\right)_{,m}\left(\sqrt{-g}\,g^{jm}\right)_{,i}$$
$$+ \delta_i^{\,j}\left[\begin{Bmatrix} h \\ mk \end{Bmatrix}\begin{Bmatrix} k \\ nh \end{Bmatrix}g^{mn}\sqrt{-g} - g^{mn}\begin{Bmatrix} h \\ mn \end{Bmatrix}\left(\sqrt{-g}\right)_{,h}\right] \qquad (1)$$

If we substitute (2.2) & (2.3) into (5.1), we can easily obtain components of energy-momentum tensor

$$t_3^{\,3} = t_3^{\,4} = -t_4^{\,3} = -t_4^{\,4} = \chi, \quad \text{other} \quad t_i^{\,j} = 0,$$

$$\text{where } \chi = \frac{\overline{A}\,\overline{B}}{16\pi\ m} \quad \text{and} \quad m = AB.$$

It means some components of Einstein pseudo-tensor survives , not vanish in general.

PSEUDO-TENSOR OF LANDAU AND LIFSHITZ

To calculate components of t^{*ij} , we use following formula proposed by Landau and Lifshitz

$$16\pi\, t^{*ij} = \left(g^{ik}g^{jl} - g^{ij}g^{kl}\right)\left[2\begin{Bmatrix} h \\ kl \end{Bmatrix}\begin{Bmatrix} m \\ hm \end{Bmatrix} - \begin{Bmatrix} m \\ kh \end{Bmatrix}\begin{Bmatrix} h \\ lm \end{Bmatrix} - \begin{Bmatrix} h \\ kh \end{Bmatrix}\begin{Bmatrix} m \\ lm \end{Bmatrix}\right]$$
$$+ g^{ik}g^{mn}\left[\begin{Bmatrix} j \\ kh \end{Bmatrix}\begin{Bmatrix} h \\ mn \end{Bmatrix} + \begin{Bmatrix} j \\ mn \end{Bmatrix}\begin{Bmatrix} h \\ kh \end{Bmatrix} - \begin{Bmatrix} j \\ nh \end{Bmatrix}\begin{Bmatrix} h \\ km \end{Bmatrix} - \begin{Bmatrix} j \\ km \end{Bmatrix}\begin{Bmatrix} h \\ nh \end{Bmatrix}\right]$$
$$+ g^{jk}g^{mn}\left[\begin{Bmatrix} i \\ kh \end{Bmatrix}\begin{Bmatrix} h \\ mn \end{Bmatrix} + \begin{Bmatrix} i \\ mn \end{Bmatrix}\begin{Bmatrix} h \\ kh \end{Bmatrix} - \begin{Bmatrix} i \\ nh \end{Bmatrix}\begin{Bmatrix} h \\ km \end{Bmatrix} - \begin{Bmatrix} i \\ km \end{Bmatrix}\begin{Bmatrix} h \\ nh \end{Bmatrix}\right]$$
$$- g^{hk}g^{mn}\left[\begin{Bmatrix} i \\ hm \end{Bmatrix}\begin{Bmatrix} j \\ kn \end{Bmatrix} - \begin{Bmatrix} i \\ hk \end{Bmatrix}\begin{Bmatrix} j \\ mn \end{Bmatrix}\right] \qquad (2)$$

Using (2.2) , (2.3) and (5.2) we obtained non-vanishing t^{*ij} are

$$t^{*33} = t^{*34} = t^{*43} = t^{*44} = \chi^*, \quad \text{other} \quad t^{*ij} = 0,$$

where
$$\chi^* = -\frac{\left[\left(\bar{A}\bar{B} + \bar{m}^2/2m\right)\right]}{m},$$

$$m = AB,$$

and
$$\bar{m} = A\bar{B} + B\bar{A}.$$

Again χ^* is not zero in general.

VI. Conclusions

We have investigated some geometrical and physical properties for Generalized Peres space-time *P** in the sense of Takeno (1961). We made following discussions:

i. If we assume that g^{ij} and F^{ij} satisfy (3.1), from theorem 2 we conclude that when the phase velocity is fundamental, *P** becomes Minkowskian,i.e the gravitational wave is apparent and the electromagnetic field F^{ij} must vanish.

ii. The result that both energy momentum pseudo *tensors(5.1)and (5.2)* survives in *P** and carries some energy or momentum in the direction of wave propogation which is striking contrast to the corresponding one for *P* where both tensor vanishes identically and does not carry energy or momentum in the direction of wave propogation but it is analogous to corresponding one for H given by Takeno(1961).

iii. When $A = B = 1$ and $E = 2f$ and when $A=A(Z)$, $B=B(Z)$ and $E = E(x,y,z,t)$ then the results for space-time P and space-time P1 consider by Lal and Pandey (1975) are the special cases of our investigation.

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Strong Summability With Respect To A Sequence Of Orlicz Functions

GJSFR Classification – F (FOR) 010106,010401,010406

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Abstract- **In this paper we define the notion of strong summability by a sequence of Orlicz functions and examine its relationship with A-statistical convergence.**

AMS subject Classification (2000): 40H05, 40F05

Keywords-Orlicz functions, strong summability, A-statistical convergence.

I. INTRODUCTION

An Orlicz function is a function , which is continuous, non-decreasing and convex with *M(0) = 0,M(x) > 0 for x > 0* and *M(x) → ∞ as x → ∞* . If convexity of M is replaced by subadditivity, then this function is called a modulus function see Maddox [11].

Lindendstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let s be the space of all real or complex sequences x = (x_k) ,

$$\ell_M = \{x \in s : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0\}$$

which is called an Orlicz sequence space. ℓM is a Banach space with the norm

$$||x|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

Also, it was shown [9] that every space ℓ_M contains a subspace isomorphic to ℓ_p (p ≥ 1).The spaces of strongly summable sequences were discussed by Maddox [10]. Parashar and Choudhary [12] defined these spaces by using the idea of Orlicz function as follows:

Let p = (p_k) be a sequence of positive real numbers and s be the space of all real sequences. Then

$$W_0(M,p) = \{x \in s : \frac{1}{n}\sum_{k=1}^{n}(M\left(\frac{|x_k|}{\rho}\right)^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0\},$$

$$W(M,p) = \{x \in s : x - \ell e \in W_0(M,p),\ \ell > 0\},$$

$$W_\infty(M,p) = \{x \in s : \sup_n \frac{1}{n}\sum_{k=1}^{n}(M\left(\frac{|x_k|}{\rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0\}.$$

If M(x) = x, then the above spaces are deduced to [C, 1, p]0, [C, 1, p] and [C, 1, p]∞ respectively. For pk = p > 0 for each k, we denote these sequence spaces by Wp 0 (M), Wp(M) ,and Wp∞ (M) respectively.

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Let X be a Banach space and s(X) denote the space of all sequences x = (x_k) in X. A scalar matrix A = (ank)∞n,k is called regular on s(X) if A maps c(X) into c(X) and limn An(x) = limk xk in X. It is known that a matrix A is regular on s(X) if and only if it is regular on s. The necessary and sufficient conditions for A to be regular [5] on s are

$$(i)\ \sup_n \sum_k |a_{nk}| < \infty,\ (ii)\ \lim_n a_{nk} = 0$$

$\lim_n$ ank = 0 for each k,

$$(iii)\ \lim_n \sum_k a_{nk} = 1.$$

and These are well-known Silverman-Toeplitz conditions see [5]. A matrix A is said to be uniformly regular if it is regular, $a_{nk} \geq 0$ and

$$\lim_n \sum_{k \geq n} |a_{nk}| = 0 \text{ uniformly on } n \in \mathbb{N}$$

An Orlicz function *M* is said to satisfy *Δ2* - condition for all values u, if there exists a constant K > 0, such that

$$M(2u) \leq KM(u), \quad (u \geq 0)$$

We define the following Definition. An Orlicz function M is said to satisfy Δ λ - condition for all values u if there exists a constant K > 0 such that

$M(\lambda u) \leq K\lambda M(u)$ for all $u \geq 0$ and $\lambda > 1$.

We define the following sequence spaces Let A = (ank)∞ n,k be a non-negative regular matrix and m = (Mk) a sequence of Orlicz functions such that each Mk satisfies Δ λ − condition. Then for p > 0

$$W_0^p(m,A,X) = \left\{x \in s(X) : \lim_n \sum_k a_{nk}\left(M_k\left(\frac{|x_k|}{\rho}\right)\right)^p = 0, \text{ for some } \rho > 0\right\},$$

$$W^p(m,A,X) = \{x \in s(X) : \text{ there exists } x_0 \in X, (x_k - x_0) \in W_0^p(m,A,X)\}.$$

For $x \in W^p(m,A,X)$, we write $x_k \to x_0\ (W^p(m,A,X))$.

If $M_k(x)$ = x for each k, then these spaces are reduced to

$W_0^p(A,X)$ and $W^p(A,X)$ respectively, where

$$W^p(A,X) = \left\{x \in s(X) : \text{there exists } x_0 \in X \text{ such that } \lim_n \sum_k a_{nk}\|x_k - x_0\|^p = 0\right\}$$

If (M_k) is replaced by (f_k) a sequence of modulus functions , then the above spaces are reduced to the spaces defined by Kolk [7].

II. INCLUSION RELATIONS

In this section, we prove the following results.

Theorem 2.1.-

$W_0^p(A,X) \subset W_0^p(m,A,X)$ if and only if

$$\lim_{t \to 0+} \sup_k M_k(t) = 0 \quad (t > 0). \tag{1}$$

Proof-Let $W_0^p(A,X) \subset W_0^p(m,A,X)$.
If we take A = I (unit matrix), then this inclusion is reduced to

$$c_0(X) \subset c_0(m,X)$$

where

$$c_0(m,X) = \left\{ x \in s(X) : \lim_n \sum_k \left(M_k \left(\frac{\|x_k\|}{\rho} \right) \right) = 0, \text{for some } \rho > 0 \right\}$$

Suppose that (2.1.1) fails to hold. Then there exists a number $\varrho_0 > 0$ and an index sequence (k_i) such that

$$M_{k_i}(t_i) \geq \epsilon_0 \quad (i = 1, 2 \cdots) \qquad (2)$$

for a positive sequence $(t_i) \in c0$. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_i y, & \text{for } k = k_i \text{ and for a fixed } y \in X \text{ with } \|y\| = 1; \\ 0, & \text{if } k \neq k_i. \end{cases}$$

Then x ∈ c0(X) since ti ∈ c0 , and hence x ∈ c0(m,X) . On the other hand, by (2.1.2) and △ λ −condition we have

$$M_{k_i}\left(\frac{\|x_{k_i}\|}{\rho} \right) = M_{k_i}\left(\frac{t_i}{\rho} \right) \geq \frac{\epsilon_0}{K\rho}, K > 0, \ \rho > 1, \quad (i = 1, 2 \cdots)$$

i.e., x /∈ c_0(m,X) , a contradiction. Therefore (2.1.1) must hold.
Conversely, suppose that (2.1.1) holds. Then for every $\varrho > 0$ there exists a number δ such that 0 < δ < 1 and

$$M_k(t) < \frac{\epsilon^{\frac{1}{p}}}{\|A\|} \quad , k = 1, 2 \cdots \quad for \ t \leq \delta \qquad (3)$$

For a sequence x = $(x_k) \in W_0^p(A,X)$ let so that limn T_n = 0. Now

$$\sum_k a_{nk} \left[M_k \left(\frac{\|x_k\|}{\rho} \right) \right]^p = \Sigma_1 + \Sigma_2 \qquad (4)$$

where Σ1 is the sum over k such that $\frac{\|x_k\|}{\rho} \leq \delta$; and Σ2 is the sum over k such that $\frac{\|x_k\|}{\rho} \leq \delta$; Since A is regular and by (2.1.3), we have

$$\Sigma_1 < \epsilon \qquad (5)$$

By (2.1.1), we have

$$\sup_k M_k(\delta) = H < \infty \text{ for } \frac{\|x_k\|}{\rho} > \delta > 0 \qquad (6)$$

Since each M_k is non-decreasing and convex, we have by (2.1.6) and △ λ −condition that for K > 0

$$M_k\left(\frac{\|x_k\|}{\rho} \right) = M_k\left(\delta\delta^{-1} \frac{\|x_k\|}{\rho} \right),$$
$$\leq K\delta^{-1} \frac{\|x_k\|}{\rho} M_k(\delta)$$
$$< K\delta^{-1} H \frac{\|x_k\|}{\rho},$$

i.e.

$$\Sigma_2 < (K\delta^{-1}H)^p T_n. \qquad (7)$$

Hence $\Sigma_2 \to 0$ as $n \to \infty$. Therefore $x \in W_0^p(m,A,X)$.

This completes the proof of the theorem.

Theorem 2.2.(a) If $W_0^p(m,A,X) \subset W_0^p(A,X)$ and (M_k) is pointwise convergent for $t > 0$ (it is not necessary that every M_k satisfies the $\triangle_\lambda$ condition) then

$$\inf_k M_k(t) > 0, \quad t > 0, \qquad (a)$$

(b) If

$$\inf_k M_k(t) > 0, \quad t > 0, \qquad (a)$$

and every Mk satisfies the △ λ condition then $W_0^p(m,A,X) \subset W_0^p(A,X)$.

Proof.-(a) Let $W_0^p(m,A,X) \subset W_0^p(A,X)$. Suppose that (2.2.1) does not hold. Then

$$\inf_k M_k(t) = 0 \quad (t > 0) \qquad (b)$$

and thus we can choose an index sequence (ki) such that

$$M_{k_i}(t_0) < \frac{1}{i} \text{ for certain } t_0 > 0 \quad (i = 1, 2, \cdots) \qquad (c)$$

Now, define a sequence x = (x_k) by

$$x_k = \begin{cases} t_0 y, & \text{for } k = k_i \text{ where } y \in X \text{ with } \|y\| = 1 \text{ and } t_0 > 0; \\ 0 & \text{, otherwise.} \end{cases}$$

Then $\|x_k\| = \|x_{k_i}\| = t_0,$ and so by (2.2.2) and (2.2.3) we get

$$\lim_k M_k\left(\frac{\|x_k\|}{\rho} \right) = 0$$

and hence

$$\lim_k \left[M_k\left(\frac{\|x_k\|}{\rho} \right) \right]^p = 0$$

Further by regularity of A, we have

$$\lim_n \sum_k a_{nk} \left[M_k\left(\frac{\|x_k\|}{\rho} \right) \right]^p = 0$$

i.e. $x = (x_k) \in W_0^p(m,A,X)$. But on the other hand

$$\lim_n \sum_k a_{nk} \|x_k\|^p = t_0^p \lim_n \sum_k a_{nk} = t_0^p,$$

Hence $x \notin W_0^p(A,X)$. Which contradicts that $W_0^p(m,A,X) \subset W_0^p(A,X)$. Hence (2.2.1)must hold.

(b) Conversely, let (2.2.1) hold and $x \in W_0^p(m,A,X)$. Suppose that $x \notin W_0(A,X)$. Then for some number $\varrho_0 > 0$ and index k0 we have $\|x_{k_i}\| > \epsilon_0 \quad (i \in \mathbb{N})$ for some subsequence of indices (ki), since A is regular. Thus

$$M_k\left(\frac{\epsilon_0}{\rho} \right) < M_k\left(\frac{\|x_{k_i}\|}{\rho} \right) \quad \text{for some } \rho > 0,$$

and further by regularity of A, we have $\lim_k M_k \left(\frac{\epsilon_0}{\rho}\right) = 0$ which contradicts (2.2.1). Hence $x \in W_0^p(A, X)$.
This completes the proof of the theorem.

III. A-STATISTICAL CONVERGENCE

In this section we find relation of A-statistical convergence with strong A-summability defined by a sequence m = (M_k) of Orlicz functions.
Let K = {ki} be an index set, i.e. precisely the sequence (ki) of indices. Let ϕ k be the characteristic sequence of K, i.e. ϕ k = (ϕ kj), where

$$\phi_j^k = \begin{cases} 1: & \text{If } j = k, \; j = 1, 2 \cdots \\ 0 & \text{, otherwise.} \end{cases}$$

If ϕ k is (C,1)-summable then the limit

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} \phi_j^k$$

is called the asymptotic density of *K* and is denoted by δ (K). An index set K = $\{k_i\}$ is said to have A-density if

$$\delta_A(K) = \lim_n A_n \phi^k$$

$$= \lim_n \sum_{k \in K} a_{nk}$$

exists, where $A = (a_{nk})_{n,k=1}^{\infty}$ is a non-negative regular matrix [cf,6].
The idea of statistical convergence was introduced by Fast [2] and studied by various authors, e.g. by ˘ Sal´at [13], Freedman and Sember [3], Fridy [4] Connor [1], and Kolk[6] .
sequence x = (x_k) ∈ s(X) is said to be A-statistically convergent to x0, [6] i.e. $x_k \to x_0(s_A(X))$ if for every ϱ > 0, $\delta_A(L_\varrho) = 0$, where $L_\varrho = \{k : \|x_k - x_0\| \geq \varrho\}$. We denote by SA(X) the set of all A-statistically convergent sequences in X. If *A* is C1−matrix, then A-statistical convergence is reduced to the statistical convergence.
Example. Define x_k if k is a square and xk = 0 otherwise. then $|\{k \leq n : x_k \neq 0\}| \leq (n)^{1/2}$, so $x = (x_k)$

Theoem 3.1.- Let A be uniformly regular matrix and the orlicz functions sequence m = (M_k) be a pointwise convergent. Then

$$x_k \to x_0(W_0^p(m, A, X)) \Rightarrow x_k \to x_0(S_A(X)),$$

if and only if

$$\lim_k M_k(t) > 0 \quad (t > 0). \tag{1}$$

Proof. Let ϱ > 0 . Then as in [8 , Theorem 3.8] we can find numbers s > 0 and r ∈ IN such that

$$\sum_{\substack{k \in L_\epsilon \\ k \geq r}} a_{nk} \leq s^{-p} \sum_{k \geq r} a_{nk} \left[M_k \left(\frac{\|x_k - x\|}{\rho} \right) \right]^p . \tag{2}$$

Where $L_\epsilon = \{k : \|x_k - x_0\| \geq \epsilon\}$. Since $x_k \to x_0(W_0^p(m, A, X))$ implies that $\delta_A(L_\epsilon) = \lim_n \sum_{k \in L_\epsilon} a_{nk} = 0$. Therefore $x_k \to x_0(S_A(X)$
Conversely, suppose that $x_k \to x_0(W_0^p(m, A, X)) \Rightarrow x_k \to x_0(S_A(X))$. If (3.1.1.) is not true, we have

$$\lim_k M_k(t_0) = 0 \text{ for some } t_0 > 0.$$

Since *A* is uniformly regular, by Lemma 2.4. of Kolk [8], there exists an infinite index set K = (k_i) with $\delta_A(K) = 0$. Define a sequence y = (y_k) by

$$y_k = \begin{cases} 0, & k \in K, \\ t_0 z, & \text{otherwise;} \end{cases}$$

where z ∈ X with llzll = 1. Then

$$\lim_k \left[M_k \left(\frac{\|y_k\|}{\rho} \right) \right]^p = 0,$$

and by the regularity of A we have

$$y_k \to 0\,(W^p(m, AX))\,.$$

But for $0 < \epsilon \leq t_0$,

$$\delta_A(k : \|y_k\| \geq \epsilon) = \; lim_n \sum_k a_{nk} - \delta_A(K) = 1 - 0 = 1.$$

Thus yk does not → $0(S_A(X))$, i.e. contradiction to the hypothesis. Hence (3.1.1) must hold. This completes the proof of the theorem.

IV. ACKNOWLEDGEMENT

The author is thankful to Prof. Mursaleen for his valuable suggestions.

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On Uniformly Spriallike Functions And A Corresponding Subclass Of Spirallike Functions

GJSFR Classification – F (FOR) 010108,010111,010105

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***Abstract*-In this paper we introduce and study the class of uniformly-spirallike functions of order and a subclass of it. Also some convolution results are obtained.**

AMS Subject Classification: 30C45.

Keywords- Uniformly convex functions, spirallike functions, parabolic region.

I. Introduction

Let A denote the class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined on the unit disk $\Delta = \{z : |z| < 1\}$ normalized by $f(0) = 0,\ f'(0) = 1.$ The function f ∈ A is spirallike if $Re\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\} > 0$ for all $z \in \Delta$ and for some a with $|\alpha| < \pi/2.$ Also f(z) is convex spirallike if if zf′ (z) is spirallike.

Definition 1.1-

(Uniformly α-spirallike functions of order β). The function f(z) of the form (1.1) is uniformly α-spirallike of order β) if the image of every circular arc Γ_z with center at ζ lying in Δ is α-spirallike of order β with respect to $f(\zeta).$ This condition is equivalent to arg $\left(\frac{z'(t)f'(z)}{f(z)-f(\zeta)} - \beta i e^{i\alpha}\right)$ lies between α and $\alpha + \pi.$

The class of all uniformly α-spirallike functions of order β is denoted by $USP(\alpha, \beta).$

Theorem 1.1.- Let $f \in \mathcal{A}.$ Then $f \in UCV(\beta)$ *iff*

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} > \left|\frac{zf''(z)}{f'(z)}\right| + \beta, z \in \Delta.$$

This class extends the class of UCV functions introduced and studied by various authors in [3, 4, 7, 12]. We now give the following analytic description for $USP(\alpha, \beta).$

Theorem 1.2- Let $|\alpha| < \pi/2,\ \beta > 0.$ A function $f \in \mathcal{A}.$ belongs to $USP(\alpha, \beta).$ iff.

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} \geq \beta, z \neq \zeta, z, \zeta \in \Delta.$$

Proof.-Describe Γ_z by $z(t) = \zeta + re^{it}, t \in [0, 2\pi].$ Then $z'(t) = i(z - \zeta).$ Now $USP(\alpha, \beta)$ *iff*

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} \geq \beta, z \neq \zeta, z, \zeta \in \Delta.$$

Since

$$\begin{aligned} arg\left(\frac{z'(t)f'(z)}{f(z)-f(\zeta)} - \beta i e^{i\alpha}\right) &= arg\left(\frac{i(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta i e^{i\alpha}\right) \\ &= \pi/2 + arg\left(\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta e^{i\alpha}\right) \\ &= \pi/2 + \alpha + arg\left(e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta\right), \end{aligned}$$

we have

$$-\pi/2 \leq arg\left(e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta\right) \leq \pi/2$$

Or

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta\right\} \geq 0$$

Or

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} \geq \beta. \tag{a}$$

This extends obviously the class of functions studied in [4]. An equivalent form of Theorem 1.2 in terms of Hadamard product is

Theorem 1.3-

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic in Δ then the Hadamard product of f and g is $f * g = \sum_{n=0}^{\infty} a_n b_n z^n.$

In particular, if f is a normalized analytic function in Δ then for all $\beta, \gamma,$ $0 \leq |\beta| \leq 1, 0 \leq |\gamma| \leq 1,$ (see [10]) we have

$$\frac{1}{\beta}f(\beta z) = f * \frac{z}{1-\beta z}$$

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and

$$\frac{f(\beta z)-f(\gamma z)}{\beta-\gamma}=f*\frac{z}{(1-\beta z)(1-\gamma z)}$$
$$zf'(\beta z)=f*\frac{z}{(1-\beta z)^2}.$$

Theorem 1.3-Let $f\in\mathcal{A}$. Then $USP(\alpha,\beta)$ iff

$$Re\left\{e^{i\alpha}\frac{f*\frac{z}{(1-bz)(1-cz)}}{f*\frac{z}{(1-bz)^2}}\right\}\leq\frac{1}{\beta}.$$

Proof-Let $w=bz,\zeta=cz.$
Now

$$f*\frac{z}{(1-bz)(1-cz)}=\frac{\frac{f(bz)-f(cz)}{b-c}}{zf'(bz)}=\frac{\frac{f(\omega)-f(\zeta)}{(\omega-\zeta)/z}}{zf'(\omega)}=\frac{f(\omega)-f(\zeta)}{(\omega-\zeta)f'(\omega)}.$$

$$Re\left\{e^{i\alpha}\frac{f*\frac{z}{(1-bz)(1-cz)}}{f*\frac{z}{(1-bz)^2}}\right\}=Re\left\{e^{i\alpha}\frac{f(\omega)-f(\zeta)}{(\omega-\zeta)f'(\omega)}\right\}\leq\frac{1}{\beta}$$ (b)

and the result follows from Theorem 1.2.

Theorem 1.4-If $f\in\mathcal{A}$ satisfies
$Re\left\{e^{i\alpha}\frac{f'(\omega)}{f'(z)}\right\}\leq\frac{1}{\beta},\ z,\omega\in\Delta$ then $f\in USP(\alpha,\beta).$

Proof-Note that

$$\frac{f(z)-f(\zeta)}{e^{-i\alpha}(z-\zeta)f'(z)}=\frac{1}{e^{-i\alpha}f'(z)}\int_0^1 f'(tz+(1-t)\zeta)dt=\int_\zeta^z e^{i\alpha}\frac{f'(\omega)}{f'(z)(z-\zeta)}d\omega,\quad\text{where }\omega=tz+(1-t)\zeta.$$

By hypothesis,

$$Re\left\{e^{i\alpha}\frac{f'(\omega)}{f'(z)}\right\}\leq\frac{1}{\beta}$$
or
$$Re\left\{\frac{f(z)-f(\zeta)}{e^{-i\alpha}(z-\zeta)f'(z)}\right\}\leq\frac{1}{\beta},$$

which is equivalent to

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\}\geq\beta.$$
$$\Rightarrow f\in USP(\alpha,\beta).$$ (a)

II. Uniformly convex α-spiral functions of order β

Let Γ_ω be the image of an arc $\Gamma_z:z=z(t),\ (a\leq t\leq b)$ under the function f(z). The arc Γ_ω is convex α-spirallike of order β if

$$arg\left(\frac{z''(t)}{z'(t)}+z'(t)\frac{f''(z)}{f'(z)}-\beta ie^{i\alpha}\right)$$

lies between α and $\alpha+\pi.$

Definition 2.1-The function f(z) represented by (1.1) is uniformly convex α-spiral of order β if the image of every circular arc Γ_z with center at ζ lying in Δ is convex α-spirallike of order β.
The class of all uniformly convex α-spiral functions of order β is denoted by $UCSP(\alpha,\beta).$
We now give an analytic description of $UCSP(\alpha,\beta).$

Theorem 2.1. A function $f(z)\in\mathcal{A}$ is in $UCSP(\alpha,\beta)$ iff

$$Re\left\{e^{-i\alpha}\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right)\right\}\geq\beta,z\neq\zeta,z,\zeta\in\Delta.$$

Proof-Let $f(z)\in UCSP(\alpha,\beta).$ Then we have

$$\alpha\leq arg\left(\frac{z''(t)}{z'(t)}+z'(t)\frac{f''(z)}{f'(z)}-\beta ie^{i\alpha}\right)\leq\alpha+\pi$$

where the curve Γ_z is given by $z(t)=\zeta+re^{it},\ 0\leq t\leq 2\pi.$
Then $f(z)\in UCSP(\alpha,\beta)$ iff

$$\alpha\leq arg\left(i+i(z-\zeta)\frac{f''(z)}{f'(z)}-\beta ie^{i\alpha}\right)\leq\alpha+\pi.$$

Since

$$arg\left(i+i(z-\zeta)\frac{f''(z)}{f'(z)}-\beta ie^{i\alpha}\right)=\frac{\pi}{2}+arg\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}-\beta e^{i\alpha}\right),$$
$$\alpha-\frac{\pi}{2}\leq arg\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}-\beta e^{i\alpha}\right)\leq\alpha+\frac{\pi}{2}.$$

(or)

$$-\frac{\pi}{2}\leq-\alpha+arg\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}-\beta e^{i\alpha}\right)\leq\frac{\pi}{2}.$$

Hence $Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}-\beta e^{i\alpha}\right)\right\}\geq 0.$
or $Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\}\geq\beta,\ z\neq\zeta,\ z,\zeta\in\Delta.$ (a)

We now prove a single variable characterization of the class $UCSP(\alpha,\beta).$

Theorem 2.2-A function f(z) ∈ A is in $UCSP(\alpha,\beta)$ iff

$$Re\left\{e^{-i\alpha}\left(1+z\frac{f''(z)}{f'(z)}\right)\right\}\geq\left|z\frac{f''(z)}{f'(z)}\right|+\beta,z\in\Delta.$$

Proof-If *f* ∈ $UCSP(\alpha,\beta)$. then we have

$$Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\}\geq\beta.$$

Equivalently,

$$Re\left\{e^{-i\alpha}\left(1+z\frac{f''(z)}{f'(z)}\right)\right\}\geq Re\left\{e^{-i\alpha}\zeta\frac{f''(z)}{f'(z)}\right\}+\beta$$

for every $z\neq\zeta\in\Delta.$

Choose ζ = $e^{ir}z$ such that

$$Re\left|\left\{e^{-i\alpha}\zeta\frac{f''(z)}{f'(z)}\right\}+\beta=\left|z\frac{f''(z)}{f'(z)}\right|+\beta.$$

Then we have

$$Re\left\{e^{-i\alpha}\left(1+z\frac{f''(z)}{f'(z)}\right)\right\}\geq\left|z\frac{f''(z)}{f'(z)}\right|+\beta.$$

Conversely assume the inequality is satisfied. Let ζ ∈ Δ be arbitrary. Since $Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\}$ is harmonic in Δ it is enough to prove that $Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\}\geq\beta$ for all $z\in\Delta$ for which $|z|>|\zeta|.$ Now

$$\begin{aligned}Re\left\{e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}&\geq\left|\frac{zf''(z)}{f'(z)}\right|+\beta\\&\geq\left|e^{-i\alpha}\zeta\frac{f''(z)}{f'(z)}\right|+\beta\\&\geq Re\left\{e^{-i\alpha}\zeta\frac{f''(z)}{f'(z)}\right\}+\beta\end{aligned}$$

$$\Rightarrow Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\}\geq\beta.$$

$$\Rightarrow f(z)\in\mathcal{A}\text{ is in }UCSP(\alpha,\beta).\qquad\text{(b)}$$

Definition 2.2-A function f(z) ∈ *A* is in $SP_p(\alpha,\beta)$ *iff*

$$Re\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\}\geq\left|\frac{zf'(z)}{f(z)}-1\right|+\beta,\ z\in\Delta.$$

Geometrically it means that $\frac{zf'(z)}{f(z)}$ lies in the parabolic region

$$\Omega_\alpha=\{\omega:Re\{e^{-i\alpha}\omega\}>|\omega-1|+\beta\}.$$

If $\omega\in\Omega_\alpha$, then $Re\{e^{-i\alpha}\omega\}\geq\frac{\cos\alpha+\beta}{2}$ and therefore the functions in the class $SP_p(\alpha,\beta)$ are α-spirallike of order $\frac{\cos\alpha+\beta}{2}$. When $\beta=0,\ SP_p(\alpha,\beta)$ reduces to $SP_p(\alpha)$ *[11].*

The class $UCSP(0,\beta)$ is the class $UCV(\beta)$. In fact, every function in the class $UCSP(\alpha,\beta)$ is related to the class U $UCV(\beta)$. as in the following theorem.

Theorem 2.3-Let f(z) ∈ A and s(z) be defined by f′(z) = $(s'(z))e^{ia\cos a}$. Then f(z) ∈ $UCSP(\alpha,\beta)$ iff s(z) $\in UCV(\beta)$.

Proof-The result follows since

$$1+\frac{zs''(z)}{s'(z)}=\frac{e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)+i\sin\alpha}{\cos\alpha}.$$

We now prove a two variable characterization of the function in the class $SP_p(\alpha,\beta)$.

Theorem 2.4-The function f is in $SP_p(\alpha,\beta)$ *iff*

$$Re\left\{e^{-i\alpha}\left((z-\zeta)\frac{f'(z)}{f(z)}+\frac{\zeta}{z}\right)\right\}\geq\beta.$$

Proof-We have f(z) is in $SP_p(\alpha,\beta)$ iff $\int_0^z\frac{f(z)}{z}dz\in UCSP(\alpha,\beta)$.

By the two variable characterization for the class $UCSP(\alpha,\beta)$ for

$$F(z)=\int_0^z\frac{f(z)}{z}dz\in UCSP(\alpha,\beta),$$

$$Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{F''(z)}{F'(z)}\right)\right\}\geq\beta.$$

$$\Rightarrow Re\left\{e^{-i\alpha}\left((z-\zeta)\frac{f'(z)}{f(z)}+\frac{\zeta}{z}\right)\right\}\geq\beta.\qquad\text{(a)}$$

Theorem 2.5- The function

$$f(z)=z+a_nz^n\ \textit{is in}\ SP_p(\alpha,\beta)\ \textit{iff}$$

$$|a_n|\leq\frac{\cos\alpha-\beta}{n(1+\cos\alpha)-1-\beta}.$$

Proof-The function $f(z)=z+a_nz^n\in SP_p(\alpha,\beta)$ iff

$$\left|\frac{zf''(z)}{f'(z)}-1\right|+\beta\leq Re\left\{e^{-i\alpha}\frac{zf''(z)}{f'(z)}\right\},|z|<1.\qquad(1)$$

It suffices to prove (2.1) for |z| = 1. Let |an| = r $a_nz^{n-1}=re^{i\theta}$. Then equation (2.1) becomes

$$\left|\frac{(n-1)re^{i\theta}}{1+re^{i\theta}}\right|+\beta\leq Re\left\{e^{-i\alpha}\frac{1+nre^{i\theta}}{1+re^{i\theta}}\right\}.\qquad(2)$$

Simplifying and separating the real part of the expression on the right hand side of (2.2) we get

$$Re\left\{e^{-i\alpha}\frac{1+nre^{i\theta}}{1+re^{i\theta}}\right\}=\frac{[1+(n+1)r\cos\theta+nr^2]\cos\alpha+\sin\alpha(n-1)r\sin\theta}{1+r^2+2r\cos\theta}.$$

Therefore inequality (2.2) gives

$$(n-1)r+\beta(1+r^2+2r\cos\theta)^{1/2}\leq\frac{(1+(n+1)r\cos\theta+nr^2)\cos\alpha+\sin\alpha(n-1)r\sin\theta}{(1+r^2+2r\cos\theta)^{1/2}}.$$

The minimum for the expression in the right hand side of the above equation occurs at $\theta=\pi$ and this minimum value is $\cos\alpha(1-nr)$. Therefore the necessary and sufficient condition for $f(z)=z+a_nz^n$ to be in $SP_p(\alpha,\beta)$ is that $(n-1)r+\beta(1-r)\leq\cos\alpha(1-nr)$.

Solving this equation for r = $|a_n|$ we have

$$|a_n|\leq\frac{\cos\alpha-\beta}{n(1+\cos\alpha)-1-\beta}. \qquad (a)$$

Since $f\in UCSP(\alpha,\beta)$ iff $zf'(z)\in SP_p(\alpha,\beta)$ we have the following.

Corollary 2.1-The function

$f(z)=z+a_nz^n$ *is in* $UCSP(\alpha,\beta)$ *iff*

$$|a_n|\leq\frac{\cos\alpha-\beta}{n[n(1+\cos\alpha)-1-\beta]}.$$

Theorem 2.6-Let $f_i(z)\in SP_p(\alpha,\beta),\ i=1,2,\ldots,n$ and F(z) be given by

$F(z)=\int_0^z\prod_{i=1}^n\left(\frac{f_i(z)}{z}\right)^{\alpha_i}dz$ *where* $\alpha_i\geq0$ *and* $\sum_{i=1}^n\alpha_i=1$. *Then* $F(z)\in UCSP(\alpha,\beta)$.

Proof.-

Since $1+\frac{zF''(z)}{F'(z)}=\sum_{i=1}^n\alpha_i\frac{zf_i'(z)}{f_i(z)}$, we have

$$\begin{aligned}Re\left\{e^{-i\alpha}\left(1+\frac{zF''(z)}{F'(z)}\right)\right\}&=\sum_{i=1}^n\alpha_iRe\left(e^{-i\alpha}\frac{f_i'(z)}{f_i(z)}\right)\\&\geq\sum_{i=1}^n\alpha_i\left[\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\beta\right]\\&\geq\left|\sum_{i=1}^n\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right|+\beta\\&\geq\left|\frac{zF''(z)}{F'(z)}\right|+\beta.\end{aligned}$$

III. Convolution Theorems

Theorem 3.1-If $f(z)=z+\sum_{n=2}^\infty a_nz^n$ and $g(z)=z+\sum_{n=2}^\infty b_nz^n$ *are* elements of $UCSP(\alpha,\beta)$ then $(f*g)(z)=h(z)=z+\sum_{n=2}^\infty a_nb_nz^n$ *is in* $UCSP(\alpha,r)$ *where*

$$r\leq\frac{\cos\alpha(32+3\cos^2\alpha-20\cos\alpha-8\beta+3\beta^2+2\beta\cos\alpha)-4\beta^2}{32+\cos^2\alpha+\beta^2-16\cos\alpha-16\beta+6\beta\cos\alpha},$$

$0\leq\alpha<1,\ \beta\geq0.$

Proof. Since $f(z)$ and $g(z)$ are in $UCSP(\alpha,\beta)$ we have $\sum_{n=2}^\infty n(2n-\cos\alpha-\beta)|a_n|\leq\cos\alpha-\beta$ and $\sum_{n=2}^\infty n(2n-\cos\alpha-\beta)|b_n|\leq\cos\alpha-\beta$. We wish to find the largest r such that $\sum_{n=2}^\infty n(2n-\cos\alpha-r)|a_nb_n|\leq\cos\alpha-r$. Equivalently we want to show that the conditions

$$\sum_{n=2}^\infty\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}n|a_n|\leq1 \qquad (1)$$

and

$$\sum_{n=2}^\infty\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}n|b_n|\leq1 \qquad (2)$$

imply that

$$\sum_{n=2}^\infty\frac{2n-\cos\alpha-r}{\cos\alpha-r}n|a_nb_n|\leq1 \qquad (3)$$

for all

$$r\leq\frac{\cos\alpha(32+3\cos^2\alpha-20\cos\alpha-8\beta+3\beta^2+2\beta\cos\alpha)-4\beta^2}{32+\cos^2\alpha+\beta^2-16\cos\alpha-16\beta+6\beta\cos\alpha}.$$

From (3.1) and (3.2) and by means of Cauchy-Schwarz inequality, we get that

$$\sum_{n=2}^\infty\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}n\sqrt{|a_nb_n|}\leq1. \qquad (4)$$

Hence it is enough if we prove

$$\frac{2n-\cos\alpha-r}{\cos\alpha-r}n|a_nb_n|\leq\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}n\sqrt{|a_nb_n|}\quad\text{for } n=2,3,\ldots$$

(or)

$$\sqrt{|a_nb_n|}\leq\left(\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}\right)\left(\frac{\cos\alpha-r}{2n-\cos\alpha-r}\right).$$

From (3.4) it follows that

$$n\sqrt{|a_nb_n|}\leq\frac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\ \text{for all } n. \qquad (5)$$

The above inequality is equivalent to

$$\frac{r+\cos\alpha}{2} \le \frac{\cos\alpha - \left[\frac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\right]^2}{1-\frac{1}{n}\left[\frac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\right]^2}. \qquad (6)$$

The right hand side of (3.6) is an increasing function of n, (n = 2, 3, . . .). By taking n = 2 in (3.6) we get

$$r \le \frac{\cos\alpha(32+3\cos^2\alpha-20\cos\alpha-8\beta+3\beta^2+2\beta\cos\alpha)-4\beta^2}{32+\cos^2\alpha+\beta^2-16\cos\alpha-16\beta+6\beta\cos\alpha}. \qquad (a)$$

Theorem 3.2-For $For\ f(z)\in UCSP(\alpha,\beta_1)$ and $g(z)\in UCSP(\alpha,\beta_2)$ we have $f(z)*g(z)\in UCSP(\alpha,r)$ where

$$r \le \frac{\cos\alpha(32-20\cos\alpha+3\cos^2\alpha-4(\beta_1+\beta_2)+3\beta_1\beta_2+(\beta_1+\beta_2)\cos\alpha)-4\beta_1\beta_2}{32+\cos^2\alpha+\beta_1\beta_2-16\cos\alpha-8(\beta_1+\beta_2)+3\cos\alpha(\beta_1+\beta_2)}.$$

Proof-Proceeding as in the proof of Theorem 3.1 we get

$$\frac{\cos\alpha+r}{2} \le \frac{\cos\alpha-\left[\frac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\frac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}{1-\frac{1}{n}\left[\frac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\frac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}. \qquad (7)$$

The right hand side of (3.7) is an increasing function of n = 2, 3, ... Setting n = 2 we get

$$r \le \frac{\cos\alpha(32-20\cos\alpha+3\cos^2\alpha-4(\beta_1+\beta_2)+3\beta_1\beta_2+(\beta_1+\beta_2)\cos\alpha)-4\beta_1\beta_2}{32+\cos^2\alpha+\beta_1\beta_2-16\cos\alpha-8(\beta_1+\beta_2)+3\cos\alpha(\beta_1+\beta_2)}. \qquad (8)$$

Theorem 3.3-Let $f(z),g(z),h(z)\in UCSP(\alpha,\beta)$. Then $Then\ f(z)*g(z)*h(z)\in UCSP(\alpha,r_1)\ where$

$$r_1 \le \frac{\begin{array}{l}6\cos^5\alpha+\cos^4\alpha(28\beta-100)+\cos^3\alpha(240-276\beta+18\beta^2)\\ +\cos^2\alpha(-672+514\beta-108\beta^2+10\beta^3)\\ +\cos\alpha(1024-768\beta+144\beta^2-28\beta^3)+16\beta^2\end{array}}{\begin{array}{l}1024-768\beta+184\beta^2-4\beta^3+10\cos^4\alpha+\cos^3\alpha(-116+18\beta)\\ +\cos^2\alpha(512-228\beta+32\beta^2)+\cos\alpha(-1280+256\beta-156\beta^2+6\beta^3)\end{array}}$$

Proof. From Theorem 3.1 we get $f(z)*g(z)\in UCSP(\alpha,r)$ where

$$r \le \frac{\cos\alpha(32+3\cos^2\alpha-20\cos\alpha-8\beta+3\beta^2+2\beta\cos\alpha)-4\beta^2}{32+\cos^2\alpha+\beta^2-16\cos\alpha-16\beta+6\beta\cos\alpha}.$$

$f(z)*g(z)*h(z)\in UCSP(\alpha,r_1)$ where

$$r_1 \le \frac{32\cos\alpha-20\cos^2\alpha-4\cos\alpha(r+\beta)+3\cos^3\alpha+(r+\beta)\cos^2\alpha+3r\beta\cos\alpha-4r\beta}{32+\cos^2\alpha+r\beta-16\cos\alpha-8(r+\beta)+3(r+\beta)\cos\alpha}. \qquad (a)$$

Substituting for r and simplifying we get the required result.

Theorem 3.4-Let

$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ $f,g\in SP_p(\alpha,\beta).\ Then\ f*g\in SP_p(\alpha,r)\ for$

$$r \le \frac{\cos\alpha[8+\cos^2\alpha+\beta^2-6\cos\alpha+\beta\cos\alpha-\beta]-2\beta^2}{2(4-2\cos\alpha-2\beta+\beta\cos\alpha)}.$$

Proof. Since f(z) and g(z) are in $SP_p(\alpha,\beta)$ we have

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)|a_n| \le \cos\alpha-\beta \text{ and}$$
$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)|b_n| \le \cos\alpha-\beta.$$

We wish to find the largest r such that

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-r)|a_nb_n| \le \cos\alpha-r.$$

Equivalently we want to show that the conditions

$$\sum_{n=2}^{\infty}\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}|a_n| \le 1 \qquad (9)$$

imply that

$$\sum_{n=2}^{\infty}\frac{2n-\cos\alpha-r}{\cos\alpha-r}|a_nb_n| \le 1$$

for all

$$r \le \frac{\cos\alpha[8+\cos^2\alpha+\beta^2-6\cos\alpha+\beta\cos\alpha-\beta]-2\beta^2}{2(4-2\cos\alpha-2\beta+\beta\cos\alpha)}.$$

From (3.8) and (3.9) and by Cauchy-Schwarz inequality, we get that

$$\sum_{n=2}^{\infty}\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}\sqrt{|a_nb_n|} \le 1. \qquad (10)$$

Hence it is enough if we prove

$$\frac{2n-\cos\alpha-r}{\cos\alpha-r}|a_nb_n| \le \frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}\sqrt{|a_nb_n|}.$$

(or)

$$\sqrt{|a_nb_n|} \le \left(\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}\right)\left(\frac{\cos\alpha-r}{2n-\cos\alpha-r}\right).$$

From (3.10) it follows that

$$\sqrt{|a_nb_n|} \le \frac{\cos\alpha-\beta}{2n-\cos\alpha-\beta} \text{ for all } n.$$

The above inequality is equivalent to

$$\frac{r+\cos\alpha}{2} \le \frac{\cos\alpha - n\left[\dfrac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\right]^2}{1-\left[\dfrac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\right]^2}. \qquad (11)$$

The right hand side of (3.11) is an increasing function of n (n = 2, 3, . . .).By taking n = 2 in (3.11) we get

$$r \le \frac{\cos\alpha[8+\cos^2\alpha+\beta^2-6\cos\alpha+\beta\cos\alpha-\beta]-2\beta^2}{2(4-2\cos\alpha-2\beta+\beta\cos\alpha)}. \qquad (a)$$

Theorem 3.5-For $f(z)\in SP_p(\alpha,\beta_1)$ *and* $g(z)\in SP_p(\alpha,\beta_2)$ and $g(z)\in SP_p(\alpha,\beta_2)$ we have

$f(z)*g(z)\in SP_p(\alpha,r)$ *where*

$$r \le \frac{\cos\alpha(8+\cos^2\alpha+\beta_1\beta_2-6\cos\alpha)-2\beta_1\beta_2}{8-4\cos\alpha-2(\beta_1+\beta_2)+(\beta_1+\beta_2)\cos\alpha}.$$

Proceeding as in the proof of Theorem 3.4 we get

$$\frac{r+\cos\alpha}{2} \le \frac{\cos\alpha - n\left[\dfrac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\dfrac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}{1-\left[\dfrac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\dfrac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}.$$

The right hand side is an increasing function of n = 2, 3, ... Setting n = 2 we get

$$r \le \frac{\cos\alpha(8+\cos^2\alpha+\beta_1\beta_2-6\cos\alpha)-2\beta_1\beta_2}{8-4\cos\alpha-2(\beta_1+\beta_2)+(\beta_1+\beta_2)\cos\alpha}.$$

Corollary 3.1-Let $f(z),g(z),h(z)\in SP_p(\alpha,\beta)$. Then $f(z)*g(z)*h(z)\in SP_p(\alpha,r_1)$ *where*

$$r_1 \le \frac{4\beta^3+\cos^4\alpha(3\beta-4)+\cos^3\alpha(32+\beta^2)+\cos^2\alpha(\beta^3-3\beta^2+60\beta-80)+\cos\alpha(-4\beta^3+2\beta^2-48\beta+64)}{64+8\beta^2-48\beta+\cos^4\alpha+\cos^3\alpha(\beta-8)+\cos^2\alpha(36-15\beta+3\beta^2)+\cos\alpha(80+50\beta-12\beta^2)}.$$

From Theorem 3.4 we get $f(z)*g(z)\in SP_p(\alpha,r)$ *where*

$$r \le \frac{\cos\alpha[8+\cos^2\alpha+\beta^2-6\cos\alpha+\beta\cos\alpha-\beta]-2\beta^2}{2(4-2\cos\alpha-2\beta+\beta\cos\alpha)}.$$

$f(z)*g(z)*h(z)\in SP_p(\alpha,r_1)$ *where*

$$r_1 \le \frac{\cos\alpha[8+\cos^2\alpha+\beta r-6\cos\alpha)-2\beta r}{8-4\cos\alpha-2\beta-2r+(\beta+r)\cos\alpha}.$$

Substituting for r and simplifying we get the required result.

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Some New Characterizations Of Strongly α-Preinvex Functions*

GJSFR Classification – F (FOR) 010301,010303,010206

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***Abstract*-In this paper, we study some properties of strongly α-preinvex functions by introducing generalized strongly α-invex functions and generalized strongly $\alpha\eta$monotonicity. Our results generalize the results obtained by Muhammad Aslam Noor, Khalida Inayat Noor [J.Math.Anal.Appl. 316(2006) 697–706].**

Keywords-strongly α- preinvex functions; directionally differentiable; generalized strongly $\alpha\eta$ monotonicity;invex functions.

MR(2000)Subject Classification: 65K99,90C46

I. Introduction

It is well known that convexity and generalized convexity play a key role in many aspects of optimization, such as optimality conditions, saddle-point theorems, duality theorems, theorems of alternatives, and convergence of optimization algorithms, so the research on convexity and generalized convexity is one of the important aspects in mathematical programming. During the past several decades, to relax convexity assumptions imposed on the functions in theorems on optimality and duality, various generalized convexity concepts have been proposed. Among them, an important and significant generalization of convex function is introduction of invexity, which was introduced by Hanson [1, 3]. This concept is particularly interesting, since it is provides a broader setting to study the optimization problems. Later, Ben-Isreal and Mond [2] introduced a class of generalized convex functions, which is called the preinvex function. Mohan and Neogy [4]have shown that the preinvex functions and invex functions are equivalent under some conditions. In [5, 6], Weir, Mond and Noor have shown that the preinvex functions preserve some nice properties that convex functions have. Jeyakumar, Mond [7] and Jeyakumar [8] introduced another class of generalized convex functions, which was called strongly α- -invex function, and some of its properties were studied. However, the concepts of the strongly α-invex function in [7, 8] is misleading. To overcome this drawback, recently, M.A.Noor and K.I.Noor [9] also introduced a new class of generalized convex condition for strongly pseudo α- -invex function. However, the functions involved in [9] are all required to be differentiable.

To relax the differentiable Assumption, in this paper, by introducing the concepts of generalized strongly α-invex functions and generalized strongly $\alpha\eta$-monotonicity, we also study some properties of strongly α- -preinvex functions. Our results generalize the results obtained in [9].

II. Preliminaries

Let K be a nonempty closed set in a real Hilbert space H. we denote by $\langle\cdot\rangle$ and $\|\cdot\|$ the inner product and norm, respectively. Let F : K −→ R and $\eta(\cdot,\cdot)$: K × K −→ H be continuous functions. Let $\alpha(\cdot,\cdot)$ K ×K −→ R\ {0} be a bifunction. First of all, we review some concepts as follows which have some relationships with this paper.

Definition 2.1-(See [9])Let u ∈ K Then the set K is said to be α-invex at u with respect to $\eta(\cdot,\cdot)$ and $\alpha(\cdot,\cdot)$ if, for all μ, ν ∈ K, t ∈ [0, 1], μ + t α (ν, μ) η (ν, μ) ∈ K. K is said to be an α- invex set with respect to η and α, if K is α -invex at each u ∈ K

Definition 2.2-(See [9])A function F on the set K is said to strongly α- preinvex , if there exists a constant μ > 0 such that

$$F(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu))\le(1-t)F(\mu)+tF(\nu)-t(1-t)\mu\|\eta(\nu,\mu)\|^2,\quad\forall\mu,\nu\in K.$$

Definition 2.3-(See [9])A differentiable function F on the set K is said to strongly α- invex if there exists a constant μ > 0 such that

$$F(\nu)-F(\mu)\ge\langle\alpha(\nu,\mu)F'(\mu),\eta(\nu,\mu)\rangle+\mu\|\eta(\nu,\mu)\|^2,\quad\forall\mu,\nu\in K,$$

where F′(μ) is the differential of a function F at μ ∈ K.

Condition A. $F(\mu+\alpha(\nu,\mu)\eta(\nu,\mu))\le F(\nu),\quad\forall\mu,\nu\in K.$

Condition C.

Let $\eta(\nu,\mu):K\times K\to H$ and $\alpha(\nu,\mu):K\times K\to R\setminus\{0\}$ satisfy the assumptions

$$\eta(\mu,\mu+t\alpha(v,\mu)\eta(v,\mu))=-t\eta(\nu,\mu),$$

$$\eta(\nu,\mu+t\alpha(v,\mu)\eta(v,\mu))=(1-t)\eta(\nu,\mu),\quad\forall\mu,\nu\in K,T\in[0,1].$$

In the sequel, we assume that F is directional differentiable, and following [10], we also assume that

$$\alpha(x,\mu)F'(x,\eta(x,\mu))=\lim_{\lambda\to0^+}\frac{F(x+\lambda\alpha(x,\mu)\eta(x,\mu))-F(x)}{\lambda},$$

*About-*Project supported by Scientific Research Fund of Hunan Provincial Education Department(09C565)*

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where $F'(\cdot,\cdot)$ is the directional derivative of F.

Definition 2.4-A directional differentiable function F on the set K is said to generalized strongly α- invex if there exists a constant μ > 0 such that

$$F(\nu) - F(\mu) \geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \mu \parallel \eta(\nu,\mu) \parallel^2, \quad \forall \mu,\nu \in K,$$

where F′(μ, η (v, μ)) is the directional derivative of a function F at μ ∈ K along the direction η (v, μ).

Remark 1-The definition generalized Definition 2.3.

Definition 2.5- (i) A directional differentiable operator T : K → H is said to generalized strongly $\alpha\eta$ -monotone, if there exists a constant α_1 > 0 such that

$$\alpha(\nu,\mu)T'(\mu,\eta(\nu,\mu)) + \alpha(\mu,\nu)T'(\nu,\eta(\mu,\nu)) \leq -\alpha_1\{\parallel \eta(\nu,\mu) \parallel^2 + \parallel \eta(\mu,\nu) \parallel^2\}, \quad \forall \mu,\nu \in K.$$

ii) A directional differentiable operator T : K → H is said to generalized strongly $\alpha\eta$-monotone, generalized $\alpha\eta$-pseudomonotone, if there exists a constant α_1 > 0 such that

$$\alpha(\mu,\nu)T'(\nu,\eta(\mu,\nu)) + \alpha_1 \parallel \eta(\nu,\mu) \parallel^2 \geq 0 \Longrightarrow -\alpha(\mu,\nu)T'(\nu,\eta(\mu,\nu)) \geq 0 \quad \forall \mu,\nu \in K.$$

Remark *2*-We may also define generalized $\alpha\eta$- monotone, generalized strictly $\alpha\eta$- monotone, generalized $\alpha\eta$ -pseudomonotone, generalized strictly $\alpha\eta$ -pseudomonotone and generalized quasi $\alpha\eta$ -monotone in the similar way.

Definition 2.6-A directional differentiable function F on the set K is said to generalized strongly pseudo $\alpha\eta$-invex, if there exists a constant α_1 > 0 such that

$$\alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \alpha_1 \parallel \eta(\mu,\nu) \parallel^2 \geq 0 \Longrightarrow F(\nu) - F(\mu) \geq 0, \quad \forall \mu,\nu \in K.$$

Remark 3- We may also define generalized strongly quasi $\alpha\eta$- invex and generalized pseudo $\alpha\eta$- invex in the similar way.

III. SOME PROPERTIES OF STRONGLY α-PREINVEX FUNCTIONS

In this section, we discuss the relationship among strongly α - Preinvex , generalized strongly α - invex and generalized strongly $\alpha\eta$ --monotone under some conditions, Our results generalize the results obtained by Muhammad Aslam Noor, Khalida Inayat Noor in [9].

Theorem 3.1-Let F be a directional differentiable function on the α- invex set K. Let Condition C holds and α (μ, *v*) = α (*v*, μ) for all μ, _ ∈ K. Then the function F is strongly α- Preinvex if and only if F is generalized strongly α-invex.

Proof- Let F be a strongly α-preinvex function on the α-invex set K. Then, for all μ, *v* ∈ K, t ∈ [0, 1], one gets

$$\nu_t = \mu + t\alpha(\nu,\mu)\eta(\nu,\mu) \in K,$$

$$F(\mu + t\alpha(\nu,\mu)\eta(\nu,\mu)) \leq (1-t)F(\mu) + tF(\nu) - t(1-t)\mu \parallel \eta(\nu,\mu) \parallel^2,$$

which can be written as

$$F(\nu) - F(\mu) \geq \frac{F(\mu + t\alpha(\nu,\mu)\eta(\nu,\mu)) - F(\mu)}{t} + (1-t)\mu \parallel \eta(\nu,\mu) \parallel^2.$$

Taking into account F be a directional differentiable function on the α- invex set K, letting t → 0+ in above inequality, we have

$$F(\nu) - F(\mu) \geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \mu \parallel \eta(\nu,\mu) \parallel^2.$$

So F is generalized strongly α- invex

Conversely, let F is generalized strongly α- invex on the α- invex K. Then, using Condition C, we obtain

$$\begin{aligned} F(\nu) - F(\nu_t) &\geq \alpha(\nu,\nu_t)F'(\nu_t,\eta(\nu,\nu_t)) + \mu \parallel \eta(\nu,\nu_t) \parallel^2 \\ &= (1-t)\alpha(\nu,\nu_t)F'(\nu_t,\eta(\nu,\mu)) + \mu(1-t)^2 \parallel \eta(\nu,\mu) \parallel^2. \end{aligned} \quad (1)$$

Similarly, we have

$$\begin{aligned} F(\mu) - F(\nu_t) &\geq \alpha(\mu,\nu_t)F'(\nu_t,\eta(\mu,\nu_t)) + \mu \parallel \eta(\mu,\nu_t) \parallel^2 \\ &= -t\alpha(\mu,\nu_t)F'(\nu_t,\eta(\nu,\mu)) + \mu t^2 \parallel \eta(\nu,\mu) \parallel^2. \end{aligned} \quad (2)$$

Multiplying (3.1) by t and (3.2) by (1 − t) and adding the resultant, we get

$$F(\mu + t\alpha(\nu,\mu)\eta(\nu,\mu)) \leq (1-t)F(\mu) + tF(\nu) - t(1-t)\mu \parallel \eta(\nu,\mu) \parallel^2.$$

This shows that F be a strongly α- preinvex function on the α-invex set K.

Lemma 3.1-Let F be a directional differentiable function on the α- invex set K, g(t) = F(μ+t α (*v*, μ) η (*v*, μ)) and $F'(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu)) + F'(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu), -\eta(\nu,\mu)) = 0,\ \forall t_0 \in [0,1]$. *Then g(t) is differentiable for all* $\mu,\nu \in K,\ t \in [0,1]$.

Proof. For all μ, *v* ∈ K, t0 ∈ [0, 1], in view of F be a directional differentiable function on the α-invex set K, we have

$$\begin{aligned} g'_+(t_0) &= \lim_{t\to t_0} \frac{F(\mu + t\alpha(\nu,\mu)\eta(\nu,\mu)) - F(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu))}{t - t_0} \\ &= \lim_{t-t_0\to 0^+} \frac{F(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu) + (t-t_0)\alpha(\nu,\mu)\eta(\nu,\mu)) - F(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu))}{t - t_0} \\ &= \alpha(\nu,\mu)F'(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu), \eta(\nu,\mu)), \end{aligned}$$

$$\begin{aligned} g'_-(t_0) &= \lim_{t\to t_0} \frac{F(\mu + t\alpha(\nu,\mu)\eta(\nu,\mu)) - F(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu))}{t - t_0} \\ &= -\lim_{t_0-t\to 0^+} \frac{F(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu) + (t_0-t)\alpha(\nu,\mu)(-\eta(\nu,\mu))) - F(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu))}{t - t_0} \\ &= -\alpha(\nu,\mu)F'(\mu + t_0\alpha(\nu,\mu)\eta(\nu,\mu), -\eta(\nu,\mu)). \end{aligned}$$

In view of

$$F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu))+F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),-\eta(\nu,\mu)) = 0,\ \forall t_0 \in [0,1].$$

This implies that g(t) is differentiable for all μ, v ∈ K, t ∈ [0, 1].

Theorem 3.2- Let F be a directional differentiable function on the α-invex set K. If F is generalized strongly α- invex set K. If F is generalized strongly α - invex and $F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu))+F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),-\eta(\nu,\mu)) = 0,\ \forall t_0 \in [0,1]$, then F is generalized strongly $\alpha\eta$-monotone. Conversely, if the function α (*v*, μ) is a symmetric, that is, α (*v*, μ) = α (μ, *v*) for all μ, *v* ∈ K, then F is generalized strongly α-invex under Condition A and C.

Proof-Let F is generalized strongly α- invex , then

$$F(\nu) - F(\mu) \geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \mu \parallel \eta(\nu,\mu) \parallel^2, \quad \forall \mu,\nu \in K. \tag{3}$$

Exchange μ and *v* in (3.3), we have

$$F(\mu) - F(\nu) \geq \alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu)) + \nu \parallel \eta(\mu,\nu) \parallel^2, \quad \forall \mu,\nu \in K. \tag{4}$$

Adding (3.3) and (3.4), one gets

$$\alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu)) \leq -\mu\{\parallel \eta(\nu,\mu) \parallel^2 + \parallel \eta(\mu,\nu) \parallel^2\}, \tag{5}$$

which implies that F is generalized strongly $\alpha\eta$-monotone

Conversely, let F be generalized strongly $\alpha\eta$-monotone. Then there exists a constant $\alpha_1 > 0$ such that

$$\alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu)) \leq -\alpha_1\{\parallel \eta(\nu,\mu) \parallel^2 + \parallel \eta(\mu,\nu) \parallel^2\},$$

which can be written as

$$F'(\nu,\eta(\mu,\nu)) \leq -F'(\mu,\eta(\nu,\mu)) - \overline{\alpha}_1\{\parallel \eta(\nu,\mu) \parallel^2 + \parallel \eta(\mu,\nu) \parallel^2\}, \tag{6}$$

Since the function α (*v*, μ) is a symmetric and $\overline{\alpha}_1 = \frac{\alpha_1}{\alpha(\mu,\nu)}$.

In view of K is a α-invex set, so, for all μ, *v*∈ K, t ∈ [0, 1], one gets

$\nu_t = \mu + t\alpha(\nu,\mu)\eta(\nu,\mu) \in K.$

Taking $v = v_t$ in (3.6) and using Condition C, we obtain

$$\begin{aligned} F'(\nu_t,\eta(\mu,\nu_t)) &\leq -F'(\mu,\eta(\nu_t,\mu)) - \overline{\alpha}_1\{\parallel \eta(\nu_t,\mu) \parallel^2 + \parallel \eta(\mu,\nu_t) \parallel^2\} \\ &= -tF'(\mu,\eta(\nu,\mu)) - 2t^2\overline{\alpha}_1 \parallel \eta(\nu,\mu) \parallel^2, \end{aligned}$$

which implies that

$$F'(\nu_t,\eta(\nu,\mu)) \geq F'(\mu,\eta(\nu,\mu)) + 2t\overline{\alpha}_1 \parallel \eta(\nu,\mu) \parallel^2 . \tag{7}$$

Let $g(t) = F(\mu + t\alpha(\nu,\mu)\eta(\nu,\mu))$ for all $\mu,\nu \in K,\ t \in [0,1]$. Then from Lemma 3.1 and (3.7), one gets

$$\begin{aligned} g'(t) &= \alpha(\nu,\mu)F'(\mu + t\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu)) \\ &\geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + 2\alpha(\nu,\mu)t\overline{\alpha}_1 \parallel \eta(\nu,\mu) \parallel^2 \qquad (3.8) \\ &= \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + 2\alpha_1 t \parallel \eta(\nu,\mu) \parallel^2 . \end{aligned}$$

Integrating (3.8) between 0 and 1, we have

$$g(1) - g(0) \geq \alpha(\nu, \mu)F'(\mu, \eta(\nu, \mu)) + \alpha_1 \parallel \eta(\nu, \mu) \parallel^2,$$

that is,

$$F(\mu + \alpha(\nu, \mu)\eta(\nu, \mu)) - F(\mu) \geq \alpha(\nu, \mu)F'(\mu, \eta(\nu, \mu)) + \alpha_1 \parallel \eta(\nu, \mu) \parallel^2 .$$

Taking into account Condition A, we have

$$F(\nu) - F(\mu) \geq \alpha(\nu, \mu)F'(\mu, \eta(\nu, \mu)) + \alpha_1 \parallel \eta(\nu, \mu) \parallel^2 .$$

This shows that the function F is generalized strongly α-invex on the α-invex set K.

Remark 4-We can also obtain corresponding results such as Theorem 3.3, Theorem 3.4 in [9] in the same way as [9].

Theorem 3.3-Let the function F be generalized strongly $\alpha\eta$-pseudomonotone and F′(μ +t0 α (ν, μ) η (ν, μ), η (ν, μ))+F′(μ+t0 α (ν, μ) η (ν, μ), η (ν, μ)) = 0, ∀t0 ∈ [0, 1]. If Condition A and C hold, then the function F is generalized strongly pseudo $\alpha\eta$invex.

Proof- Let the function F be generalized strongly $\alpha\eta$-pseudomonotone. Then, for all μ, ν ∈ K,

$$\alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu)) + \alpha_1 \parallel \eta(\nu,\mu) \parallel^2 \geq 0 \Longrightarrow -\alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu)) \geq 0 \quad \forall \mu,\nu \in K. \qquad (9)$$

Since K is an α-invex set, for all μ, ν∈ K, t ∈ [0, 1], $\nu_t = \mu + t\,\alpha(\nu, \mu)\,\eta(\nu, \mu) \in K$. Taking $\nu = \nu_t$ in (3.9) and using Condition C, we have

$$F'(\nu_t, \eta(\nu, \mu)) \geq 0. \qquad (10)$$

Let g(t) = F(ν_t) = F(μ + t α (ν, μ) η (ν, μ)) for all μ, ν ∈ K, t ∈ [0, 1]. Then using (3.10) and Lemma 3.1, one gets

$$g'(t) = \alpha(\nu, \mu)F'(\nu_t, \eta(\nu, \mu)) \geq 0.$$

Integrating the above relation between 0 and 1, we have g(1) − g(0) ≥ 0, that is, F(ν_t) − F(μ) ≥ 0. Using Condition A, we have F(ν)−F(μ) ≥ 0. This shows that the function F is generalized strongly pseudo $\alpha\eta$-invex.

Remark 5-We can also obtain corresponding results such as Corollary 3.1–Corollary 3.4 in [9] in the same way as [9].

IV. Concluding Remarks

In this paper, to relax the differentiable Assumption, by introducing the concepts of generalized strongly α-invex functions and generalized strongly $\alpha\eta$-monotonicity, we study some properties of strongly α-preinvex functions. Our results generalize the results obtained by Muhammad Aslam Noor, Khalida Inayat Noor in [9].

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Oscillation criteria for second-order half-linear dynamic equations on time scales

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GJSFR Classification – F (FOR) 010203,010204,010109

***Abstract*-In this paper, some oscillation criteria are established for second-order half-linear dynamic equations on time scales. The results obtained essentially improve and extend those of Agarwal et al. [R. P. Agarwal, D. O'Regan, S. H. Saker, Philos-type oscillation criteria for second-order half linear dynamic equations, Rocky Mountain J. Math. 37 (2007) 1085–1104], Saker [S. H. Saker, Oscillation criteria of second order half-linear dynamic equations on time scales, J. Comput. Appl. Math. 177 (2005) 375–387], Hassan [T. S. Hassan, Oscillation criteria for half-linear dynamic equations on time scales, J. Math. Anal. Appl. 345 (2008) 176–185]. Some examples are given to illustrate the main results.**

Keywords-Oscillation; Half-linear dynamic equations; Time scales

Mathematics Subject Classification 2010: 34C10, 34K11, 34N05

I. Introduction

The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger [1], is an area of mathematics that has recently received a lot of attention. Several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal et al. [2] and the references cited therein. For an excellent introduction to the calculus on time scales, see Bohner and Peterson [3].

Further information on working with dynamic equations on time scales can be found in [4].Recently, much attention is attracted by questions of the oscillation and nonoscillation of different classes of dynamic equations on time scales, we refer the reader to the papers [5–17] and the references therein.

We are concerned with the oscillation of the second-order half-linear dynamic equations

$$\left(r(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+p(t)x^{\gamma}(t)=0, \qquad (1)$$

on a time scale T; where $\gamma > 0°$ is a quotient of odd positive integers, r and p are rd-continuous positive functions defined on T:

Since we are interested in oscillatory behavior, we assume throughout this paper that the given time scale T is unbounded above. We assume t0 2 T and it is convenient to assume t0 > 0: We define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty)\cap\mathbb{T}$.

Agarwal et al. [5], Grace et al. [8], Hassan [10], Saker et al. [14] studied (1.1), and established some oscillation criteria under the case

$$\int_{t_0}^{\infty}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}=\infty, \qquad (2)$$

and the authors obtained some sufficient conditions which guarantee that every solution x of (1.1) oscillates or $\lim_{t\to\infty}x(t)=0$ under the case

$$\int_{t_0}^{\infty}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}<\infty. \qquad (3)$$

Our aim of this paper is to establish some new oscillation criteria for (1.1) under the case when (1.3). The paper is organized as follows: In Section 2, we shall establish several new oscillation criteria for (1.1). In Section 3, some applications and examples are given to illustrate the main results.

II. Main Results

In this section, by employing a Riccati transformation technique we establish oscillation criteria for Eq. (1.1). To prove our main results, we will use the formula

$$(x^{\gamma}(t))^{\Delta}=\gamma\int_0^1[hx^{\sigma}(t)+(1-h)x(t)]^{\gamma-1}\mathrm{d}hx^{\Delta}(t),$$

which is a simple consequence of Keller's chain rule [3, Theorem 1.90].In the following, we denote

$$d_+(t):=\max\{0,d^{\Delta}(t)\},\quad R(t):=\int_t^{\infty}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},\quad \alpha(t):=(r(t))^{\frac{1}{\gamma}}R(t),\quad \beta(t):=\frac{\alpha(t)-\mu(t)}{\alpha(t)}.$$

Theorem 2.1-Assume that (1.3) holds, $\gamma\le 1,\ \beta(t)>0.$ Further, there exists a positive function $\eta\in C^1_{rd}(\mathbb{T},\mathbb{R})$. *If for* $T\ge$ sufficiently large, such that

$$\limsup_{t\to\infty}\int_T^t\left[\eta(s)p(s)-\frac{r(s)((\eta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\eta(s))^{\gamma}}\right]\Delta s=\infty, \qquad (1)$$

and

$$\limsup_{t\to\infty}\int_T^t\left[p(s)R^{\gamma\sigma}(s)-(\frac{\gamma}{\gamma+1})^{\gamma+1}\frac{1}{R^{\sigma}(s)\beta^{\gamma^2}(s)(r(s))^{\frac{1}{\gamma}}}\right]\Delta s=\infty, \qquad (2)$$

then (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t)>0$ for $t\ge t_0$. In view of (1.1),

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This research is supported by the Natural Science Foundation of China (60774004, 60904024), China Postdoctoral Science Foundation funded project (20080441126, 200902564), Shandong Postdoctoral funded project (200802018) and supported by the Natural Science Foundation of Shandong (Y2008A28, ZR2009AL003), also supported by University of Jinan Research Funds for Doctors (XBS0843).
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we get

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} = -p(t)(x(t))^{\gamma} < 0, \quad t \geq t_0. \tag{3}$$

Therefore, $r(t)(x^{\Delta}(t))^{\gamma}$ is an eventually strictly decreasing function, and there exists $t_1 \geq t_0$ such that $x^{\Delta}(t) > 0, t \geq t_1$ or $x^{\Delta}(t) < 0, t \geq t_1$.

Case 1- $x^{\Delta}(t) > 0, t \geq t_1$. Define

$$\omega(t) = \eta(t)\frac{r(t)(x^{\Delta}(t))^{\gamma}}{(x(t))^{\gamma}}, \quad t \geq t_1, \tag{4}$$

then $\omega(t) > 0$. By the product rule and then the quotient rule, we have

$$\omega^{\Delta}(t) = (r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}(\frac{\eta(t)}{(x(t))^{\gamma}})^{\Delta} + \frac{\eta(t)[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta}}{(x(t))^{\gamma}}$$

$$= (r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}\frac{\eta^{\Delta}(t)(x(t))^{\gamma} - \eta(t)[(x(t))^{\gamma}]^{\Delta}}{(x(t))^{\gamma}(x(\sigma(t)))^{\gamma}} + \frac{\eta(t)[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta}}{(x(t))^{\gamma}}.$$

From (2.4) and (2.3), we obtain

$$\omega^{\Delta}(t) = \frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \eta(t)p(t) - \frac{\eta(t)(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}[(x(t))^{\gamma}]^{\Delta}}{(x(t))^{\gamma}(x(\sigma(t)))^{\gamma}}. \tag{5}$$

In view of Keller's chain rule, we find

$$[(x(t))^{\gamma}]^{\Delta} \geq \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t). \tag{6}$$

It follows from (2.5) and (2.6) that

$$\omega^{\Delta}(t) \leq \frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \eta(t)p(t) - \gamma\frac{\eta(t)(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{x^{\gamma}(t)x(\sigma(t))}. \tag{7}$$

From (2.3), we have that

$$x^{\Delta}(t) \geq (\frac{r^{\sigma}(t)}{r(t)})^{\frac{1}{\gamma}}x^{\Delta}(\sigma(t)). \tag{8}$$

Note that $x^{\Delta}(t) > 0$, we obtain $x(t) \leq x(\sigma(t))$. Thus, by (2.7), (2.8) and (2.4), we get

$$\omega^{\Delta}(t) \leq -\eta(t)p(t) + \frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \gamma\frac{\eta(t)}{\eta^{\sigma}(t)}(r(t))^{-\frac{1}{\gamma}}(\eta^{\sigma}(t))^{-\frac{1}{\gamma}}(\omega^{\sigma}(t))^{\frac{\gamma+1}{\gamma}}. \tag{9}$$

Set

$$\lambda = \frac{\gamma+1}{\gamma}, \quad X = \left(\gamma\frac{\eta(t)}{\eta^{\sigma}(t)}\right)^{\frac{\gamma}{\gamma+1}}\frac{\omega^{\sigma}(t)}{(r(t)\eta^{\sigma}(t))^{\frac{1}{\gamma+1}}}, \quad Y = (\frac{\gamma}{\gamma+1})^{\gamma}\left(\frac{r(t)\eta^{\sigma}(t)}{(\gamma\frac{\eta(t)}{\eta^{\sigma}(t)})^{\gamma}}\right)^{\frac{\gamma}{\gamma+1}}\left(\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}\right)^{\gamma}$$

Using the inequality

$$X^{\lambda} - \lambda XY^{\lambda-1} + (\lambda-1)Y^{\lambda} \geq 0, \quad \lambda > 1,$$

we get

$$\frac{(\eta^{\Delta}(t))_{+}}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \gamma\frac{\eta(t)}{\eta^{\sigma}(t)}(r(t))^{-\frac{1}{\gamma}}(\eta^{\sigma}(t))^{-\frac{1}{\gamma}}(\omega^{\sigma}(t))^{\frac{\gamma+1}{\gamma}} \leq \frac{r(t)((\eta^{\Delta}(t))_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\eta(t))^{\gamma}}$$

By (2.9), it is easy to see that

$$\omega^{\Delta}(t) \leq -\eta(t)p(t) + \frac{r(t)((\eta^{\Delta}(t))_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\eta(t))^{\gamma}}. \tag{10}$$

Integrating (2.10) from t1 to t; we have

$$0 < \omega(t) \leq \omega(t_1) - \int_{t_1}^{t}\left[\eta(s)p(s) - \frac{r(s)((\eta^{\Delta}(s))_{+})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\eta(s))^{\gamma}}\right]\Delta s,$$

which is a contradiction to (2.1).

Case2 $x^{\Delta}(t) < 0, t \geq t_1$. Define

$$\omega(t) = \frac{r(t)(x^{\Delta}(t))^{\gamma}}{(x(t))^{\gamma}}. \tag{11}$$

Then $\omega(t) < 0$ for $t \geq t_1$. In view of (2.3), $r(t)(x^{\Delta}(t))^{\gamma}$ is an eventually strictly decreasing function, hence

$$(r(s))^{\frac{1}{\gamma}}x^{\Delta}(s) \leq (r(t))^{\frac{1}{\gamma}}x^{\Delta}(t), \quad s \geq t.$$

Integrating it from t to τ, we have

$$x(\tau) \leq x(t) + (r(t))^{\frac{1}{\gamma}}x^{\Delta}(t)\int_{t}^{\tau}\frac{\Delta s}{(r(s))^{\frac{1}{\gamma}}}, \quad \tau \geq t.$$

Taking $\tau \to \infty$ in the above inequality, we get

$$x(t) + (r(t))^{\frac{1}{\gamma}}x^{\Delta}(t)R(t) \geq 0, \quad t \geq t_1.$$

Thus

$$(r(t))^{\frac{1}{\gamma}}R(t)\frac{x^{\Delta}(t)}{x(t)} \geq -1. \tag{12}$$

By the above inequality and (2.11), we have

$$-1 \leq R^{\gamma}(t)\omega(t) \leq 0. \tag{13}$$

Δ-differentiating (2.11) and by (2.3), we obtain

$$\omega^{\Delta}(t) = -p(t) - \frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}[(x(t))^{\gamma}]^{\Delta}}{(x(t))^{\gamma}(x(\sigma(t)))^{\gamma}}.$$

In view of Keller's chain rule, we see that

$$[(x(t))^{\gamma}]^{\Delta} \leq \gamma(x(t))^{\gamma-1}x^{\Delta}(t).$$

Thus

$$\omega^{\Delta}(t) \leq -p(t) - \gamma\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{x(t)x^{\gamma}(\sigma(t))}.$$

On the other hand, from $x^{\Delta}(t) < 0$, we have $x(t) \geq x^{\sigma}(t)$, and

$$-\gamma\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{x(t)x^{\gamma}(\sigma(t))} \leq -\gamma\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{(x(t))^{\gamma+1}} = -\gamma\left(\frac{\omega(t)}{r(t)}\right)^{\frac{1}{\gamma}}\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}}{(x(t))^{\gamma}}$$

$$= \gamma\left(\frac{-\omega(t)}{r(t)}\right)^{\frac{1}{\gamma}}\frac{(x^{\sigma}(t))^{\gamma}}{(x(t))^{\gamma}}\omega^{\sigma}(t). \tag{14}$$

From (2.14), we find that $\omega^{\Delta}(t) < 0$, then $\omega^{\sigma}(t) \leq \omega(t)$. Hence, we get

$$-\gamma\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{x(t)x^{\gamma}(\sigma(t))} \leq \gamma\left(\frac{-\omega(t)}{r(t)}\right)^{\frac{1}{\gamma}}\frac{(x^{\sigma}(t))^{\gamma}}{(x(t))^{\gamma}}\omega^{\sigma}(t) \leq -\gamma\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\omega^{\frac{\gamma+1}{\gamma}}(t)\frac{(x^{\sigma}(t))^{\gamma}}{(x(t))^{\gamma}}.$$

Noting that

$$\frac{x^{\sigma}(t)}{x(t)} = \frac{x(t) + \mu(t)x^{\Delta}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}(t)}{x(t)},$$

by (2.12), we find that

$$\frac{x^{\sigma}(t)}{x(t)} \geq 1 - \frac{\mu(t)}{(r(t))^{\frac{1}{\gamma}}R(t)} = \frac{\alpha(t) - \mu(t)}{\alpha(t)}.$$

Hence, by (2.14), we have

$$\omega^{\Delta}(t) + p(t) + \gamma(r(t))^{-\frac{1}{\gamma}}\beta^{\gamma}(t)(\omega(t))^{\frac{\gamma+1}{\gamma}} \leq 0, \quad t \geq t_1. \tag{15}$$

Multiplying (2.15) by $R^{\gamma\sigma}(t)$, we obtain

$$R^{\gamma\sigma}(t)\omega^{\Delta}(t)+p(t)R^{\gamma\sigma}(t)+\gamma R^{\gamma\sigma}(t)(r(t))^{-\frac{1}{\gamma}}\beta^{\gamma}(t)(\omega(t))^{\frac{\gamma+1}{\gamma}}\le 0,\quad t\ge t_1.$$

Integrating it from t1 to t; we get

$$\int_{t_1}^{t}R^{\gamma\sigma}(s)\omega^{\Delta}(s)\Delta s+\int_{t_1}^{t}p(s)R^{\gamma\sigma}(s)\Delta s+\gamma\int_{t_1}^{t}R^{\gamma\sigma}(s)(r(s))^{-\frac{1}{\gamma}}\beta^{\gamma}(s)(\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s\le 0. \tag{16}$$

Integrating by parts, we have

$$\int_{t_1}^{t}R^{\gamma\sigma}(s)\omega^{\Delta}(s)\Delta s=R^{\gamma}(t)\omega(t)-R^{\gamma}(t_1)\omega(t_1)-\int_{t_1}^{t}[R^{\gamma}(s)]^{\Delta}\omega(s)\Delta s. \tag{17}$$

In view of Keller's chain rule, we obtain

$$[R^{\gamma}(t)]^{\Delta}=\gamma\int_0^1[hR^{\sigma}(t)+(1-h)R(t)]^{\gamma-1}\mathrm{d}hR^{\Delta}(t),$$

note that $R^{\Delta}(t)=-(1/r(t))^{\frac{1}{\gamma}}<0$, we have

$$-\int_{t_1}^{t}[R^{\gamma}(s)]^{\Delta}\omega(s)\Delta s\ge\gamma\int_{t_1}^{t}(\frac{1}{r(s)})^{\frac{1}{\gamma}}(R^{\sigma}(s))^{\gamma-1}\omega(s)\Delta s. \tag{18}$$

By (2.18), (2.17) and (2.16), we obtain that

$$R^{\gamma}(t)\omega(t)-R^{\gamma}(t_1)\omega(t_1)+\int_{t_1}^{t}p(s)R^{\gamma\sigma}(s)\Delta s+\gamma\int_{t_1}^{t}(\frac{1}{r(s)})^{\frac{1}{\gamma}}(R^{\sigma}(s))^{\gamma-1}\omega(s)\Delta s$$
$$+\gamma\int_{t_1}^{t}R^{\gamma\sigma}(s)(r(s))^{-\frac{1}{\gamma}}\beta^{\gamma}(s)(\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s\le 0. \tag{19}$$

Define $p=(\gamma+1)/\gamma,\ q=\gamma+1$ and

$$A=-(\gamma+1)^{\frac{\gamma}{\gamma+1}}\beta^{\frac{\gamma^2}{\gamma+1}}(t)\left(\frac{R^{\gamma\sigma}(t)}{(r(t))^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\gamma+1}}\omega(t),\quad B=\frac{\gamma}{\gamma+1}(\gamma+1)^{\frac{1}{\gamma+1}}\left(\frac{1}{(r(t))^{\frac{1}{\gamma}}}\right)^{\frac{1}{\gamma+1}}\frac{1}{(R^{\sigma}(t))^{\frac{1}{\gamma+1}}}\frac{1}{\beta^{\frac{\gamma^2}{\gamma+1}}(t)}.$$

Then by the inequality

$$\frac{A^p}{p}+\frac{B^q}{q}\ge AB,\quad \frac{1}{p}+\frac{1}{q}=1,$$

we have

$$\gamma R^{\gamma\sigma}(t)(r(t))^{-\frac{1}{\gamma}}\beta^{\gamma}(t)(\omega(t))^{\frac{\gamma+1}{\gamma}}$$
$$+(\frac{\gamma}{\gamma+1})^{\gamma+1}\frac{1}{\beta^{\gamma^2}(t)R^{\sigma}(t)(r(t))^{\frac{1}{\gamma}}}\ge-\gamma(\frac{1}{r(t)})^{\frac{1}{\gamma}}(R^{\sigma}(t))^{\gamma-1}\omega(t). \tag{20}$$

Thus, by (2.20) and (2.19), we have

$$R^{\gamma}(t)\omega(t)-R^{\gamma}(t_1)\omega(t_1)+\int_{t_1}^{t}\left[p(s)R^{\gamma\sigma}(s)-(\frac{\gamma}{\gamma+1})^{\gamma+1}\frac{1}{R^{\sigma}(s)\beta^{\gamma^2}(s)(r(s))^{\frac{1}{\gamma}}}\right]\Delta s\le 0.$$

Therefore, by (2.2), we have

$$R^{\gamma}(t)\omega(t)\to-\infty,\quad t\to\infty,$$

which contradicts (2.13). This completes the proof.

Theorem 2.2-Assume that (1.3) holds, $\gamma\ge 1,\ \beta(t)>0.$ Further, there exists a positive function $\eta\in C^1_{rd}(\mathbb{T},\mathbb{R})$. *If for* $T\ge t_0$, sufficiently large, such that (2.1) holds and

$$\limsup_{t\to\infty}\int_T^t\left[p(s)R^{\gamma\sigma}(s)-(\frac{\gamma}{\gamma+1})^{\gamma+1}\frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2}\beta^{\gamma^2}(s)(r(s))^{\frac{1}{\gamma}}}\right]\Delta s=\infty, \tag{21}$$

then (1.1) is oscillatory.

Proof-Let x be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t)>0$ for $t\ge t_0$. In view of (1.1), we obtain (2.3). Therefore, $r(t)(x^{\Delta}(t))^{\gamma}$ is an eventually strictly decreasing function, and there exists a $t_1\ge t_0$ such that $x^{\Delta}(t)>0,\ t\ge t_1$ or $x^{\Delta}(t)<0,\ t\ge t_1$.

Case 1 $x^{\Delta}(t)>0,\ t\ge t_1$. Define ω as (2.4). Proceeding as in the proof of Case 1 in Theorem 2.1, we can obtain a contradiction to (2.1).

Case 2- $x^{\Delta}(t)<0,\ t>t_1$. Define ω as (2.11), we have that (2.12) and (2.13) hold. Δ-differentiating (2.11) and by (2.3), we obtain

$$\omega^{\Delta}(t)=-p(t)-\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}[(x(t))^{\gamma}]^{\Delta}}{(x(t))^{\gamma}(x(\sigma(t)))^{\gamma}}.$$

In view of Keller's chain rule, we get

$$[(x(t))^{\gamma}]^{\Delta}\le\gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t).$$

Thus

$$\omega^{\Delta}(t)\le-p(t)-\gamma\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{(x(t))^{\gamma}x(\sigma(t))}. \tag{22}$$

On the other hand, from $x^{\Delta}(t)<0$, we have $x(t)\ge x^{\sigma}(t)$,

$$-\gamma\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{(x(t))^{\gamma}x(\sigma(t))}\le-\gamma\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{(x(t))^{\gamma+1}}=-\gamma\left(\frac{\omega(t)}{r(t)}\right)^{\frac{1}{\gamma}}\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}}{(x(t))^{\gamma}}$$
$$=\gamma\left(\frac{-\omega(t)}{r(t)}\right)^{\frac{1}{\gamma}}\frac{(x^{\sigma}(t))^{\gamma}}{(x(t))^{\gamma}}\omega^{\sigma}(t).$$

From (2.22), we have $\omega^{\Delta}(t)<0$, then $\omega^{\sigma}(t)\le\omega(t)$. So, we get

$$-\gamma\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}x^{\Delta}(t)}{(x(t))^{\gamma}x(\sigma(t))}\le\gamma\left(\frac{-\omega(t)}{r(t)}\right)^{\frac{1}{\gamma}}\frac{(x^{\sigma}(t))^{\gamma}}{(x(t))^{\gamma}}\omega^{\sigma}(t)\le-\gamma\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\omega^{\frac{\gamma+1}{\gamma}}(t)\frac{(x^{\sigma}(t))^{\gamma}}{(x(t))^{\gamma}}.$$

Noting that

$$\frac{x^{\sigma}(t)}{x(t)}=\frac{x(t)+\mu(t)x^{\Delta}(t)}{x(t)}=1+\mu(t)\frac{x^{\Delta}(t)}{x(t)},$$

by (2.12), we have

$$\frac{x^{\sigma}(t)}{x(t)}\ge 1-\frac{\mu(t)}{(r(t))^{\frac{1}{\gamma}}R(t)}=\frac{\alpha(t)-\mu(t)}{\alpha(t)}.$$

Hence, by (2.22), we get (2.15), then we obtain that (2.16) and (2.17) hold. In view of Keller's chain rule, we have

$$[R^{\gamma}(t)]^{\Delta}=\gamma\int_0^1[hR^{\sigma}(t)+(1-h)R(t)]^{\gamma-1}\mathrm{d}hR^{\Delta}(t),$$

note that $R^{\Delta}(t)=-(1/r(t))^{\frac{1}{\gamma}}<0$, we see that

$$-\int_{t_1}^{t}[R^{\gamma}(s)]^{\Delta}\omega(s)\Delta s\ge\gamma\int_{t_1}^{t}(\frac{1}{r(s)})^{\frac{1}{\gamma}}(R(s))^{\gamma-1}\omega(s)\Delta s. \tag{23}$$

From (2.23), (2.17) and (2.16), we obtain that

$$R^{\gamma}(t)\omega(t)-R^{\gamma}(t_1)\omega(t_1)+\int_{t_1}^{t}p(s)R^{\gamma\sigma}(s)\Delta s+\gamma\int_{t_1}^{t}(\frac{1}{r(s)})^{\frac{1}{\gamma}}(R(s))^{\gamma-1}\omega(s)\Delta s$$
$$+\gamma\int_{t_1}^{t}R^{\gamma\sigma}(s)(r(s))^{-\frac{1}{\gamma}}\beta^{\gamma}(s)(\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s\le 0. \tag{24}$$

Define $p=(\gamma+1)/\gamma$, $q=\gamma+1$ and

$$A=-(\gamma+1)^{\frac{\gamma}{\gamma+1}}\beta^{\frac{\gamma^2}{\gamma+1}}(t)\left(\frac{R^{\gamma\sigma}(t)}{(r(t))^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\gamma+1}}\omega(t),\quad B=\frac{\gamma}{\gamma+1}(\gamma+1)^{\frac{1}{\gamma+1}}\left(\frac{1}{(r(t))^{\frac{1}{\gamma}}}\right)^{\frac{1}{\gamma+1}}\frac{R^{\gamma-1}(t)}{(R^\sigma(t))^{\frac{\gamma^2}{\gamma+1}}}\frac{1}{\beta^{\frac{\gamma^2}{\gamma+1}}(t)}.$$

Then by the inequality

$$\frac{A^p}{p}+\frac{B^q}{q}\geq AB,\quad \frac{1}{p}+\frac{1}{q}=1,$$

we have

$$\gamma R^{\gamma\sigma}(t)(r(t))^{-\frac{1}{\gamma}}\beta^\gamma(t)(\omega(t))^{\frac{\gamma+1}{\gamma}}$$
$$+(\frac{\gamma}{\gamma+1})^{\gamma+1}\frac{R^{\gamma^2-1}(t)}{\beta^{\gamma^2}(t)(R^\sigma(t))^{\gamma^2}(r(t))^{\frac{1}{\gamma}}}\geq-\gamma(\frac{1}{r(t)})^{\frac{1}{\gamma}}(R(t))^{\gamma-1}\omega(t). \quad (25)$$

Thus, by (2.25) and (2.24), we have

$$R^\gamma(t)\omega(t)-R^\gamma(t_1)\omega(t_1)+\int_{t_1}^t p(s)R^{\gamma\sigma}(s)\Delta s+\gamma\int_{t_1}^t(\frac{1}{r(s)})^{\frac{1}{\gamma}}(R(s))^{\gamma-1}\omega(s)\Delta s$$
$$+\gamma\int_{t_1}^t R^{\gamma\sigma}(s)(r(s))^{-\frac{1}{\gamma}}\beta^\gamma(s)(\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s\leq 0.$$

It follows from (2.21) that

$$R^\gamma(t)\omega(t)\to-\infty,\quad t\to\infty,$$

which contradicts (2.13). This completes the proof.

Theorem 2.3-Assume that (1.3) holds, $\gamma>0,\ \beta(t)>0.$ Further, there exists a positive function $\eta\in C^1_{rd}(\mathbb{T},\mathbb{R})$. *If for* $T\geq t_0,$ sufficiently large, such that (2.1) holds, and

$$\limsup_{t\to\infty}\int_T^t p(s)R^{\gamma+1}(\sigma(s))\Delta s=\infty, \quad (26)$$

then (1.1) is oscillatory.

Proof-Let x be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t)>0$ for $t\geq t_0$.

As in the proof of Theorem 2.1 or Theorem 2.2, we consider two cases.

Case 1- $x^\Delta(t)>0,\ t\geq t_1.$ By (2.1), this case is not true.

Case 2-When $\gamma\leq 1,$ proceeding as in the proof of Case 2 of Theorem 2.1, we have that (2.15) holds. Multiplying (2.15) by $R^{\gamma+1}(\sigma(t)),$ and integrating it from t1 to t; we get

$$\int_{t_1}^t R^{\gamma+1}(\sigma(s))\omega^\Delta(s)\Delta s+\int_{t_1}^t p(s)R^{\gamma+1}(\sigma(s))\Delta s+\gamma\int_{t_1}^t R^{\gamma+1}(\sigma(s))(r(s))^{-\frac{1}{\gamma}}\beta^\gamma(s)(\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s\leq 0.$$

Integrating by parts, we see that

$$\int_{t_1}^t R^{\gamma+1}(\sigma(s))\omega^\Delta(s)\Delta s=R^{\gamma+1}(t)\omega(t)-R^{\gamma+1}(t_1)\omega(t_1)-\int_{t_1}^t[R^{\gamma+1}(s)]^\Delta\omega(s)\Delta s,$$

in view of Keller's chain rule, we obtain

$$[R^{\gamma+1}(t)]^\Delta=(\gamma+1)\int_0^1[hR^\sigma(t)+(1-h)R(t)]^\gamma dhR^\Delta(t),$$

note that $R^\Delta(t)=-(1/r(t))^{\frac{1}{\gamma}}<0,$

$$-\int_{t_1}^t[R^{\gamma+1}(s)]^\Delta\omega(s)\Delta s\geq(\gamma+1)\int_{t_1}^t(\frac{1}{r(s)})^{\frac{1}{\gamma}}(R(s))^\gamma\omega(s)\Delta s.$$

Thus, from (2.27), we get

$$R^{\gamma+1}(t)\omega(t)-R^{\gamma+1}(t_1)\omega(t_1)+\int_{t_1}^t p(s)R^{\gamma+1}(\sigma(s))\Delta s+(\gamma+1)\int_{t_1}^t(\frac{1}{r(s)})^{\frac{1}{\gamma}}(R(s))^\gamma\omega(s)\Delta s$$
$$+\gamma\int_{t_1}^t R^{\gamma+1}(\sigma(s))(r(s))^{-\frac{1}{\gamma}}\beta^\gamma(s)(\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s\leq 0. \quad (28)$$

By (2.13), we find that $-R^\gamma(t)\omega(t)\leq 1,$ then

$$-R^{\gamma+1}(t)\omega(t)\leq R(t)<\infty,\quad t\to\infty,$$

and

$$-\int_{t_1}^\infty(\frac{1}{r(s)})^{\frac{1}{\gamma}}(R(s))^\gamma\omega(s)\Delta s\leq\int_{t_1}^\infty(\frac{1}{r(s)})^{\frac{1}{\gamma}}\Delta s<\infty.$$

Noting that $\beta(t)\leq 1,\ R^\sigma(t)/R(t)\leq 1,$ we obtain

$$\int_{t_1}^\infty R^{\gamma+1}(\sigma(s))(r(s))^{-\frac{1}{\gamma}}\beta^\gamma(s)(\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s=\int_{t_1}^\infty(r(s))^{-\frac{1}{\gamma}}\beta^\gamma(s)(\frac{R^\sigma(s)}{R(s)})^{\gamma+1}(R^\gamma(s)\omega(s))^{\frac{\gamma+1}{\gamma}}\Delta s$$
$$\leq\int_{t_1}^\infty(r(s))^{-\frac{1}{\gamma}}\Delta s<\infty.$$

Thus, from (2.28), we get

$$\int_{t_1}^\infty p(s)R^{\gamma+1}(\sigma(s))\Delta s<\infty,$$

which contradicts (2.26).

When $\gamma\geq 1,$ the proof is similar to the case $\gamma\leq 1,$ so we omit it. This completes the proof.

III. Applications and Examples

Agarwal et al. [5], Grace et al. [8], Hassan [10], Saker [14] considered Eq. (1.1), and established some oscillation criteria for (1.1). We introduce some results as follows.

Theorem 3.1-(Saker [14]) Assume (1.3) holds and $\gamma>1.$ Furthermore, assume that there exists a positive function $\eta\in C^1_{rd}(\mathbb{T},\mathbb{R})$ such that (2.1) holds, and

$$\int_{t_0}^{\infty}\left[\frac{1}{r(t)}\int_{t_0}^{t}p(s)\Delta s\right]^{\frac{1}{\gamma}}\Delta t=\infty. \quad (1)$$

Then every solution of (1.1) oscillates or converges to zero.

Theorem 3.2-(Hassan [10]) Assume (1.3) and (3.1) hold, $\gamma \le 1$, Furthermore, assume that there exists a positive function $\eta \in C^1_{rd}(\mathbb{T},\mathbb{R})$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{\infty}\left[\alpha^{\gamma}(s)\eta^{\sigma}(s)p(s)-\frac{r(s)((\eta^{\Delta}(s))_+)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\alpha(s)\eta^{\sigma}(s))^{\gamma}}\right]\Delta s=\infty, \quad (2)$$

where

$$\alpha(t):=\frac{R(t)}{R(t)+\mu(t)},\ R(t):=r^{\frac{1}{\gamma}}(t)\int_{t_0}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

Then every solution of (1.1) oscillates or converges to zero.

In the following, we shall give some examples to illustrate the main results.

Example 3.1 -Consider the second-order sub-linear dynamic equation

$$((t\sigma(t))^{\gamma})^{\Delta}+\frac{\sigma^{\gamma^2+\gamma}(t)}{t^{\gamma^2+1}}x^{\gamma}(t)=0,\quad t\in[1,\infty)_{\mathbb{T}}, \quad (3)$$

where

$$\alpha(t):=\frac{R(t)}{R(t)+\mu(t)},\ R(t):=r^{\frac{1}{\gamma}}(t)\int_{t_0}^{t}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

We see that

$$R(t)=\int_t^{\infty}\frac{1}{s\sigma(s)}\Delta s=\frac{1}{t},\ \alpha(t)-\mu(t)=t\sigma(t)R(t)-\mu(t)=t,\ \beta(t)=\frac{t}{\sigma(t)}.$$

Let $\eta(t)=1$: We get that (2.1) holds. On the other hand

$$\limsup_{t\to\infty}\int_T^t\left[p(s)R^{\gamma\sigma}(s)-(\frac{\gamma}{\gamma+1})^{\gamma+1}\frac{1}{R^{\sigma}(s)\beta^{\gamma^2}(s)(r(s))^{\frac{1}{\gamma}}}\right]\Delta s$$

$$=\limsup_{t\to\infty}\int_T^t\left(1-(\frac{\gamma}{\gamma+1})^{\gamma+1}\right)\frac{\sigma^{\gamma^2}(s)}{s^{\gamma^2+1}}\Delta s=\infty.$$

Hence, from Theorem 2.1, every solution

Example 3.2-Consider the second-order super-linear dynamic equation

$$((t\sigma(t))^{\gamma})^{\Delta}+\frac{(\sigma(t))^{2\gamma^2+\gamma-1}}{t^{2\gamma^2}}x^{\gamma}(t)=0,\quad t\in[1,\infty)_{\mathbb{T}}, \quad (4)$$

where

$$\gamma\ge 1,\ r(t)=(t\sigma(t))^{\gamma},\ p(t)=\frac{(\sigma(t))^{2\gamma^2+\gamma-1}}{t^{2\gamma^2}}.$$

We find

$$R(t)=\int_t^{\infty}\frac{1}{s\sigma(s)}\Delta s=\frac{1}{t},\ \alpha(t)-\mu(t)=t\sigma(t)R(t)-\mu(t)=t,\ \beta(t)=\frac{t}{\sigma(t)}.$$

Let $\eta(t)=1$.We get that (2.1) holds. On the other hand,

$$\limsup_{t\to\infty}\int_T^t\left[p(s)R^{\gamma\sigma}(s)-(\frac{\gamma}{\gamma+1})^{\gamma+1}\frac{R^{\gamma^2-1}(s)}{(R^{\sigma}(s))^{\gamma^2}\beta^{\gamma^2}(s)(r(s))^{\frac{1}{\gamma}}}\right]\Delta s$$

$$=\left(1-(\frac{\gamma}{\gamma+1})^{\gamma+1}\right)\limsup_{t\to\infty}\int_T^t\frac{\Delta s}{s^{2\gamma^2}(\sigma(s))^{1-2\gamma^2}}\ge\left(1-(\frac{\gamma}{\gamma+1})^{\gamma+1}\right)\limsup_{t\to\infty}\int_T^t\frac{\Delta s}{s}=\infty.$$

Hence, from Theorem 2.2, every solution x of (3.4) oscillates.

Example 3.3-Consider the second-order sub-linear dynamic equation

$$\left(\frac{(t\sigma(t))^{\gamma^2}}{((t^{\gamma})^{\Delta})^{\gamma}}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{\sigma^{\gamma^2+\gamma}(t)}{t}x^{\gamma}(t)=0,\quad t\in[1,\infty)_{\mathbb{T}}, \quad (5)$$

where

$$\gamma\le 1,\ r(t)=\frac{(t\sigma(t))^{\gamma^2}}{((t^{\gamma})^{\Delta})^{\gamma}},\ p(t)=\frac{\sigma^{\gamma^2+\gamma}(t)}{t},\ \frac{\sigma^{\gamma}(t)}{\gamma t^{\gamma-1}}-\mu(t)>0.$$

We have

$$R(t)=\int_t^{\infty}(\frac{-1}{s^{\gamma}})^{\Delta}\Delta s=\frac{1}{t^{\gamma}},\ \alpha(t)-\mu(t)=\frac{(t\sigma(t))^{\gamma}}{(t^{\gamma})^{\Delta}}R(t)-\mu(t)=\frac{\sigma^{\gamma}(t)}{(t^{\gamma})^{\Delta}}-\mu(t)\ge\frac{\sigma^{\gamma}(t)}{\gamma t^{\gamma-1}}-\mu(t).$$

Thus, $\beta(t)>0$. Let $\eta(t)=1$. We get that (2.1) holds. On the other hand,

Example 3.4-Consider the second-order super-linear dynamic equation.

$$\left(\frac{(t\sigma(t))^{\gamma^2}}{((t^{\gamma})^{\Delta})^{\gamma}}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{\sigma^{\gamma^2+\gamma}(t)}{t}x^{\gamma}(t)=0,\quad t\in[1,\infty)_{\mathbb{T}}, \quad (6)$$

where

$$\gamma\ge 1, r(t)=\frac{(t\sigma(t))^{\gamma^2}}{((t^{\gamma})^{\Delta})^{\gamma}},\ p(t)=\frac{\sigma^{\gamma^2+\gamma}(t)}{t},\ \frac{\sigma(t)}{\gamma}-\mu(t)>0.$$

We obtain

$$R(t)=\int_t^{\infty}(\frac{-1}{s^{\gamma}})^{\Delta}\Delta s=\frac{1}{t^{\gamma}},\ \alpha(t)-\mu(t)=\frac{(t\sigma(t))^{\gamma}}{(t^{\gamma})^{\Delta}}R(t)-\mu(t)=\frac{\sigma^{\gamma}(t)}{(t^{\gamma})^{\Delta}}-\mu(t)\ge\frac{\sigma(t)}{\gamma}-\mu(t).$$

Hence, $\beta(t)>0$. Let $\eta(t)=1$. We have that (2.1) holds. On the other hand,

$$\limsup_{t\to\infty}\int_T^t p(s)R^{\gamma+1}(\sigma(s))\Delta s=\limsup_{t\to\infty}\int_T^t\frac{\Delta s}{s}=\infty.$$

Therefore, by Theorem 2.3, every solution x of (3.6) oscillates.

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Subclass Of K−Uniformly Starlike Functions Associated With Wright Generalized Hypergeometric Functions

GJSFR Classification – F (FOR) 010111,010106,010204

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Abstract- **In this paper we consider the class of functions of the form**

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \ (a_n \geq 0),$$

that are satisfying the condition

$$(1-\mu)\frac{f(z_0)}{z_0} + \mu f'(z_0) = 1, \ (0 \leq \mu \leq 1; \ -1 < z_0 < 1 \text{ and } z_0 \neq 0).$$

We obtain coefficient bounds, distortion theorem and extreme points of the subclass of starlike functions defined by linear operator. Furthermore, we discuss radius of convexity and closure properties.

Keywords-Univalent, convex, starlike, uniformly convex, uniformly starlike, Linear operator.

2000 Mathematics Subject Classification: 30C45.

I. Introduction

Let S be the class of functions $f(z)$ that are analytic in the unit disc $U = \{z : |z| < 1\}$ with f(0) = 0. Denote by T, the subclass of *S* consists of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1}$$

and also denote T_1 ,the subclass of *S* consisting of functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \ (a_n \geq 0, \ a_1 > 0).$$

$$\text{where, either } f(z_0) = z_0 \ (-1 < z_0 < 1; z_0 \neq 0) \text{ or, } f'(z_0) = 1 \ (-1 < z_0 < 1). \tag{2}$$

Let T_μ be the subclass of T_1 satisfying

$$(1-\mu)\frac{f(z_0)}{z_0} + \mu f'(z_0) = 1 \text{ where } (-1 < z_0 < 1), \ 0 \leq \mu \leq 1. \tag{3}$$

Following Goodman [9, 10] , Rønning[15] defined two subclasses of S,

(i)for the functions f in S is said to be k− uniformly starlike functions of order if γ if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in U, \ -1 < \gamma \leq 1, \text{ and } k \geq 0$$

and (ii)for the functions f in S is said to be k− uniformly convex functions of order γ if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|, \ z \in U, \ -1 < \gamma \leq 1, \text{ and } k \geq 0.$$

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_p, A_p$ and $\beta_1, B_1 \ldots, \beta_q, B_q$ $(p, q \in N = 1, 2, 3, .$

that

$$1 + \sum_{n=1}^{q} B_n - \sum_{n=1}^{p} A_n \geq 0. \quad z \in U. \tag{4}$$

The Wright generalized hypergeometric function[20]

$${}_p\Psi_q[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z] = {}_p\Psi_q[(\alpha_n, A_n)_{1,p}(\beta_n, B_n)_{1,q}; z]$$

is defined by

$${}_p\Psi_q[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z] = \sum_{n=0}^{\infty}\{\prod_{t=0}^{p}\Gamma(\alpha_t + nA_t)\}\{\prod_{t=0}^{q}\Gamma(\beta_t + nB_t)\}^{-1}\frac{z^n}{n!}, z \in U.$$

If $A_t = 1(t = 1, 2, ..., p)$ and $B_t = 1(t = 1, 2, ..., q)$we have the relationship:

$$\Omega_p\Psi_q[(\alpha_t, 1)_{1,p}(\beta_t, 1)_{1,q}; z] \equiv {}_pF_q(\alpha_1, \ldots \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n}\frac{z^n}{n!} \tag{5}$$

$(p \leq q + 1; \ p, q \in N_0 = N \cup \{0\}; z \in U)$ is the generalized hypergeometric function(see for details[8]) where N denotes the set of all positive integers and $(\alpha)_n$ is the Pochhammer symbol and

$$\Omega = \left(\prod_{t=0}^{p}\Gamma(\alpha_t)\right)^{-1}\left(\prod_{t=0}^{q}\Gamma(\beta_t)\right). \tag{6}$$

By using the generalized hypergeometric function Dziok and Srivastava [8] introduced the linear operator. In[4] Dziok and Raina extended the linear operator by using

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Wright generalized hypergeometric function . First we define a function

$$ {}_p\phi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] = \Omega z_p\Psi_q[(\alpha_t, A_t)_{1,p}(\beta_t, B_t)_{1,q}; z]. $$

Let $W[(\alpha_n, A_n)_{1,p}; (\beta_n, B_n)_{1,q}] : S \to S$ be a linear operator defined by

$$ W[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}]f(z) := z\ {}_p\phi_q m[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] * f(z) $$

We observe that , for f(z) of the form(1.1),we have

$$ W[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}]f(z) = z + \sum_{n=2}^{\infty} \Omega\sigma_n(\alpha_1)\, a_n z^n \qquad (7) $$

where Ω is given by (1.6)and $\sigma_n(\alpha_1)$is defined by

$$ \sigma_n(\alpha_1) = \frac{\Gamma(\alpha_1 + A_1(n-1))\ldots\Gamma(\alpha_p + A_p(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1))\ldots\Gamma(\beta_q + B_q(n-1))}. \qquad (8) $$

If, for convenience, we write

$$ W[\alpha_1]f(z) = W[(\alpha_1, A_1),\ldots,(\alpha_l, A_p);(\beta_1, B_1),\ldots,(\beta_q, B_q)]f(z) \qquad (9) $$

introduced by Dziok and Raina[4]. In view of the relationship (1.5) the linear operator(1.7) and by setting $A_t = 1(t = 1,...,q)$ and $B_t = 1(t = 1,...,s)$ we are led immediately to the aforementioned Dziok- Srivastava operator which contains, as its further special cases, such other linear operators of Geometric Function Theory as the Hohlov operator, the Carlson-Shaffer operator[3], the Ruscheweyh derivative operator[16], the generalized Bernardi-Libera-Livingston operator[2], the fractional derivative operator[7], and so on (see, for the precise relationships,Dziok and Srivastava [4, 5, 6, 8]).

For $-1 \le \gamma < 1$, we let $W_a^p([\alpha_1], \gamma, k, z_0)$ denote the subclass of starlike functions corresponding to the family UCV for functions f(z) of the form (??) such that

$$ \mathrm{Re}\left\{\frac{z(W_q^p[\alpha_1]f(z))'}{W_q^p[\alpha_1]f(z)} - \gamma\right\} \ge k\left|\frac{z(W_q^p[\alpha_1]f(z))'}{W_q^p[\alpha_1]f(z)} - 1\right| \qquad (10) $$

For $k \ge 0$ and $-1 \le \gamma < 1$, we let

$$ TW_q^p([\alpha_1], \gamma, k, z_0) = W_q^p([\alpha_1], \gamma, k, z_0) \cap T_\mu $$

the subclass of Tμ consisting of functions of the form (1.2) and satisfying the analytic criterion(1.3).

Using the techniques of Silverman [17] and motivated by the earlier works [12, 11, 14] and [19], in this paper we obtain the coefficient bounds, distortion bounds, extreme points, radius of starlikeness and closure theorems for the functions belong to the class $TW_q^p([\alpha_1], \gamma, k, z_0)$.

II. Main Results

Theorem 1. A function f(z) of the form (1.2) is in the class $W_q^p([\alpha_1], \gamma, k, z_0)$ if and only if

$$ \sum_{n=2}^{\infty}[n(1+\beta) - (\alpha+\beta)]\Omega\sigma_n(\alpha_1)a_n \le a_1(1-\alpha), \qquad (1) $$

$-1 \le \gamma < 1, k \ge 0.$

The proof of the Theorem 1 is similar to that of Theorem 2.2, in [1], hence we omit the details.

Theorem 2-Let f(z) be defined by (1.2). Then $f(z) \in TS(\mu, \alpha, \beta, z_0)$ if and only if

$$ \sum_{n=2}^{\infty}\left\{\frac{[n(1+k) - (k+\gamma)]}{1-\alpha}\Omega\sigma_n(\alpha_1) - [(1-\mu) + n\mu]z_0^{n-1}\right\}a_n \le 1, \qquad (2) $$

$-1 \le \gamma < 1, k \ge 0.$

Proof- Since

$(1-\mu)\frac{f(z_0)}{z_0} + \mu f'(z_0) = 1,\ (0 \le \mu \le 1;\ -1 < z_0 < 1 \text{ and } z_0 \ne 0)$, we get

$$ a_1 = 1 + \sum_{n=2}^{\infty}[(1-\mu) + n\mu]a_n z_0^{n-1}. $$

Substituting for a_1 in (2.1) we get (2.2). (a)

Corollary 1-Let the function f(z) defined by (1.2) belongs $TW_q^p([\alpha_1], \gamma, k, z_0)$. *Then*

$$ a_n \le \left\{\frac{[n(1+k) - (k+\gamma)]}{1-\alpha}\Omega\sigma_n(\alpha_1) - [(1-\mu) + n\mu]z_0^{n-1}\right\}^{-1} $$

$n \ge 2,\ -1 \le \alpha < 1,\ \beta \ge 0$ with equality for

$$ f(z) = \frac{[n(1+k) - (k+\gamma)]\Omega\sigma_n(\alpha_1)z - (1-\gamma)z^n}{[n(1+k) - (k+\gamma)]\Omega\sigma_n(\alpha_1) - (1-\gamma)[(1-\mu) + n\mu]z_0^{n-1}} \qquad (3) $$

Theorem 3-Let

$$ f_1(z) = z \text{ and } f_n(z) = \frac{[n(1+k) - (k+\gamma)]\Omega\sigma_n(\alpha_1)z - (1-\gamma)z^n}{[n(1+k) - (k+\gamma)]\Omega\sigma_n(\alpha_1) - (1-\gamma)[(1-\mu) + n\mu]z_0^{n-1}}\ (n \ge 2). \qquad (4) $$

Then $f(z) \in TW_q^p([\alpha_1], \gamma, k, z_0)$ if and only if it can be expressed in the form

$$ f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z),\ \text{ where } \lambda_n \ge 0 \text{ and } \sum_{n=1}^{\infty}\lambda_n = 1. $$

Proof-The proof of the Theorem 3, follows on line similar to the proof of the theorem on extreme points given in Silverman [17] (b)

III. A Distortion Theorem

Theorem 4-Let the function f(z) defined by (1.2) belong to $TW_q^p([\alpha_1], \gamma, k, z_0)$.

$$ |f(z)| \ge a_1|z|\left\{1 - \frac{(1-\alpha)}{\Omega\sigma_2(\alpha_1)(2+k+\gamma)}|z|\right\} \qquad (1) $$

$$ |f(z)| \le a_1|z|\left\{1 + \frac{(1-\alpha)}{\Omega\sigma_2(\alpha_1)(2-k-\gamma)}|z|\right\} \qquad (2) $$

for $z \in U$.

Proof-In the view of (2.1) and the fact that $\Omega\sigma_n(\alpha_1)$ is non-decreasing for $n \geq 2$, we have

$$\Omega\sigma_2(\alpha_1)(2-\alpha+\beta)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty}[n(1+k)-(k+\gamma)]\Omega\sigma_n(\alpha_1)a_n \leq a_1(1-\gamma) \quad (3)$$

which is equivalent to,

$$\sum_{n=2}^{\infty} a_n \leq \frac{a_1(1-\gamma)}{\Omega\sigma_2(\alpha_1)(2+k-\gamma)}. \quad (4)$$

Using (1.2) and (3.4), we obtain

$$\begin{aligned}|f(z)| &\geq a_1|z| - |z|^2\sum_{n=2}^{\infty} a_n \\ &\geq a_1|z| - |z|^2\frac{a_1(1-\gamma)}{\Omega\sigma_2(\alpha_1)(2+k-\gamma)} \\ &\geq a_1|z|\left\{1-\frac{(1-\gamma)}{\Omega\sigma_2(\alpha_1)(2+k-\gamma)}|z|\right\}\end{aligned}$$

and

$$|f(z)| \leq a_1|z|\left\{1+\frac{(1-\gamma)}{\Omega\sigma_2(\alpha_1)(2+k-\gamma)}|z|\right\}. \quad (5)$$

IV. Closure Theorems

Let the functions fj(z) be defined for j = 1, 2, . . .m by

$$f_j(z) = a_{1,j}\, z - \sum_{n=2}^{\infty} a_{n,j}\, z^n \quad (1)$$

$$a_{1,j} > 0,\ a_{n,j} \geq 0,\ z \in U.$$

Theorem 5. Let fj(z) defined by (4.1) be in the class $TW_q^p([\alpha_1],\gamma,k,z_0)$. Then the function h(z) defined by

$$h(z) = \sum_{j=1}^{m} d_j f_j(z),\quad d_j \geq 0 \quad (2)$$

also in the same class $TW_q^p([\alpha_1],\gamma,k,z_0)$, where

$$\sum_{j=1}^{m} d_j = 1. \quad (3)$$

Proof-From (4.2) we have

$$h(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n \quad (4)$$

where

$b_1 = \sum_{j=1}^{m} d_j a_{1,j}$ and $b_n = \sum_{j=1}^{m} d_j a_{n,j}\ (n = 2,3,\dots)$.

Since $f_j(z) \in TW_q^p([\alpha_1],\gamma,k,z_0)\ (j = 1,2,\dots m)$ and by applying Theorem 2, we get

$$\sum_{n=2}^{\infty}\left\{\frac{[n(1+k)-(k+\gamma)]}{1-\alpha}\Omega\sigma_n(\alpha_1) - [(1-\mu)+n\mu]z_0^{n-1}\right\}a_{n,j} \leq 1,\ (j = 1,2,\dots m).$$

Therefore, we have

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{\frac{[n(1+k)-(k+\gamma)]}{1-\alpha}\Omega\sigma_n(\alpha_1) - [(1-\mu)+n\mu]z_0^{n-1}\right\}\left(\sum_{j=1}^{m} d_j a_{n,j}\right) \\ &= \sum_{j=1}^{m} d_j\left[\sum_{n=2}^{\infty}\left\{\frac{[n(1+k)-(k+\gamma)]}{1-\alpha}\Omega\sigma_n(\alpha_1) - [(1-\mu)+n\mu]z_0^{n-1}\right\}a_{n,j}\right] \\ &\leq \sum_{j=1}^{m} d_j = 1 \quad \text{(by Theorem 2 and by (4.3))}.\end{aligned}$$

Which implies that h $h(z) \in TW_q^p([\alpha_1],\gamma,k,z_0)$ and so the proof is complete.

V. Radius of Convexity and Starlikeness

In this section we obtain the radius of starlikeness of order $\delta\ (0 \leq \delta < 1)$, radius of convexity of order $\delta\ (0 \leq \delta < 1)$, for the class $TW_q^p([\alpha_1],\gamma,k,z_0)$.

Theorem 6-Let $f \in TS(\mu,\alpha,\beta,z_0)$. Then

1) f is starlike of order $\delta\ (0 \leq \delta < 1)$, in the disc $|z| < r_1$; that is, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta,\ (|z| < r_1\ ;\ 0 \leq \delta < 1)$, where

$$r_1 = \inf_{n \leq 2}\left\{\Omega\sigma_n(\alpha_1)\frac{1-\delta}{n-\delta}\frac{[n(1+k)-(k+\gamma)]}{1-\alpha}\right\}^{\frac{1}{n-1}}.$$

2) f is convex of order $\delta\ (0 \leq \delta < 1)$, in the unit disc $|z| < r_2$, that is Re $\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, $(|z| < r_2; 0 \leq \delta < 1)$, where

$$r_2 = \inf_{n \leq 2}\left\{\Omega\sigma_n(\alpha_1)\frac{(1-\delta)[n(1+k)-(k+\gamma)]}{n(n-\delta)(1-\gamma)}\right\}^{\frac{1}{n-1}}.$$

Each of these results are sharp for the extremal function f(z) given by (2.4).

Proof-Given $f \in T_1$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\delta. \quad (1)$$

For the left hand side of (5.1) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n\,|z|^{n-1}}{a_1 - \sum_{n=2}^{\infty} a_n\,|z|^{n-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}a_n\,|z|^{n-1} < a_1. \quad (2)$$

Substituting $a_1 = 1+\sum_{n=2}^{\infty}[(1-\mu)+n\mu]a_n z_0^{n-1}$ in (5.2), we have

$$\sum_{n=2}^{\infty}\left\{\frac{n-\delta}{1-\delta}|z|^{n-1}-[(1-\mu)+n\mu]z_0^{n-1}\right\}a_n \le 1 \qquad (3)$$

Using the fact, that $f \in TW_q^p([\alpha_1],\gamma,k,z_0)$ if and only if

$$\sum_{n=2}^{\infty}\left\{\frac{[n(1+k)-(k+\gamma)]}{1-\alpha}\Omega\sigma_n(\alpha_1)-[(1-\mu)+n\mu]z_0^{n-1}\right\}a_n < 1.$$

We can say (5.1) is true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1}-[(1-\mu)+n\mu]z_0^{n-1} \le \frac{[n(1+k)-(k+\gamma)]}{1-\alpha}\Omega\sigma_n(\alpha_1)-[(1-\mu)+n\mu]z_0^{n-1}.$$

Or, equivalently,

$$|z|^{n-1} < \frac{(1-\delta)[n(1+k)-(k+\gamma)]}{(n-\delta)(1-\gamma)}\Omega\sigma_n(\alpha_1)$$

which yields the starlikeness of the family. (2) Using the fact that f is convex if and only if zf0 is starlike, we can prove (2), on lines similar the proof of (1). (a)

Remark-We note that the radius of starlikeness and convexity are independent of the fixed point z0.

VI. Convex Families

Suppose B is nonempty subset of the real interval (0, 1),we define $TW_q^p([\alpha_1],\gamma,k,B)$ by

$$TW_q^p([\alpha_1],\gamma,k,B)=\bigcup_{z_i\in B} TW_q^p([\alpha_1],\gamma,k,z_i).$$

If B consists of a single element say z0 then $TW_q^p([\alpha_1],\gamma,k,z_0)$ is a convex family. Because if f1(z) and f2(z) are in $TW_q^p([\alpha_1],\gamma,k,z_0)$, then it can be seen that for $0\le\lambda\le1$, $\lambda f_1(z)+(1-\lambda)f_2(z)$ is in $TW_q^p([\alpha_1],\gamma,k,z_0)$. To examine this class for other subsets of B, we prove the following lemma

Lemma 1.- If $f(z)\in TW_q^p([\alpha_1],\gamma,k,z_0)\cap TW_q^p([\alpha_1],\gamma,k,z_1)$, where z0 and z1 are distinct positive numbers then f(z) = z.

Proof-For the functions of the form (1.2), we have

$$a_1=1+\sum_{n=2}^{\infty}a_n[(1-\mu)+n\mu]z_0^{n-1}$$

and

$$a_1=1+\sum_{n=2}^{\infty}a_n[(1-\mu)+n\mu]z_1^{n-1}.$$

That is,

$$a_n[(1-\mu)+n\mu][z_1^{n-1}-z_0^{n-1}]=0.$$

Hence $a_n\equiv 0$ for $n\ge2$, and so the results follows. (a)

Theorem 7.- If B is contained in the interval (0, 1) and $0\le\alpha<1,\beta\ge0$, then $TW_q^p([\alpha_1],\gamma,k,B)$ is a convex family if and only if B is connected.

Proof-Let B be connected. Suppose of the form (1.2) is in $z_0,z_1\in B$ with $z_0\le z_1$. If $f(z)$ $TW_q^p([\alpha_1],\gamma,k,z_0)$ and $g(z)=b_1z-\sum_{n=2}^{\infty}b_nz^n$ is in $TW_q^p([\alpha_1],\gamma,k,z_1)$ then for $0\le\lambda\le1$, we shall prove that there exists a z_2 $(z_0\le z_2\le z_1)$ such that $h(z)=\lambda f(z)+(1-\lambda)g(z)$ is in $TW_q^p([\alpha_1],\gamma,k,z_2)$. Set

$$\begin{aligned} t(z) &= [(1-\mu)\frac{h(z}{z}+\mu h'(z)] \\ &= \lambda\left[a_1-\sum_{n=2}^{\infty}a_nz^{n-1}((1-\mu)+n\mu)\right]+(1-\lambda)\left[b_1-\sum_{n=2}^{\infty}b_nz^{n-1}((1-\mu)+n\mu)\right] \\ t(z) &= 1+\lambda\sum_{n=2}^{\infty}\left[a_n\{(1-\mu)+n\mu\}(z_0^{n-1}-z^{n-1})\right] \\ &\quad +(1-\lambda)\sum_{n=2}^{\infty}\left[b_n\{(1-\mu)+n\mu\}(z_1^{n-1}-z^{n-1})\right] \end{aligned} \qquad (6)$$

and we observe that f(z) is real when z is real with $t(z_0)\ge1$ and $t(z_1)\le1$.

Hence for some z_1, $z_0\le z_2\le z_1$, we have $t(z_2) = 1$. Since z_1, z_2 and γ are arbitrary, the family $TS(\mu,\alpha,\beta,B)$ Conversely, suppose B is not connected. Then we can take z0, z1 z0, z1 2 B, z_2 /2 B such that $z_0 < z_2 < z_1$. Let us assume f(z) and g(z) are not both identity function. Then using (6.1)

fixing z = z_2 and allow γ to vary,

$$\begin{aligned} t(z) &= t(z_2,\lambda) \\ &= 1+\lambda\sum_{n=2}^{\infty}\left[a_n\{(1-\mu)+n\mu\}(z_0^{n-1}-z_2^{n-1})\right] \\ &\quad +(1-\lambda)\sum_{n=2}^{\infty}\left[b_n\{(1-\mu)+n\mu\}(z_1^{n-1}-z_2^{n-1})\right]. \end{aligned}$$

Since $t(z_2,0)>1$ and $t(z_2,1)<1$, there must exists λ_0, $0<\lambda_0<1$, for which $t(z_2,\lambda_0)=1$. Hence $h(z)\in TW_q^p([\alpha_1],\gamma,k,z_2)$ for $\lambda=\lambda_0$. Since $z_2\notin B$

Since $z_2\notin B$ from the Lemma 1, it follows that $h(z)\in TW_q^p([\alpha_1],\gamma,k,B)$.

Therefore $\in TW_q^p([\alpha_1],\gamma,k,B)$. is not a convex family. (b)

Concluding Remarks

Observe that, if A_t = 1(t = 1, 2, ..., p) and B_t = 1(t = 1, 2, ..., q) specializing the parameters p, q, α_1, α2, . . . , αp, and $\beta_1,\beta_2,\ldots,\beta_q,\gamma$, in the class $TW_q^p([\alpha_1],\gamma,k,z_0)$ we obtain various classes introduced and studied in the literature (see [12, 14, 17, 19]).

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Graph Transformations,Interpolation And Extremal Theorems For Graph Parameters

GJSFR Classification – F (FOR) 010104,010105,010404

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Abstract-Let $\mathcal{G}$ be the class of all graphs and $\mathcal{J} \subseteq \mathcal{G}$. A graph parameter π is called an interpolation graph parameter over $\mathcal{J}$ if there exist integers a and b such that

$$\{\pi(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : a \le k \le b\}.$$

Thus if π is an interpolation graph parameter over $\mathcal{J}$ then $\{\pi(G) : G \in \mathcal{J}\}$ is uniquely determined by

$\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\}$ and $\max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in \mathcal{J}\}$.

The problem of finding $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$ is called the extremal problem in graph theory. We will discuss our results which have been done, in this direction, in the past ten years. Some open problems are also reviewed.

AMS Subject Classification 2000: 05C07, 05C35

Keywords -Interpolation theorem, extremal problem, graph parameter.

I. Introduction

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [1].

Let $\mathcal{J}$ be a class of non-isomorphic graphs. A graph transformation on $\mathcal{J}$ is a subset of $\mathcal{J} \times \mathcal{J}$ Let _ be a graph transformation on $\mathcal{J}$ We can define the *p*-graph having $\mathcal{J}$ as its vertex set and there is a directed edge from G to H if and only if (G;H) $(G, H) \in \rho$ If *p* is symmetric, it yields an undirected graph and otherwise a directed graph.

Harary [10] used a graph transformation called a fundamental exchange or an edge exchange as follows: Let G be a connected graph of order $n > 3$ The tree graph, T(G), of G is defined by specifying V (T(G)) as the set of all spanning trees of G, and two vertices $T_1, T_2 \in V(\mathbf{T}(G))$ are adjacent in T(G) if and only ifT1 and T2 have exactly $n-2$ edges in common. This is an example of an undirected p.graph. It was proved by Harary [10] that the tree graph T(G) is connected.

A non-increasing sequence d = (d_1; d_2; : : : ; d_n) of non-negative integers is a graphic degree sequence if it is a degree sequence of some graph G. In this case, G is called a realization of d:A degree sequence of an r-regular graph of order n is denoted by r^n.

Let G be a graph. For the distinct vertices a, b, c, and d in V (G) such that ab and cd are edges in G while ac and bd are not edges in G. Define $G^{\sigma(a,b;c,d)}$, simply written G^σ to be the graph obtained from G by deleting the edges ab and cd and adding the edges ac and bd.

The operation σ (a; b; c; d) is called a switching operation. For a graphic degree sequence d,let R(d) and CR(d) be the sets of non-isomorphic realizations and connected realizations of d,respectively. The $\Sigma(\mathbf{d})$ is defined as a relation on R(d) as $(G, H) \in \Sigma(\mathbf{d})$ if $G \ncong H$ and there is a switching σ on G such that H = G^σ Thus the $\Sigma(\mathbf{d})$ -graph is simple. The concept of $\Sigma(\mathbf{d})$ -graph was introduced and developed in a joint paper by Eggleton and Holton [7]. It provides A structured way to examine all the graphs which \realize" a given degree sequence. The $\Sigma(\mathbf{d})$-graph and the subgraph induced by CR(d) are connected as a consequence of Taylor [22, 23].For positive integers m and n with $0 \le m \le \binom{n}{2}$, let G(m; n) and CG(m; n) be the sets of all non-isomorphic graphs and the set of connected graphs of order n and size m, respectively. Let and $f \notin E(G)$. Define $G^{t(e,f)}$ to be a graph with $V(G^{t(e,f)}) = V(G)$ and $E(G^{t(e,f)}) = E(G-e+f)$. A transformation $t(e, f)$ is called an edge jump. Now let $\mathcal{T}(m,n)$ be a relation on G(m; n) defined by (G;H) $\in \mathcal{T}(m,n)$ if $G \ncong H$ and H can be obtained from G by an edge jump. Since T (m; n) is symmetric, it follows that the T (m; n)-graph is simple.

II. The interpolation graph parameters

Let $\mathcal{G}$ be the class of all graphs. A function $\pi : \mathcal{G} \to \mathbb{Z}$ is called a graph parameter if $\pi(G) = \pi(H)$ for all isomorphic graphs G and H. A graph parameter π is called an interpolation graph parameter over $\mathcal{J} \subseteq \mathcal{G}$ if there exist integers x and y such that

$$\{\pi(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : x \le k \le y\}.$$

If π is an interpolation graph parameter over $\mathcal{J}$ then $\{\pi(G) : G \in \mathcal{J}\}$ is uniquely determined by min $\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\}$ and $\max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in \mathcal{J}\}$.

In 1964, Erd}os and Gallai [8] proved that any regular graph on n vertices has chromatic number $k \le \frac{3n}{5}$ unless the graph is complete. Commenting on their result in a personal communication, Erd}os wrote to Pullman \probably such a graph exists for every $k \le \frac{3n}{5}$ except possibly for trivial exceptional cases."

Caccetta and Pullman [3] confirmed and strengthened Erd}os' conjecture by showing that if $k > 1$, then for every $n \ge \frac{5k}{3}$, there exists a connected, regular, k-chromatic graph of order n. This is an example of interpolation graph parameter *X* over the class of all connected regular graphs of order n.

A. The $\Sigma(\mathbf{d})$-graphs

We will review in this subsection the interpolation properties of various graph parameters over *R(d) and CR(d).* We first prove a general result as follows:

*About-*This work was carried out with financial support by the Thailand Research Fund.*

Theorem 2.1- *Let* $\mathcal{J} \subseteq \mathcal{R}(\mathrm{d})$ and the subgraph of $\Sigma(\mathrm{d})$-graph induced by $\mathcal{J}$ be connected. Let π be a graph parameter. For any graph G of degree sequence d and any switching σ If $|\pi(G) - \pi(G^{\sigma})| \leq 1,$ then π an interpolation graph parameter over $\mathcal{J}$.

Proof-Let $H, K \in \mathcal{J}$ such that $\pi(H) = \min\{\pi(G) : G \in \mathcal{J}\}$ and $\pi(K) = \max\{\pi(G) : G \in \mathcal{J}\}.$

Since the subgraph of $\Sigma(\mathrm{d})$ -graph induced by $\mathcal{J}$ is connected, there exists a path P : H = G_1;G_2; : : : ;G_t = K in $\mathcal{J}$ Thus there exists a sequence $\sigma_1, \sigma_2, \ldots, \sigma_{t-1}$ such that $G_{i+1} = G_i^{\sigma_i}$ Since $|\pi(G_i) - \pi(G_{i+1})| = |\pi(G_i) - \pi(G_i^{\sigma_i})| \leq 1,$ it follows that $\{\pi(G_i) : i = 1, 2, \ldots, t\} = \{k \in \mathbb{Z} : \pi(H) \leq k \leq \pi(K)\}.$ Thus π is an interpolation graph parameter over $\mathcal{J}$.

The following result can be obtained as consequences of Taylor [22, 23].

Corollary 2.2-Let π be a graph parameter. For a graph G of degree sequence d and A switching σ $|\pi(G) - \pi(G^{\sigma})| \leq 1.$ then π is an interpolation graph parameter over R(d) and CR(d).

We will now review interpolation results on graph parameters over R(d) and CR(d). Here we use $\omega(G)$ and $\alpha(G)$ for the clique and independent number of a graph G respectively.

We proved in [13] and [14] the following result.

Theorem 2.3-Let G be a graph and σ be a switching on G. *If* $\pi \in \{\chi, \omega\},$ *then* $|\pi(G) - \pi(G^{\sigma})| \leq 1.$

Note that $\alpha(G) = \omega(\overline{G})$ for any graph G and G and $\overline{G}^{\sigma(a,b;c,d)} = \overline{G^{\sigma(a,c;b,d)}}.$

Thus we have the following corollary.

Corollary 2.4-Let G be a graph and σ be a switching on G. Then $|\alpha(G) - \alpha(G^{\sigma})| \leq 1.$ For the matching number $\alpha'(G)$ of a graph G we obtained in [18] the following result.

Theorem 2.5-If σ is a switching on G; then $|\alpha'(G) - \alpha'(G^{\sigma})| \leq 1.$ The following results were obtained by Gallai [9] showing a relationship between the independence and covering number. Here we use $\beta(G)$ and $\beta'(G)$ for the covering and edge covering number of a graph G respectively.

Theorem 2.6- For a graph G of order n; $\alpha(G) + \beta(G) = n.$

Theorem 2.7-For a graph G of order n and $\delta \geq 1, \alpha'(G) + \beta'(G) = n.$

As a consequence we obtain the following result.

Theorem 2.8-Let G be a graph, $\delta(G) \geq 1$ and σ be a switching on G. If $\pi \in \{\beta, \beta'\},$ then $|\pi(G) - \pi(G^{\sigma})| \leq 1.$

Let G be a graph and $F \subseteq V(G).$ Then F is called an induced forest of G if G[F] contains no cycle. For a graph G; we define, f(G) as:

$f(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$

The graph parameter f is called the forest number. The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph G is known as the decycling number of G, and is denoted by $\phi(G)$ Thus for a graph G of order n, $\phi(G) + f(G) = n.$ We proved in [15] the following results on f and ϕ

Theorem 2.9-If S is any subset of vertices of G such that G[S] is a forest, and ϕ is any switching on G; then $G^{\sigma}[S]$ contains at most one cycle.

Proof-Let $S \subseteq V(G)$ and $G[S]$ Let a; b; c; d $\in V(G)$ with ab; cd $\in$ E(G) and ac; bd $\notin E(G)$ Since G[S] contains no cycle, it follows that G[S] + ac and G[S] + bd contains at most one cycle. Thus if $|S \cap \{a, b, c, d\}| \leq 3,$ then G_[S] contains at most one cycle. Now suppose that {a; b; c; d} $\subseteq S.$ Since G[S] is a forest, for any two vertices $u, v \in S$ there is at most one (u; v)-path in G[S]. In particular, if there is an (a; c)-path in G[S], then there is no (b; d)-path in G[S]. Thus $G^{\sigma}[S]$ contains at most one cycle, where $\sigma = \sigma(a, b; c, d).$

The following corollary can be obtained as a consequence of above theorem.

Corollary 2.10-Let G be a graph and ϕ be a switching on G. If $\pi \in \{f, \phi\},$ *then* $|\pi(G) - \pi(G^{\sigma})| \leq 1.$

A dominating set of a graph G = (V;E) is a subset D of V such that each vertex of V -D is adjacent to at least one vertex of D: The domination number $\gamma(G)$ of a graph G is the cardinality of a minimal dominating set with the least number of elements. We proved in [17] the following results.

Theorem 2.11-If G is a graph with $\gamma(G)$ - γ and ϕ is a switching on G; then $\gamma(G^{\sigma}) \leq \gamma + 1.$

Proof-Let D be a minimum dominating set of G. Let a; b; c; d $\in$ V (G) with ab; cd $\in$ E(G) and ac; bd $\in$ E(G). Put ϕ = ϕ (a; b; c; d). If {a; b; c; d} $\cap D = \emptyset$ or $\{a, b, c, d\} \subseteq D,$ then D is a dominating set of $G^{\sigma}.$ If $a, b \in D$ or $c, d \in D,$ then D is a dominating set of G^{σ} Finally if $a \in D$ or $c \in D,$ then $D \cup \{b\}$ or $D \cup \{d\}$ is a respective dominating set of $G^{\sigma}.$Thus $\gamma(G^{\sigma}) \leq \gamma + 1.$

By the fact that a switching is symmetric we obtain the following result.

Corollary 2.12-If ϕ is a switching on G; then $|\gamma(G) - \gamma(G^{\sigma})| \leq 1.$

Combining the results in this subsection we can conclude the following theorem.

Theorem 2.13-Let d = (d_1; d_2; : : : ; dn); $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence. Then $\chi, \omega, f, \phi, \alpha, \alpha', \beta, \beta'$ *and* γ are interpolation graph parameters over R(d) and CR(d).

B. *The T (m; n)-graphs*

We recently proved in [21] that the T (m; n)-graph and the subgraph of the T (m; n)-graph induced by CG(m; n) are connected. We also obtained in the same paper the following results.

Theorem 2.14- $\pi \in \{\chi, \omega, f, \phi, \alpha, \alpha', \beta, \beta', \gamma\}.$ Then for any G $\in \mathcal{G}(m, n)$ and an edge jump t(e; f) on G, $|\pi(G) - \pi(G^{t(e,f)})| \leq 1.$

Theorem 2.15- $\pi \in \{\chi, \omega, f, \phi, \alpha, \alpha', \beta, \beta', \gamma\}.$ Then π is an interpolation graph parameter over G(m; n) and CG(m; n):

III. THE EXTREMAL PROBLEMS

An extremal problem asks for minimum and maximum values of a function π over a class of objects. In our context we consider the problem of determining $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J}),$ where π is a graph parameter and

$\mathcal{J}$ is a class of graphs We emphasize on the graph parameters as stated in Section 2 and the classes of graphs $\mathcal{J} \in \{\mathcal{R}(r^n), \mathcal{CR}(r^n), \mathcal{G}(m,n), \mathcal{CG}(m,n)\}$. Therefore we use the following notation.

- $\min(\pi, r^n) = \min\{\pi(G) : G \in \mathcal{R}(r^n)\}$,
- $\max(\pi, r^n) = \max\{\pi(G) : G \in \mathcal{R}(r^n)\}$,
- $\text{Min}(\pi, r^n) = \min\{\pi(G) : G \in \mathcal{CR}(r^n)\}$,
- $\text{Max}(\pi, r^n) = \max\{\pi(G) : G \in \mathcal{CR}(r^n)\}$,
- $\min(\pi; m, n) = \min\{\pi(G) : G \in \mathcal{G}(m,n)\}$,
- $\max(\pi; m, n) = \max\{\pi(G) : G \in \mathcal{G}(m,n)\}$,
- $\text{Min}(\pi; m, n) = \min\{\pi(G) : G \in \mathcal{CG}(m,n)\}$, and
- $\text{Max}(\pi; m, n) = \max\{\pi(G) : G \in \mathcal{CG}(m,n)\}$.

A. *R(r^n) and CR(r^n)*

We will focus on the extremal problem over the classes of regular graphs and some other related classes in this subsection. A classical result of Erd}os and Gallai [8] gives a motivation to the extremal problem.

Theorem 3.1-An r-regular graph G of order $n > r+1$ has chromatic number $k \le \frac{3n}{5}$, with equality if and only if the complementary graph $\overline{G}$ *of* G is the union of disjoint 5-cycles.

We obtained in [13] the extremal values of X

Theorem 3.2- $r \ge 2$ *and* $n \ge 2r$,

$$\min(\chi, r^n) = \begin{cases} 2 & if\ n\ is\ even, \\ 3 & if\ n\ is\ odd. \end{cases}$$

Theorem 3.3- *If* $r \ge 2$, *then*

i. $\min(\chi, r^{r+1}) = \max(\chi, r^{r+1}) = r+1$, *and*

ii. $\min(\chi, r^{r+2}) = \max(\chi, r^{r+2}) = (r+2)/2$.

Theorem 3.4-For any $r \ge 4$ and odd integer s such that $3 \le s \le r$, et q and t be integers satisfying Then

$$\min(\chi, r^{r+s}) = \begin{cases} q & if\ t = 0, \\ q+1 & if\ 1 \le t \le s-2, \\ q+2 & if\ t = s-1. \end{cases}$$

Theorem 3.5-For any even integer $r \ge 6$ and any even number s such that $4 \le s \le r$, *let* q and t be integers satisfying $r + s = sq + t,\ 0 \le t < s$. then

$$\min(\chi, r^{r+s}) = \begin{cases} q & if\ t = 0, \\ q+1 & if\ t \ge 2. \end{cases}$$

By using Brooks' theorem [2] and some graph construction we obtained the following theorems in [13].

Theorem 3.6- *Let* $r \ge 2$. Then

i. $\max(\chi, r^{2r}) = r$,

ii. $\max(\chi, r^{2r+1}) = \begin{cases} 3 & if\ r = 2, \\ r & if\ r \ge 4, \end{cases}$

iii. $\max(\chi, r^n) = r+1$ *for* $n \ge 2r+2$.

Theorem 3.7-For any r and s such that $3 \le s \le r-1$, we have

i. $\max(\chi, r^{r+s}) \ge (r+s)/2$ *if* $r+s$ *is even, and*

ii. $\max(\chi, r^{r+s}) \ge (r+s-1)/2$ *if* $r+s$ *is odd.*

The exact values of max (χ, r^n) are not easy to obtain if $r+3 \le n \le 2r-1$. Result of Theorem 3.1 gives an upper bound for χ in the class of connected regular graphs of order n but the bound can be very far from the actual value depending on the regularity. We were able to improve the bound in [12] by introducing the de_nition of F(j)-graph.

Let $\mathcal{J}$ be a positive integer. An F(j)-graph is a $(j-1)$-regular graph G of minimum order $f(j)$ with the property that $\chi(\overline{G}) > f(j)/2$. It is easy to see that F(3)-graph is C_5 and f(3) = 5.

We found *F(j)-graphs* for all odd integers j as stated in the following theorems.

Theorem 3.8-For odd integer $j \ge 3$, we have $f(j) = \frac{5}{2}(j-1)$ *if* $j \equiv 3 (\text{mod } 4)$ *and* $f(j) = 1 + \frac{5}{2}(j-1)$ *if* $j \equiv 1 (\text{mod } 4)$.

Theorem 3.9-[12]- Any r-regular graph of order n with $n - r = j$ *odd* odd and $j \ge 3$ has chromatic number at most $\frac{f(j)+1}{2f(j)} \cdot n$, and this bound is achieved precisely for those graphs with complement equal to a disjoint union of F(j)-graphs.

Problem 1.- Find an F(j)-graph for even integer $j \ge 4$.

Problem 2.- Find max (χ, r^{r+j}) if j is even and $4 \le j \le r-2$.

The extremal problem for ω has been completely answered in [14]. Since K_{r+1} is the only r-regular graph of order $r+1$, it follows that min $(\omega, r^{r+1}) = \max(\omega, r^{r+1}) = r+1$. Given positive integers n and k with $k \le n$, there exists a connected graph G of order n with $\omega(G) = k$. As we shall see in the next theorem that there is no regular graph G of order n having $\omega(G)$ strictly lies between $\frac{n}{2}$ and n.

Theorem 3.10-Let d = r^n be a graphic degree sequence with $r+2 \le n \le 2r+1$. Then $\max(\omega, r^n) = \lfloor \frac{n}{2} \rfloor$.

The idea of obtaining min (ω, r^n) is similar to what we have done for min (χ, r^n) and we have $\min(\omega, r^n) = \min(\chi, r^n)$ in all situations.

Problem 3.- We have obtained $\min(\omega, r^n) = \min(\chi, r^n)$ in all situations. It is interesting to find $\min(\omega, r^n) = \min(\chi, r^n)$

Problem 4.- By using the relation $\alpha(G) = \omega(\overline{G})$, can we obtain min (α, r^n), $\max(\alpha, r^n)$, $\text{Min}(\alpha, r^n)$ and $\text{Max}(\alpha, r^n)$?

For the graph parameter f, we found in [16] a lower bound of min(f; d) by using the probabilistic method. In particular, we proved the following theorem.

Theorem 3.11-Let G be a graph having degree sequence d = $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, $d_1 \ge d_2 \ge \ldots \ge d_n \ge 1$. *Then*

$$f(G) \ge 2\sum_{i=1}^{n} \frac{1}{d_i + 1}.$$

The value of min (f, r^n) is not easy to obtain if we work on r-regular graphs. It is reasonable to extend the class of r-regular graphs of order n to a larger class $\mathcal{G}_\Delta(n)$. Let n and positive integers with $n > \Delta$. Let $\mathcal{G}_\Delta(n)$ be the class of all graphs G of order n and $\Delta(G) = \Delta$. Let d = (d1; d2; : : : ; dn) be a sequence of non-negative integers. Define $\overline{d}$ a degree sequence $(\overline{d_1}, \overline{d_2}, \ldots, \overline{d_n})$, where $\overline{d_i} = n - d_i - 1$, for $i = 1, 2, \ldots, n$. It is clear that d is graphic if and only if d is. We proved in [16] the following results.

Theorem 3.12-Let d = $\mathbf{d} = (d_1, d_2, \ldots, d_n), d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be a graphic degree sequence and $and\ d_1 + 1 \leq n \leq 2d_1 + 1.\ Then$

i. min(f; d) = 2 if and only $d_1 = d_2 = d_3 = \cdots = d_n\ and\ n = d_1 + 1\ and$

if d does not have a complete graph as its realization, then min(f; d) = 3 if and only if $\overline{d}$ has a disjoint union of stars as its realization.

Theorem 3.13-Let n = $Let\ n = (\Delta + 1)q + t, 0 \leq t \leq \Delta.$ then

1. $\min(f, \mathcal{G}_\Delta(n)) = 2q,\ if\ t = 0,$
2. $\min(f, \mathcal{G}_\Delta(n)) = 2q + 1,\ if\ t = 1,\ and$
3. $\min(f, \mathcal{G}_\Delta(n)) = 2q + 2,\ if\ 2 \leq t \leq \Delta.$

With some modi_cation of Theorem 3.13 in the class of r-regular graphs of order n and some properties of F(j)-graph, we found min (f, r^n) in all situations as stated in the following theorems in [19].

Theorem 3.14- $For\ r \geq 3,\ and\ n = r + j,\ 1 \leq j \leq r + 1$

i. min(f; r^n) = 2, if and only if n = r + 1,
ii. min(f; r^n) = 3, if and only if n = r + 2,
iii. min(f; r^n) = 4, for all even integers n $r + 3 \leq n,$
iv. min(f;r^n) = 4, for all odd integers n, $n,\ r + 3 \leq n\ and\ n \geq f(j),$
v. min(f; r^n) = 5, for all odd integers n, $r + 3 \leq n\ and\ n < f(j),$ where

$f(j) = \frac{5}{2}(j-1)\ if\ j \equiv 3 (\mathrm{mod}\ 4),\ and\ f(j) = 1 + \frac{5}{2}(j-1)\ if\ j \equiv 1 (\mathrm{mod}\ 4).$

Theorem 3.15-

$For\ n \geq 2r + 2\ and\ r \geq 3,\ write\ n = (r+1)q + t, q \geq 2\ and\ 0 \leq t \leq r.\ Then$

i. min(f; r^n) = 2q if t = 0,
ii. min(f; r^n) = 2q + 1 if t = 1,
iii. min(f; r^n) $= 2q + 2\ \ if\ 2 \leq t \leq r - 1,$
iv. min(f; r^n) = 2q + 3 if t = r:

We obtained in [15] the values of max(f; r^n), for all r and n as stated in the following theorems.

Theorem 3.16-
$$\max(f, r^n) = \begin{cases} n - r + 1 & if\ r + 1 \leq n \leq 2r - 1, \\ \lfloor \frac{nr-2}{2(r-1)} \rfloor & if\ n \geq 2r. \end{cases}$$

Note that $\text{if } r \geq 2,$ then max(f; r^n) = Max(f; rn): The investigation of Min(f; rn) was considered in [20] and we settled almost all cases as stated in the following results.

Theorem 3.17-Let n be an even integer $n \geq 12.\ Then$

$$\mathrm{Min}(f, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & if\ n \equiv 2 (\mathrm{mod}\ 8), \\ \lfloor \frac{5}{8}n \rfloor & otherwise. \end{cases}$$

Theorem 3.18-Let n and r be integers with $r \geq 4.\ Then$

$$\mathrm{Min}(f, r^n) \geq \lceil \frac{2n}{r} \rceil.$$

Let $n = rq + t,\ 0 \leq t \leq r - 1,\ r \geq 4.$ Then $\mathrm{Min}(f, r^n) \geq 2q + \lceil \frac{2t}{r} \rceil.$ By construction we have the following results.

$$\mathrm{Min}(f, r^n) = \begin{cases} 2q & \text{if } t = 0, \\ 2q + 1 & \text{if } t = 1, 2, \\ 2q + 2 & \text{if } t > \frac{r}{2}. \end{cases}$$

Problem 5.- Find Min(f; r^n) $\text{if } 3 \leq t \leq \frac{r}{2}.$

Let $\mathcal{B}(r^{2n})$ be the class of r-regular bipartite graphs of order 2n. It was shown in [27], page 53 that the subgraph of the $\Sigma(r^{2n})$-graph induced by $\mathcal{B}(r^{2n})$ is connected. Therefore f is an interpolation graph parameter over We write $\min(f, \mathcal{B}(r^{2n}))$ for $\min\{f(G) : G \in \mathcal{B}(r^{2n})\}$ and $\max(f, \mathcal{B}(r^{2n}))$ for $\max\{f(G) : G \in \mathcal{B}(r^{2n})\}$. Thus $f(\mathcal{B}(r^{2n}))$ is uniquely determined by $\min(f, \mathcal{B}(r^{2n}))$ and $\max(f, \mathcal{B}(r^{2n}))$. Evidently, $\min(f, \mathcal{B}(r^{2n})) = \max(f, \mathcal{B}(r^{2n})) = 2n$ if $r \in$ {0,1} $\max(f, \mathcal{B}(2^{2n})) = 2n - 1$ and $\min(f, \mathcal{B}(2^{2n})) = \lceil \frac{3n}{2} \rceil.$ We proved in [5] the following theorems.

Theorem 3.19- $If\ r \geq 2,\ then\ \max(f, \mathcal{B}(r^{2n})) = \max(f, r^{2n}) = \lfloor \frac{nr-1}{r-1} \rfloor.$

Theorem 3.20- $\min(f, \mathcal{B}(3^{2n})) = n + \lceil \frac{n}{4} \rceil.$

Theorem 3.21- $\min(f, \mathcal{B}(4^{2n})) = n + \lceil \frac{n}{7} \rceil.$

The problem of determining min $(f, \mathcal{B}(r^{2n}))$ is not easy if $r \geq 5.$

Problem 6.-Find $(f, \mathcal{B}(r^{2n}))$ if $r \geq 5.$

Problem 7-Let CB(r^{2n}) be the class of connected r-regular bipartite graphs of order 2n and $r \geq 2.$ It is clear that $\max(f, \mathcal{CB}(r^{2n})) = \max(f, \mathcal{B}(r^{2n})).$ Find $\min(f, \mathcal{CB}(r^{2n})).$

Problem 8-The hypercube Qn is a connected n-regular bipartite graph of order 2n. The exact values of f(Qn) have been obtained when n is a power of 2. Details can be found in [6].Find f(Q_n) for other values of n.

In [18], we determined the values of $\min(\alpha', r^n)$ and $\max(\alpha', r^n)$ for all r and n. Since

$\min(\alpha', 0^n) = \max(\alpha', 0^n) = 0$ and $\min(\alpha', 1^{2n}) = \max(\alpha', 1^{2n}) = n,$

we can assume that $r \geq 2$ and $n > r + 1.$ An existence of an r-regular Hamiltonian graph of order n implies that max $(\alpha', r^n) = \lfloor \frac{n}{2} \rfloor.$ A component of a graph is odd or even according as it has odd or even number of vertices. We denote by o(G) the number of odd components of G. Tutte [25] proved the following theorem.

Theorem 3.22-The number of edges in a maximum matching of a graph $G\ is\ \frac{1}{2}(|V(G)| - d),$ where $d = \max_{S \subseteq V(G)}\{o(G - S) - |S|\}.$

Let F(r; d) be the minimum order of an r-regular graph G with $\alpha'(G) = \frac{1}{2}(|V(G)| - d).$ It is clear that $|V(G)| \equiv d (\mathrm{mod}\ 2).$ Wallis [26] found $F(r, 2)$ for all $r \geq 3.$ More precisely, he proved the following theorem.

Theorem 3.23-Let G be an r-regular graph with no 1-factor and no odd component. Then

$$|V(G)| \geq \begin{cases} 3r + 7 & if\ r\ is\ odd,\ r \geq 3, \\ 3r + 4 & if\ r\ is\ even,\ r \geq 6, \\ 22 & if\ r = 4. \end{cases}$$

Furthermore, no such graphs exist for r = 1 or 2.

If G is an r-regular graph with $\alpha'(G) = \frac{1}{2}(|V(G)| - d),$ there exists a k-subset K of V (G) such that $o(G - K) = k + d.$ If $k = 0,$ then r is even, G contains d odd components,and each component of G has order at least r +

1. Suppose that $k \geq 1$ and $G - K$ has an odd component with p vertices where $p \leq r$. Thus the number of edges within the component is at most $\frac{1}{2}p(p-1)$. This means that the sum of degrees of these p vertices in G - K is at most But G is an r-regular graph, so the sum of degrees of these p vertices in G is pr: Hence the number of edges joining K to the component must be at least pr - p(p - 1). For a fixed integer r and an integer p satisfying $1 \leq p \leq r$, the function $f(p) = pr - p(p-1),\ 1 \leq p \leq r$ has minimum value f(1) = f(r) = r: So any odd component with r or less vertices is joined to K by r or more edges. Suppose that there are o+ odd components of G - K with more than r vertices and o- odd components with less than or equal to r vertices. Thus

$$o_+ + o_- = k + d \quad (1)$$

$$o_+ + ro_- \leq kr. \quad (2)$$

From these 2 relations, we have

$o_+ \geq \lceil \frac{rd}{r-1} \rceil = d + \lceil \frac{d}{r-1} \rceil$ and $k \geq \lceil \frac{d}{r-1} \rceil$.

We obtained the following results in [18].

Theorem 3.24- Let r be an even integer, $r \geq 2$. Then $F(r,d) = d(r+1)$. then

$$\min(\alpha', r^n) = \frac{dr}{2} + \lfloor \frac{1+e}{2} \rfloor.$$

Suppose that r is odd and $r \geq 3$. Let G be an r-regular graph of order n such that $\alpha'(G) = \frac{1}{2}(n-d)$. Then d must be even. Put d = 2q. There exists a nonempty subset K of V (G) of cardinality k such that $o(G-K) = k + 2q$. By (1) and (2), we have

$$n \geq k + (r+2)o_+ \geq \lceil \frac{2q}{r-1} \rceil + (r+2)(2q + \lceil \frac{2q}{r-1} \rceil) = \lceil \frac{2q}{r-1} \rceil (r+3) + 2q(r+2).$$

Wallis [26] de_ned G(x; y) to be a graph with x+y vertices, x and y being of degree x+y-3 and x + y - 2, respectively. Thus G(x; y) exists if and only if y is even and $y \geq 2$. It is noted that for any graph G(x; y), it has y vertices of degree r and x vertices of degree r - 1. Let Let $x_i, y_i, i = 1, 2, \ldots, m$, be integers such that G(x_i; y_i) exists for all i = 1; 2; : : : ; m: We then construct a graph $G(x_1, y_1) * G(x_2, y_2) * \ldots * G(x_m, y_m)$ from disjoint copies of the graphs by inserting a new vertex, say u, by joining u to all vertices of G(xi; yi) which have the smallest degree, for i = 1; 2; : : : ; m: With this notion we see that for an odd integer $r \geq 3,\ q = 1, 2, \ldots, \frac{r-1}{2}$ and for any odd positive integers ai; i = 1; 2; : : : ; 1 + 2q whose sum is r, it follows that

$$G_q = G(a_1, r+2-a_1) * G(a_2, r+2-a_2) * \ldots * G(a_{1+2q}, r+2-a_{1+2q})$$

is an r-regular graph on (r + 2)(1 + 2q) + 1 vertices with $\alpha'(G_q) = \frac{1}{2}(|V(G_q)| - 2q)$. We have the following results.

Theorem 3.26-For an odd integer $r \geq 3$. then

i. $F(r, 2q) = (r+2)(1+2q) + 1,\ for\ q = 1, 2, \ldots, \frac{r-1}{2},$

ii. $if\ q = \frac{r-1}{2}s + t, 0 \leq t < \frac{r-1}{2},$ then $F(r, 2q) = sF(r, r-1) + F(r, 2t)$, where F(r; 0) = 0:

Corollary 3.27- Let r be an odd integer, $r \geq 3.\ If\ F(r, 2q) \leq n < F(r, 2(q+1))$, then $\min(\alpha', r^n) = \frac{1}{2}(n - 2q)$.

Problem 9.- It is clear that α' is an interpolation graph parameter over CR(rn) and it is easy to see that $\text{Max}(\alpha', r^n) = \lfloor \frac{n}{2} \rfloor$. Find $\text{Min}(\alpha', r^n)$. Find $\text{Min}(\alpha', r^n)$.

B. $\mathcal{G}(m,n)$ and $\mathcal{CG}(m,n)$

We will discuss in this subsection the extremal problem for graph parameters over $\mathcal{G}(m,n)$ and $\mathcal{CG}(m,n)$ Mantel's theorem [11] provides the maximum number of edges that a 2-chromatic graph of order n can have. On the other hand the minimum number of edges in a 2-chromatic graph of order $n \geq 2$ is 1 and the minimum number of edges in a 2-chromatic connected graph of order $n \geq 2$ is 1 and the minimum number of edges in a 2-chromatic connected graph of order $n \geq 2$ is $n-1$. Turán [24] extended the result of Mantel by introducing the Turán graph. This result of Turán is viewed as the origin of extremal graph theory. The Turán graph $T_{n;r}$ is the complete r-partite graph of order n whose partite sets differ in cardinality by at most 1.

Theorem 3.28-Among the graphs of order n containing no complete subgraph of order r+1,$T_{n;r}$ has the maximum number of edges.

In order to apply Turán theorem in our context, we would like to state the following facts.

i. If $n = rq + t,\ 0 \leq t < r$, then $T_{n,r}$ then Tn;r consists of t partite sets of cardinality $\lceil \frac{n}{r} \rceil$ and $r - t$ partite sets of cardinality $\lceil \frac{n}{r} \rceil$

ii. Let $G \in \mathcal{G}(m,n)$. If $\omega(G) \leq r$, then $m \leq \varepsilon(T_{n,r})$.

iii. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

iv. Let t(n; r) $= \varepsilon(T_{n,r})$. Then for a fixed n, we get $t(n, r-1) < t(n, r)$ for all $r,\ 2 \leq r \leq n$. In fact $t(n,r) - t(n, r-1) \geq \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.

We obtained in [21] the following theorems.

Theorem 3.29-Let m; n and k be positive integers with $n \geq k \geq 3\ and\ \binom{k}{2} \leq m < \binom{k+1}{2}$. Then $\max(\chi; m, n) = k$.

Theorem 3.30-Let m; n and $k \geq 2$ be positive integers satisfying $t(n, k-1) < m \leq t(n, k)$. Then $\max(\chi; m, n) = k$.

We now conclude the following corollary.

Corollary 3.31-Let m; n and k be positive integers.

i. $If\ n \geq k\ and\ \binom{k}{2} \leq m < \binom{k+1}{2},\ then\ \max(\omega; m, n) = k$

ii. $If\ t(n, k-1) < m \leq t(n,k),\ then\ \min(\omega; m, n) = k.$

iii. $If\ t(n, k-1) < m \leq t(n,k),\ then\ \text{Min}(\chi; m, n) = k.$

iv. $If\ k \geq 3\ and\ t(n, k-1) < m \leq t(n,k),\ then\ \text{Min}(\omega; m, n) = k.$

Results on $\text{Max}(\chi; m, n)$ and $\text{Max}(\omega; m, n)$ be obtained similarly as stated in the following theorems.

Theorem 3.32-Let n;m and k be positive integers with $n \geq k \geq 3\ and\ \binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1.$ then $\text{Max}(\chi; m, n) = k.$

Theorem 3.33-Let n;m and k be positive integers with $n \geq k \geq 3\ and\ \binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1.$ then $\max(\omega; m, n)$

Thus all extreme values of χ and ω over $\mathcal{G}(m,n)$ and $\mathcal{CG}(m,n)$ are obtained in all situations.The extremal values of the graph parameter f over $\mathcal{G}(m,n)$ and $\mathcal{CG}(m,n)$ were obtained in [4].

Let G be a graph and X; Y be disjoint nonempty subsets of V (G). Denote by $\varepsilon(X)$ the number of edges in $G[X]$ and $\varepsilon(X,Y)$ the number of edges in G connecting vertices in X to vertices in Y:

Let Let $G \in \mathcal{G}(m,n)$ and F be a maximum induced forest of G. Let jFj = a: Therefore G - F has order n - a. An upper bound for m can be obtained by the following inequality.

$$m = \varepsilon(G-F) + \varepsilon(G-F,F) + \varepsilon(F) \le \binom{n-a}{2} + a(n-a) + (a-1).$$

Let a = n - i for any $i \in \{1,2,\ldots,n-2\}$. Then $m \le (i+1)n - \frac{i^2+3i+2}{2}$.

For an integer i = 1; 2; . . . ; n - 2, let

$$\mathrm{M}_n(n-i) := (i+1)n - \frac{i^2+3i+2}{2}.$$

It is clear that $\mathrm{M}_n(n-i)$ is an integer. We showed in [4] that $\max(f;m,n) = n-i$ if and only if $\mathrm{M}_n(n-i+1) < m \le \mathrm{M}_n(n-i)$

And $\mathrm{Max}(f;m,n) = n-i$ if and only if $m \ge n-1$ and $\mathrm{M}_n(n-i+1) < m \le \mathrm{M}_n(n-i)$.

In order to obtain the values of min(f; m; n), we _rst _nd the minimum number of edges of a graph of order n having the forest number a. Let G(n; f = a) be the set of graphs of order n having the forest number a. It is clear that $\mathcal{G}(n; f=a) \ne \emptyset$ if and only if $2 \le a \le n$. For integers n and a, let

$$\mathrm{m}_n(a) := \min\{\varepsilon(G) : G \in \mathcal{G}(n; f=a)\}.$$

Thus $\mathrm{m}_n(n) = 0$, $\mathrm{m}_n(n-1) = 3$ and $\mathrm{m}_n(2) = \binom{n}{2}$. It is easy to see that for a graph G of order $n \ge 2$, $f(G) = 2$ if and only if $G \cong K_n$. We now find $\mathrm{m}_n(a)$ for $2 < a < n$. Theorem 3.12 gives a characterization of graphs having forest number 3. Thus $\mathrm{m}_n(3) = \binom{n}{2} - n + 1$, for all $n \ge 4$. We proved in [4] the following lemma.

Lemma 3.35-If G is a graph of order n with $\Delta(G) = \Delta$ *and* $f(G) = 2q+1$ for some integer q, *then* $n \le (\Delta+1)q+1$.

By Lemma 3.35, we have a lower bound for the maximum degree of a given graph in terms of its order and its forest number. In other words, if G is a graph of order n, then $\Delta(G) \ge \lceil \frac{2n}{f(G)} \rceil - 1$. In particular, if f(G) = 2q for some integer q, then $n \le (\Delta+1)q+1$.

By Lemma 3.35, we have a lower bound for the maximum degree of a given graph in terms of its order and its forest number. In other words, if G is a graph of order n, then $\Delta(G) \ge \lceil \frac{2n}{f(G)} \rceil - 1$. In particular, if f(G) = 2q for some integer q, then $\Delta(G) \ge \lceil \frac{n}{q} \rceil - 1$ By Lemma 3.35 the lower bound for $\Delta(G)$ can be improved if f(G) is odd. That is, if f(G) = 2q + 1 for some integer q, then $n \le (\Delta(G)+1)q+1$ which is equivalent to $\Delta(G) \ge \lceil \frac{n-1}{q} \rceil - 1$. We have the following corollary.

Corollary 3.36-Let G be a graph of order n and q be a positive integer. If f(G) = 2q, then

$$\Delta(G) \ge \lceil \tfrac{n}{q} \rceil - 1, \textit{ and if } f(G) = 2q+1, \textit{ then } \Delta(G) \ge \lceil \tfrac{n-1}{q} \rceil - 1.$$

Let $\mathcal{G}^*(n; f=a) = \{G \in \mathcal{G}(n; f=a) :$ is a union of $\lceil \frac{a}{2} \rceil$ cliques}It is clear that $\mathcal{G}^*(n; f=a) \subset \mathcal{G}(n; f=a)$. We have the following theorem.

Theorem 3.37-Let G be a graph of order n with f(G) = a. Then there exists a graph $H \in \mathcal{G}^*(n; f=a)$ such that $\varepsilon(H) \le \varepsilon(G)$.

By Theorem 3.37, we know the structure of graphs of order n with prescribed the forest number. In general, for a graph $G \in \mathcal{G}(n; f=a)$, there may be many such graphs H ε $\mathcal{G}^*(n; f=a)$. We now seek for such a graph H with minimum number of edges.

By using the results of Mantel [11] and Turán [24] as mentioned in the previous subsection,we have the following results.

i. Let- $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \cdots \cup p_kK_k$. Then the order of G is G is p1+3p3+4p4+. . .+kpk and f(G) = p1 + 2(p3 + p4 + . . . + pk). Suppose that $p_1 \ge 2$, $p_k \ge 1$ and $k \ge 4$. Then,by replacing $2K_1 \cup K_k$ by $K_3 \cup K_{k-1}$ we obtain a graph H with $\varepsilon(H) \le \varepsilon(G)$.Further, $\varepsilon(H) = \varepsilon(G)$ if and only if k= 4.
ii. $\mathrm{m}_n(n-1) = 3$ if $n \ge 4$. Let $G \in \mathcal{G}(n; f=n-1)$. Then "(G) = 3 if and only if $n \ge 4$ and $G = (n-3)K_1 \cup K_3$.
iii. Let a be an integer with $\frac{2n}{3} \le a \le n-1$. If (p; q) is the solution of $p+3q=n$ and $p+2q=a$, then $G = pK_1 \cup qK_3$ satisfies $f(G) = a$.
iv. Let a be an integer with $\frac{2n}{3} \le a \le n-1$. and $G \in \mathcal{G}(n; f=a)$ such that $\varepsilon(G) = \mathrm{m}_n(a)$. Then by Theorem 3.37, we can choose $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \cdots \cup p_kK_k \in \mathcal{G}^*(n; f=a)$ and $k \le 4$. If $k=4$, then $p_1 \ge 2$. Thus, there exists a graph $H = pK_1 \cup qK_3$ such that $p+3q=n$, $p+2q=a$ and $\varepsilon(H) = \varepsilon(G) = \mathrm{m}_n(a)$.
v. Let a be an integer with $a < \frac{2n}{3}$ and $G \in \mathcal{G}(n; f=a)$ and $\Delta(G) \ge 3$. Thus if G = $p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \cdots \cup p_kK_k$, $f(G) = a < \frac{2n}{3}$ and $\varepsilon(G) = \mathrm{m}_n(a)$, then $p_1 \le 1$ and $k \ge 4$.
vi. If $n = rq+t$, $0 \le t < r$, then $T_{n,r}$ consists of t partite sets of cardinality $\lceil \frac{n}{r} \rceil$ and r – t partite sets of cardinality $\lceil \frac{n}{r} \rceil$.
vii. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$. where $a = \lfloor \frac{n}{r} \rfloor$.
viii. Let $t(n,r) = \varepsilon(T_{n,r})$. Then for a fixed n, by using elementary arithmetic, we get t(n; r -1) $1) < t(n,r)$ for all r, $2 \le r \le n$. In fact $t(n,r) - t(n,r-1) \ge \binom{a+1}{2}$, where $a = \lfloor \frac{n}{r} \rfloor$.
ix. Let $\bar{t}(n,r) = \binom{n}{2} - \varepsilon(T_{n,r})$. Summarizing the results, we have the following theorems.

Theorem 3.38-Let n and a be integers with $2 \le a \le n-1$. then

i. $\mathrm{m}_n(n) = 0$,
ii. $\mathrm{m}_n(n-1) = 3$ if $n \ge 3$ and $G = (n-3)K_1 \cup K_3$ is the only graph of order n satisfying $f(G) = n-1$ *and* $\varepsilon(G) = 3$,
iii. $\mathrm{m}_n(n-i) =$ $3i$ *if* $1 \le i \le \lceil \frac{n}{3} \rceil$,

iv. Suppose $4 \le a < \frac{2n}{3}$. Then $\mathbf{m}_n(a) = \bar{t}(n,q)$ *if* $a = 2q$, and $\mathbf{m}_n(a) = \bar{t}(n-1,q)$ *if* $a = 2q+1$, for some integer q, and

v. mn(3) = $\binom{n-1}{2}$ *if* $n \ge 3$, and $\mathbf{m}_n(2) = \binom{n}{2}$ *if* $n \ge 2$.

Theorem 3.39-Let n and m be integers with $2 \le a \le n-1$. then

i. min(f; m; n) = max(f; m; n) = n if and only if $\in \{0,1,2\}$,

ii. min(f; m; n) = max(f; m; n) = 2 if and only if m = $m = \binom{n}{2}$,

iii. for $3 \le a \le n-1$, $\min(f;m,n) = a$ if and only if $\mathbf{m}_n(a) \le m < \mathbf{m}_n(a-1)$.

We now find the minimum number of edges of a connected graph order n having the forest number a. Let Let $\mathcal{CG}(n; f = a)$ be the set of all connected graphs of order n having the forest number a. For integers n and a, let

$\mathbf{cm}_n(a) = \min\{\varepsilon(G) : G \in \mathcal{CG}(n; f = a)\}$.

Further, $\mathbf{cm}_n(n) = n-1$, $\mathbf{cm}_n(2) = \binom{n}{2}$. We now find $\mathbf{cm}_n(a)$ for $2 < a < n$.

Let $\mathcal{CG}^*(n; f = a) = \{G \in \mathcal{CG}(n; f = a) : G$ G is obtained from $\lceil \frac{a}{2} \rceil$ disjoint cliques and $\lceil \frac{a}{2} \rceil - 1$ edgesg. We have the following theorem.

Theorem 3.40- Let G be a connected graph of order n with f(G) = a. Then there exists A graph $H \in \mathcal{CG}^*(n; f = a)$

By Theorem 3.40 we know that for a graph $G \in \mathcal{CG}(n; f = a)$, there may be many such graphs $H \in \mathcal{CG}^*(n; f = a)$ We now seek for such a graph H with minimum number of edges. By applying Turán Theorem once again, we have the following theorems.

Theorem 3.41- Let n and a be integers with $2 \le a \le n-1$. then

i. cm$_n$(n) = n - 1,

Suppose that $4 \le a \le n-1$. Then

ii. $\mathbf{cm}_n(a) = \bar{t}(n,q) + q - 1$ *if* $a = 2q$, and $\mathbf{cm}_n(a) = t(n-1,q) + q$ *if* $a = 2q+1$, for some integer q, and

iii. cm$_n$(3)= $\mathbf{cm}_n(3) = \binom{n-1}{2} + 1$ *if* $n \ge 3$, $\mathbf{cm}_n(2) = \binom{n}{2}$ *if* $n \ge 2$.

Theorem 3.42-Let n and m be integers with $n-1 \le m \le \binom{n}{2}$. Then

i. Min(f; m; n) = Max(f; m; n) = n if and only if m = n - 1,

ii. Min(f; m; n) = Max(f; m; n) = 2 if and only if m = $\binom{n}{2}$

iii. for $3 \le a \le n-1$, $\mathrm{Min}(f;m,n) = a$ Min(f; m; n) = a if and only if $\mathbf{cm}_n(a) \le m < \mathbf{cm}_n(a-1)$

Problem 10.- Several graph parameters have been proved to interpolate over $\mathcal{G}(m,n)$ and $\mathcal{CG}(m,n)$ as stated in Theorem 2.15. Find $\min(\pi;m,n)$, $\max(\pi;m,n)$, $\mathrm{Min}(\pi;m,n)$ and $\mathrm{Max}(\pi;m,n)$ where $\pi \in \{\alpha, \alpha', \gamma\}$.

IV. References

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Convergence Of Some Doubly Sequences Iterations With Errors In Banach Spaces

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GJSFR Classification – F (FOR) 010112,010108,010109

***Abstract*-In this paper, we obtain convergence theorems of *G*-iteration process with errors in the doubly sequences settings. Furthermore, we give some examples to support our results. Finally, we apply this G-iteration process to obtain a solution of a nonlinear equation. AMS: 47H10, 54H25**

Keywords-Mann iteration, G-iteration, fixed points.

I. Introduction

Let *X* be a real Banach space and *X** its dual. For $1 < p < \infty$, the duality mapping $J_p : X \to 2^{X^*}$, is defined by
$J_p(x) = \{f^* \in X^* : \langle x\ ,\ f^*\rangle = \|x\|^p, \|f^*\|^p = \|x\|^{p-1}\}, \quad x \in X,$
where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between *X and X** Recall that a mapping $A : X \to X$ is said to be accretive if $\forall\ x, y\ \in D(A)\ \exists\ j_p(x-y)\ \in\ J_p(x-y)$ such that $\langle Ax - Ay\ ,\ j_p(x-y)\rangle \geq 0$ and is said to be strongly accretive if if $A - kI$ is accretive where $k \in (0, 1)$ is a constant and I denotes the identity operator on X: Let $S(T) = \{x^* \in D(A) : Ax^* = f\} \neq \emptyset$ denote the solution set of the equation $Ax = f$. If $\langle Ax - Ay\ ,\ j_p(x-y)\rangle \geq 0$ for all $x \in D(A)$ for all $x \in D(A)$ and $y = x^* \in S(T)$, then A is said to be quasiaccretive. The notion of strongly quasi-accretive is similarly defined. A mapping $T : X \to X$ is said to be pseudo-contractive if $\forall\ x, y\ \in D(T), \exists j_p(x-y) \in J_p(x-y)$ such that $\langle (I-T)x - (I-T)y\ ,\ j_p(x-y)\rangle \geq 0.$ Observe that T is pseudo-contractive if and only if $A = (I-T)$ is accretive. A map T is called hemicontractive if and only if $A = (I-T)$ is quasi-accretive.

Let *X* be a real Banach space of dimension dimX > 2: The modulus of smoothness of *X* is defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

If $\rho_X(\tau) > 0\ \forall\ \tau > 0$ then X is said to be smooth. If these exist a constant c > 0 and a real number $1 < p < \infty$ such that $\rho_X(\tau) \leq C\tau^p$, then X is said to p-uniformly smooth Banach space, then the following geometric inequality holds (see e.g., [4, 5]):

$$\|x+y\|^p \leq \|x\|^p + p\,\langle y\ ,\ j_p(x+y)\rangle + C_p\|y\|^p, \quad x, y \in X, \qquad (1)$$

and some real positive constant $C_p \geq 1$. If T is a self-mapping of a closed convex subset *E* of *X* and I the identity of *X:* Then, *T* is a nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \ \text{ for all } x, y \in E.$$

Krasnoselskii [11] proved that the sequence of iteration $\{T^n(x_0)\}$, starting from a given point $x_0 \in E$, does not converge necessarily to a fixed point of T; whereas the sequence $\{T^n{}_\lambda(x_0)\}$, where $T_\lambda = (1 - \lambda)I + \lambda T,\ 0 < \lambda \leq 1,$ may converges to a fixed point of T, as shown by Krasnoselskii [11] which assumed $\lambda = \frac{1}{2}$, E compact and X uniformly convex. The above scheme has been extended by means of so-called,Mann iterative process (see [14]), associated with T and described in the following way:

Let $x_0 \in E$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n, \qquad (2)$$

for n = 0; 1; 2; : : : ; where

i. $0 \leq c_n < 1, \quad n \geq 0,$
ii. $\lim_{n\to\infty} c_n = 0,$
iii. $\sum_{n=1}^{\infty} c_n = \infty.$

The scheme (2) has been studied by many authors (see for example [4, 10, 12, 15, 16, 19, 23, 25, 26, 27, 29]) and others. In 1986, Pathak [20] introduced a generalization of the scheme (2) for two self-mappings S;T on a closed subset E of a Banach space X which described in the following way:

Let $x_1 \in E$ and $\{Sx_n\}$ and $\{Sx_n\}$ be a sequence defined by

$$Sx_{n+1} = (1-t)Sx_n + tTx_n, \text{ for } n = 1, 2, 3, \ldots, \text{ where } t \in (0, 1), \qquad (3)$$

and he proved that under certain conditions if the sequence $\{Sx_n\}$ converges, then it converges to the unique common fixed point of S and T: *If S = I*; the identity mapping, the iteration (3) is identical with the iteration (2).

The scheme (2) has been extended by means of the so-called G-iteration process (see [21, 22]) associated with a single mapping T and described in the following manner:

Let $x_0 \in E$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = (\mu_n - \lambda_n)x_n + \lambda_nTx_n + (1 - \mu_n)Tx_{n-1} \text{ for } n \geq 0, \qquad (4)$$

Where $\{\mu_n\}$ and $\{\lambda_n\}$

i. $\lambda_0 = \mu_0 = 1,$
ii. $0 < \lambda_n < 1,\ 0 \leq \mu_n \leq 1$ such that $\mu_n \geq \lambda_n,\ n > 0,$
iii. $\lim_{n\to\infty} \lambda_n = h > 0,$
iv. $\lim_{n\to\infty} \mu_n = 1.$

We note that when $\{\mu_n\} = 1,$ the G-iteration process reduces to Mann iteration (2).The idea of considering fixed point iteration procedures with errors comes from practical numerical computations. This topic of research play important role in the stability problem of fixed point iterations. In 1995, Liu [13] initiated a study of fixed point iterations with errors. Several authors have proved some fixed point theorems for Mann type iteration with errors using several classes of mappings (see [4, 6, 7],

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[17, 18, 25, 28] and others). On the other hand there are some attempts in the the doubly sequence setting see [1, 2, 2, 18].

To introduce the meaning of doubly sequence iterations, we introduce the following concepts.

Definition 1.1- (see e.g [18]) Let N denote the set of all natural numbers and let N be a normed linear space. By a double sequence in N is meant a function f : $\mathbb{N}\times\mathbb{N}\to N$ defined by f(*n;m)* $f(n,m)=x_{n,m}\in N.$

The double sequence $\{x_{n,m}\}$ is said to converge strongly to x^* if for a given $\epsilon>0$ there exist integers M;M1 > 0 such that $\forall\, n\geq M,\ \ m\geq M_1,$ we have that

$\|x_{n,m}-x^*\|<\epsilon.$

If $\forall n,r\geq M,\ \ m,t\geq M_1,$ we have that

$\|x_{n,m}-x^*\|<\epsilon.$

then the double sequence is said to be Cauchy. Furthermore, if for each fixed $n\ x_{n,m}\to x_n^*$ as $m\to\infty$ and then $x_n^*\to x^*$ as $n\to\infty,$ so $x_{n,m}\to x^*$ as $n,m\to\infty.$

Let E1 be a nonempty closed convex subset of the Banach space X and T : E1 ! E1: Then, the Mann double sequence $\{x_{k,n}\}_{k>0,n>0}$ generated from an arbitrary $x_{0,0}\in E_1$ is defined by $x_{k,n+1}=(1-\alpha_n)x_{k,n}+\alpha_nTx_{k,n}+(1-\alpha_n)u_{k,n}\,;\ \ k,n\geq 0,$ (see [3]), where $\alpha_n\in[0,1].$ Now, we define the double Ishikawa iteration process with errors as follows:

$$\begin{cases} y_{k,n+1}=(1-b_n)x_{k,n}+b_nTx_{k,n}+c_{k,n} \\ x_{k,n+1}=(1-a_n)x_{k,n}+a_nTy_{k,n}+d_{k,n}, k,n\geq 0 \end{cases} \quad (5)$$

In the next, we give the equivalence between Mann and Ishikawa iterates with errors in the doubly sequence setting.

Theorem 1.1-Let X be a real Banach space with uniformaly convex dual and E1 be a nonempty closed convex subset of X: $Let\ T:E_1\to E_1$ be a continuous and strongly pseudo-contractive mapping. Then for $x_{0,0}\in E_1,$ the following assertions are equivalent:

i. Doubly Mann iterates with errors converges to the fixed point of T:
ii. Doubly Ishikawa iterates with errors converges to the fixed point of T:

Proof-The proof is very similar to the proof of Theorem 2.1 in [24] with some simple modifications, so we will omit it.

II. Fixed Points and Doubly G-Iteration Process

In this section we consider the G-iteration process associated with two self-mappings *S; T* on a closed convex subset of a normed space $(N,\|\cdot\|)$ as given below. For $x_0\in\mathbb{N}$, set

$$Sx_{n+1,k+1}=(\mu_n-\lambda_n)Sx_{n,k}+\lambda_nTx_{n,k}+(1-\mu_n)Tx_{n-1,k-1}+(1-\mu_n)u_{n,k} \quad (6)$$

For $n,k>0,$ where $\{\mu_n\}$ and $\{\lambda_n\}$ satisfy (i), (ii),(iii) and (iv). If $k=0$, $\{\mu_n\}=1,$ we obtain the iteration (4). Also, when $k=0$, $\{\mu_n\}=1$ and S = I; the identity mapping, the iteration (6) is identical with the Mann iteration (2).

In this section, it is proved that for two mappings S and T which satisfy conditions (6) and (7) below, if the sequence of G-iteration associated with S; T as in (6), then it converges to a common fixed point of S and T:

The contractive conditions to be used are the following:

For all $x,y\in N,$

$$\|Tx-Ty\|\leq\alpha\|Sx-Sy\|+\beta\|Sx-Tx\|+\gamma\|Sy-Tx\|+\delta\max\{\|Sy-Ty\|,\|Sx-Ty\|\}, \quad (7)$$

where, $\alpha,\beta,\gamma\geq 0$ with $0<\alpha+\beta+\gamma+\delta<1.$

The second contraction condition is the following:

$$\|Tx-Ty\|^p+a_1[\|Sx-Tx\|^p+\|Sx-Ty\|^p]+a_2[\|Sy-Ty\|^p+\|Sy-Tx\|^p]$$
$$\leq q\max\{c\|Sx-Sy\|^p,\|Sx-Tx\|^p+\|Sy-Ty\|^p,\|Sx-Ty\|^p+\|Sy-Tx\|^p\}, \quad (8)$$

where p > 0 and $0<q-(a_1+a_2)<1.$

First of all we prove the following theorem:

Theorem 2.1 K be a nonempty closed convex subset of a normed space N. Let $S,T:K\to K$ be mappings satisfying condition (7) and the following condition:

$S^2=T^2=I;$ where I denotes the identity mapping: (9)

Let $\{Sx_{n,k}\}$ be the sequence of G-iteration as defined by (6). If the sequence $\{Sx_{n,k}\}$ converges to a point z 2 K; then z is the unique common fixed point of S and T:

Proof-For each; $n\geq 0,$ we have

$$\begin{aligned}\|Sx_{n+1,k+1}-TSz\| &\leq [(\mu_n-\lambda_n)\|Sx_{n,k}-TSz\|+\lambda_n\|Tx_{n,k}-TSz\| \\ &+ (1-\mu_n)\|Tx_{n-1,k-1}-TSz\|]+(1-\mu_n)\|u_{n,k}\|.\end{aligned} \quad (10)$$

Since S and T satisfy (7), then by using (9) we have

$$\begin{aligned}\|Tx_{n,k}-TSz\| &\leq \alpha\|Sx_{n,k}-SSz\|+\beta\|Sx_{n,k}-Tx_{n,k}\|+\gamma\|SSz-Tx_{n,k}\| \\ &+ \delta\max\{\|SSz-TSz\|,\|Sx_{n,k}-TSz\|\}.\end{aligned} \quad (11)$$

From (10) and (11), we obtain

$$\begin{aligned}&\|Sx_{n+1,k+1}-TSz\|\leq(\mu_n-\lambda_n)\|Sx_{n,k}-TSz\|+(1-\mu_n)\|Tx_{n-1,k-1}-TSz\| \\ &+\alpha\lambda_n\|Sx_{n,k}-z\|+\beta\lambda_n\|Sx_{n,k}-Tx_{n,k}\|+\gamma\lambda_n[\|z-Sx_{n,k}\|+\|Sx_{n,k}-Tx_{n,k}\|] \\ &+\delta\lambda_n\max\{\|z-TSz\|,\|Sx_{n,k}-TSz\|\}+(1-\mu_n)\|u_{n,k}\|.\end{aligned} \quad (12)$$

From (6), one gets

$$\|Sx_{n,k}-Tx_{n,k}\|\leq\frac{1}{\lambda_n}[\|\mu_nSx_{n,k}-Sx_{n+1,k+1}\|]+\frac{1-\mu_n}{\lambda_n}\|Tx_{n-1,k-1}+u_{n,k}\|. \quad (13)$$

Substituting (13) in (12) and letting n; $k\to\infty.$ we obtain $\|z-TSz\|\leq(1-h(1-\delta))\|z-TSz\|.$

Since $0<[1-h(1-\delta)]<1,$ then we have

$$TSz=z, \quad (14)$$

i.e., z is a fixed point of TS.

Now using (9), we have

$$Tz=T^2Sz=Sz\ \text{ and hence, }\ STz=SSz=z. \quad (15)$$

i.e.,

$$TSz=STz=z, \quad (16)$$

Again using (7), (9), (14), (15) and (16), we have

$$\begin{aligned}\|z-Tz\| &= \|T(Tz)-Tz\| \\ &\leq \alpha\|STz-Sz\|+\beta\|STz-T^2z\|+\gamma\|Sz-TSz\| \\ &+ \delta\max\{\|Sz-Tz\|,\|STz-Tz\|\} \\ &\leq (\alpha+\gamma+\delta)\|z-Tz\|,\end{aligned} \quad (17)$$

which implies that,

$\|z-Tz\|\leq\eta\|z-Tz\|$

a contradiction, Since $0<\eta=(\alpha+\gamma+\delta)<1,$ it follows that $Tz=z.$ i.e., z is a fixed point of T; but $Tz=Sz.$So, we have

$Sz = Tz = z.$

i.e., z is a common fixed point of S and T:

Now to prove the uniqueness of z, let $w(w \neq z)$ be another common fixed point of S and T: Then, we have

$$\begin{aligned} \|z-w\| &= \|T(Tz)-T(Tw)\| \leq \alpha\|STz-STw\| + \beta\|STz-T^2z\| \quad (18) \\ &+ \gamma\|STw-T^2z\| + \delta\max\{\|STw-T^2w\|, \|STz-T^2w\|\} \\ &\leq \eta\|z-w\|, \end{aligned} \quad (19)$$

a contradiction, since $0 \leq \eta < 1$, then $z = w$. This completes the proof of the theorem

Theorem 2.2- Let K be a closed convex subset of a normed space N: Let $S, T : K \to K$ be mappings satisfying the conditions (8) and (9) as given before. Let $\{Sx_{n,k}\}$ be the sequence of G-iteration process as given by (6).If $\{Sx_{n,k}\}$ converges to z in K and if

$\max\{cq, 2q\} < 1 + a_1 + a_2,$

then z is the unique common fixed point of S and T:

Proof- We consider two cases.

Case (I)-When p is a positive integer. Consider,

$$\|Sx_{n+1,k+1} - TSz\|^p \leq \left[(\mu_n - \lambda_n)\|Sx_{n,k} - TSz\| + \lambda_n\|Tx_{n,k} - TSz\| + (1-\mu_n)\|Tx_{n-1,k-1} - TSz\|\right]^p. \quad (20)$$

Since S and T satisfy (8), then by using (9), we have

$$\begin{aligned} &\|Tx_{n,k} - TSz\|^p \leq q\max\{c\|Sx_{n,k} - SSz\|^p, [\|Sx_{n,k} - Tx_{n,k}\|^p + \|SSz - TSz\|^p], \\ &d[\|Sx_{n,k} - TSz\|^p + \|SSz - Tx_{n,k}\|^p]\} - a_1[\|Sx_{n,k} - Tx_{n,k}\|^p + \|Sx_{n,k} - TSz\|^p] \\ &-a_2[\|SSz - TSz\|^p + \|SSz - Tx_{n,k}\|^p]\} \\ &\leq q\max\{c\|Sx_{n,k} - z\|^p, \|Sx_{n,k} - Tx_{n,k}\|^p + \|z - TSz\|^p, \|Sx_{n,k} - TSz\|^p \\ &+[\|z - Sx_{n,k}\| + \|Sx_{n,k} - Tx_{n,k}\|]^p\} - a_1[\|Sx_{n,k} - Tx_{n,k}\|^p + \|Sx_{n,k} - TSz\|^p] \\ &-a_2[\|z - TSz\|^p + [\|z - Sx_{n,k}\| + \|Sx_{n,k} - Tx_{n,k}\|]^p]. \end{aligned} \quad (21)$$

Letting $n, k \to \infty$ in (21), we obtain

$$\lim_{n,k\to\infty} \|Tx_{n,k} - TSz\|^p \leq q\max\{0, \|z-TSz\|^p, \|z-TSz\|^p\} - (a_1+a_2)\|z-TSz\|^p, \quad (22)$$

which implies that,

$$\lim_{n,k\to\infty} \|Tx_{n,k} - TSz\|^p \leq (q - (a_1+a_2))\|z-TSz\|^p, \quad (23)$$

From (21), we obtain

$$\begin{aligned} \|Sx_{n+1,k+1} - TSz\|^p &\leq \Phi^p + \lambda_n\binom{p}{1}\Phi^{p-1}\|Tx_{n,k} - TSz\| + \lambda_n^2\binom{p}{2}\Phi^{p-2}\|Tx_{n,k} - TSz\|^2 \\ &+ \ldots + \lambda_n^p\|Tx_{n,k} - TSz\|^p, \end{aligned} \quad (24)$$

where,

$$\Phi^p = \left[(\mu_n - \lambda_n)\|Sx_{n,k} - TSz\| + (1-\mu_n)\|Tx_{n-1,k-1} - TSz\| + (1-\mu_n)\|u_{n,k}\|\right]^p. \quad (25)$$

Hence, as n; $n, k \to \infty$ (25) and using (24), we obtain

$$\begin{aligned} &\|z - TSz\|^p \leq \lim_{n,k\to\infty}\Phi^p + \binom{p}{1}h(q-(a_1+a_2))^{\frac{1}{p}}\|z-TSz\| \lim_{n,k\to\infty}\Phi^{p-1} \\ &+\binom{p}{2}h^2(q-(a_1+a_2))^{\frac{2}{p}}\|z-TSz\|^2 \lim_{n,k\to\infty}\Phi^{p-2} + \ldots + h^p(q-(a_1+a_2))\|z-TSz\|^p. \end{aligned} \quad (26)$$

Now, we shall compute

$$\lim_{n,k\to\infty}\Phi^p, \lim_{n,k\to\infty}\Phi^{p-1}, .$$

From (25) if we letting $n, k \to \infty$ we obtain

$$\lim_{n,k\to\infty}\Phi^p = (1-h)^p\|z-TSz\|^p. \quad (27)$$

Substituting from (27) in (26), we thus obtain

$$\begin{aligned} &\|z-TSz\|^p \leq (1-h)^p\|z-TSz\|^p + \binom{p}{1}h(1-h)^{p-1}(q-(a_1+a_2))^{\frac{1}{p}}\|z-TSz\|^p \\ &+\binom{p}{2}h^2(1-h)^{p-2}(q-(a_1+a_2))^{\frac{2}{p}}\|z-TSz\|^p + \ldots + h^p(q-(a_1+a_2))\|z-TSz\|^p \\ &= \|z-TSz\|^p\left[(1-h) + h(q-(a_1+a_2))^{\frac{1}{p}}\right]^p, \end{aligned}$$

this implies that

$\|z-TSz\|^p \leq \lambda\|z-TSz\|^p,$

then TSz = z, i.e., z is a fixed point of TS; where

$0 < \lambda = \left[(1-h) + h(q-(a_1+a_2))^{\frac{1}{p}}\right]^p < 1.$

Similarly, we can prove that STz = z. Now, let $Tz \neq z$, then using (7), we have

$$\begin{aligned} \|z-Tz\|^p &= \|T(Tz)-Tz\|^p \leq q\max\{c\|STz-Sz\|^p, \|STz-T^2z\|^p + \|Sz-Tz\|^p, \\ &\quad \|STz-Tz\|^p + \|Sz-T^2z\|^p\} - a_1[\|STz-T^2z\|^p + \|STz-Tz\|^p] \\ &- a_2[\|Sz-Tz\|^p + \|Sz-T^2z\|^p] \\ &\leq q\max\{c\|z-Tz\|^p, 0, 2\|z-Tz\|^p\} - (a_1+a_2)\|z-Tz\|^p. \end{aligned}$$

If $\max\{cq, 2q\} < 1 + a_1 + a_2$, then

$$\|z-Tz\|^p \leq \frac{\max\{cq, 2q\}}{1+(a_1+a_2)}\|z-Tz\|^p,$$

a contradiction, since $\frac{\max\{cq,2q\}}{1+(a_1+a_2)} < 1$, hence $Tz = z$. i.e., z is a fixed point of T. But Tz = Sz; so

$Sz = Tz = z.$

i.e., z is a common fixed point of S and T.

Case (II). When p is any fraction. Then, we have

$$\|Sx_{n+1,k+1} - TSz\|^p \leq \Phi^p\left[1 + \frac{\lambda_n\|Tx_{n,k} - TSz\|}{\Phi}\right]^p, \quad (28)$$

where Φ is given by (25). By the same argument as given in Case (I), we see as $n, k \to \infty$

$\|Tx_{n,k} - TSz\|^p \leq (q-(a_1+a_2))\|z-Tz\|^p.$

Therefore, as $n, k \to \infty$ in (28), we obtain

$$\begin{aligned} \|z-TSz\|^p &\leq \lim_{n,k\to\infty}\Phi^p\left[1 + \frac{h(q-(a_1+a_2))^{\frac{1}{p}}\|z-TSz\|}{\lim_{n,k\to\infty}\Phi}\right]^p \\ &\leq (1-h)^p\|z-TSz\|^p\left[1 + \frac{h(q-(a_1+a_2))^{\frac{1}{p}}\|z-TSz\|}{(1-h)\|z-TSz\|}\right]^p, \end{aligned}$$

which implies that,

$\|z-TSz\|^p \leq \left[(1-h) + h(q-(a_1+a_2))^{\frac{1}{p}}\right]^p\|z-TSz\|^p,$

this implies that,

$\|z-TSz\|^p \leq \lambda\|z-TSz\|^p,$

then TSz = z, i.e., z is a fixed point of TS, where

$0 < \lambda = \left[(1-h) + h(q-(a_1+a_2))^{\frac{1}{p}}\right]^p < 1.$

Similarly, as in Case (I), we can prove that

$Tz = Sz = z.$

i.e., z is a common fixed point of S and T. Now, we let $w(w \neq z)$ be another fixed point of S and T:

Then using (8) and (9), we obtain

$$\begin{aligned} \|z-w\|^p &= \|T(Tz)-T(Tw)\|^p \leq q\max\{c\|STz-STw\|^p, \\ &\quad \|STz-T^2z\|^p + \|STw-T^2w\|^p, \|STz-T^2w\|^p + \|STw-T^2z\|^p\} \\ &- a_1[\|STz-T^2z\|^p + \|STz-T^2w\|^p] - a_2[\|STw-T^2w\|^p + \|STw-T^2z\|^p]. \end{aligned} \quad (29)$$

Hence,

$$\|z-w\|^p \le \frac{q}{1+a_1+a_2}\max\{c\|z-w\|^p, 0, 2\|z-w\|^p\}$$
$$= \|z-w\|^p \frac{1}{1+a_1+a_2}\max\{cq, 2q\},$$

a contradiction. Hence z = w: This completes the proof of the theorem.

III. Examples

In this section, we give some examples to discuss the validity of the hypothesis and degree of generality of some of our theorems.

Example 3.1 Let N = R^2; the set of all 2-tuples i.e., $x=(x_1,x_2)$ of real numbers and the norm $\|x\|$ is defined by

$$\|x\| = \left(\sum_{i=1}^{2}|x_i|^2\right)^{\frac{1}{2}}, x \in \mathbb{R}^2.$$

Further, let $K=\{x:\|x-O\|\le 1, O,x\in\mathbb{R}^2\}$ and define the mappings $S,T:K\to K$ such that for arbitrary $x=(x_1,x_2)\in K$,

$$Sx=(x_2,x_1),$$

and

$$Tx=(-x_1,-x_2).$$

Suppose $\{Sx_{n,k}\}$ be a sequence of elements of K satisfying condition (6) with $u_{n,k}=0$, where

$$\lambda_n = 1-\frac{n}{2n+1} \text{ and } \mu_n = \frac{n+3}{2n+3} \text{ for } n\ge 0.$$

Consider, x1;1 = (0:5; 0) $\in K$, then it is easy to see that x2;2 = (0:166667;¡0:333333) and x3;3 = (0:21903;¡0:2810981) ect:

Now it is easy to see that all conditions of Theorem 2.1 are satisfied, for instance taking x = x1;1 and y = x2;2 then, we have 0:4713996 $\le \alpha+\gamma+\delta<1$, which is true, since $0\le\alpha+\gamma+\delta<1$. Also, 0 is the unique common fixed point of S and T:

Example 3.2-Let K = f1; 2; 3g be a finite set with a metric d given by

d(1; 1) = d(2; 2) = d(3; 3) = 0;

d(1; 3) = d(2; 3) = 1;

d(1; 2) = 2:

Define S and T on K by,

$$S(1)=S(2)=2,\ \ S(3)=3,\ \ T(1)=T(2)=T(3)=2.$$

It is easy to see that K is closed convex and bounded. Further, an easy routine calculation shows that condition (8) holds for the points x; y $\in K=\{1,2,3\}$ and for instance $0<q-(a_1+a_2)<1$ and $\max\{cq,2q\}<1+a_1+a_2$ when $u_{n,k}=0$.

Therefore all conditions of Theorem 2.2 are satisfied and 2 is the unique common fixed point of S and T:

Example 3.3-Let K = (1; 2) with the discrete metric. Define the mappings S and T on K by

$$S1=S2=1,\ \ T1=T2=2.$$

It is easy to see that all conditions of Theorem 2.2 are satisfied but S and T have no common fixed points

IV. An Application

In this section we will apply the G-iteration process as defined below to find the solution of the equation Tx = f: For this purpose we let X be a Banach space and let $T:D(T)\subset X\to X$ be locally Lipschitz and strongly quasi-accretive. It is proved that a G-iteration process converges strongly to the unique solution of the equation $Tx=f,\ f\in R(T)$.

Theorem 4.1-Let X be a real p¡uniformly smooth Banach space and let $T:D(T)\subset X\to X$ be locally Lipschitz and strongly quasi-accretive operator with open domain D(T) in X such that the equation Tx = f has a solution $x^*\in D(T)$ *for* $f\in R(T)$ arbitrary but fixed. Define $T_\lambda : D(T)\to X$

$$T_\lambda x = x-\lambda(Tx-f)\ \ \forall\ x\in D(T)$$

Then there exist a neighborhood B of x* and a real number $\lambda\in(0,1)$ such that starting with an arbitrary $x_{0,0}\in B$ the G-iteration sequence (x_n) generated by

$$x_{n+1,k+1} = (\mu_n-\lambda_n)x_{n,k}+\lambda_n Tx_{n,k}+(1-\mu_{n,k})Tx_{n-1,k-1}+(1-\mu_n)u_{n,k} \quad for\ n,k\ge 0,$$

where $\{\mu_n\}$ *and* $\{\lambda_n\}$ satisfy

i. $\lambda_0=\mu_0=1$

ii. $0<\lambda_n<1,\ \ 0\le\mu_n\le 1$ *such that* $\mu_n\ge\lambda_n,\ n>0,$

iii. $\lim_{n\to\infty}\lambda_n = h>0,$

iv. $\lim_{n\to\infty}\mu_n = 1.$

remains in B and converges strongly to x¤ with convergence being at least as fast as geometric progression.

Proof- Since T is locally Lipschitz, there is an r > 0 such that T is Lipschitz on

$$B=\bar{B}_r(x_0)=\{x\in X:\|x-x^*\|\le r\}\subset D(T)$$

Let $k\in(0,1)$ and $L>1$ denote the strong accretivity and Lipschitz constant of A respectively. Observe that $f=Tx^*$.

$$h\lambda = \left(\frac{k}{L^pC_p}\right)^{\frac{1}{p-1}}$$

and generate the sequence $x_{n,k}\in B$ as in (6). We now prove that $x_{n,k}\in B,\ \forall\ n,k>0$. Suppose that $x_{n,k}\in B$. Then

$$\|x_{n+1,k+1}-x^*\|^p = \|(\mu_n-\lambda_n)x_{n,k}+\lambda_nTx_{n,k}+(1-\mu_n)(Tx_{n-1}+u_{n,k})-x^*\|^p$$
$$= \|(\mu_n-\lambda_n)x_{n,k}+\lambda_n[x_{n,k}-\lambda(Tx_{n,k}-f)]+(1-\mu_n)[x_{n,k}-\lambda(Tx_{n,k}-f)+u_{n,k}]-x^*\|^p$$
$$= \|x_{n,k}-\lambda\lambda_{n,k}(Tx_{n,k}-Tx^*)-(1-\mu_n)\lambda(Tx_{n,k}-Tx^*)+(1-\mu_n)u_{n,k}-x^*\|^p$$
$$= \|x_{n,k}-x^*-(\lambda\lambda_n+(1-\mu_n)\lambda)(Tx_{n,k}-Tx^*)+(1-\mu_n)u_{n,k}\|^p.$$

Now using (1), we obtain

$$\|x_{n+1,k+1}-x^*\|^p$$
$$\le \|x_{n,k}-x^*+(1-\mu_n)u_{n,k}\|^p + p[(\mu_n-1)\lambda-\lambda\lambda_n]\langle Tx_{n,k}-Tx^*, J_p(x_{n,k}-x^*)\rangle$$
$$+C_p[\lambda\lambda_n+(1-\mu_n)\lambda]^p\|Tx_{n,k}-Tx^*\|^p$$
$$= (1-ph\lambda k+L^pC_p(h\lambda)^p)\|x_{n,k}-x^*\|^p+(1-\mu_n)K(u_{n,k},p)$$
$$\le (1-(pk-L^pC_p(h\lambda)^{p-1})h\lambda)\|x_{n,k}-x^*\|^p+(1-\mu_n)K(u_{n,k},p)$$
$$= (1-(p-1)k(\frac{k}{L^pC_p})^{\frac{1}{p-1}})\|x_{n,k}-x^*\|^p+(1-\mu_n)K(\|u_{n,k}\|,p)\le r^p+(1-\mu_n)K(u_{n,k},p),$$

Where $K(u_{n,k},p)$ is a function depends on $u_{n,k}$ and p. Now, since $x_{0,0}\in$ B by choice of the initial guess, it follows by the inductive hypothesis that the sequence $\{x_{n,k}\}$ remains in B: Set $\delta^*=\left(1-(p-1)k(\frac{k}{L^pC_p})^{\frac{1}{p-1}}\right)^{\frac{1}{p}}$ and observe that $\delta^*\in(0,1)$, since $k<\frac{LC_p^{\frac{1}{p}}}{(p-1)^{\frac{1}{p-1}}},\ \forall\ 1<p<\infty$. Hence, we obtain

$$\|x_{n,k}-x^*\|^p\le(\delta^*)^{np}\|x_{0,0}-x^*\|^p+(1-\mu_n)K(u_{n,k},p),$$

sinc $\delta^{*np}\to 0$ as $n\to\infty$ the assertions of the theorem follows and the proof is complete.

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Are Kepler's elliptical orbits right?

Necat Taşdelen

***Abstract*: Canonically, it is difficult to change the perception of the community when a new theory is launched in Physics. It seems Kepler's laws about the orbit of the planets around the Sun need some corrections .It is said that Newton's gravitational law predicts Kepler's laws. Does Newton confirmed Kepler personally or some other mathematicians, after Newton, have tricked to make these predictions? I don't know. But, using the same Newton's laws I find the contrary: orbits are not elliptical but expanding and then after compressing spirals along the orbit of the Sun in its galaxy: a spiral on a paraboloid.**

Keywords: Planet's orbit, Newton's laws, Spirals

I. INTRODUCTION

Consider Newton's well-known law F*dt=m*dv expression. (1)

From there we write the energy conservation equation: F*r*dt=m*r*dv (2)

The total energy is: 1/2*m*V^2+m*g*r+1/2*I*w^2=Ct (3)

Where,

I*w^2=Ct as innate.(since the existence)
V is variable
g is variable

Then, at positions #1 and #2 (or #n)

1/2*m*V1^2+m*g1*r1= 1/2*m*V2^2+m*g2*r2=Constant (4)

V is a vector

V^2=Vr^2+Vp^2

Where,

Vr is the radial component
Vp is the perpendicular component, so that

Vorbital^2=Vradial^2+Vperpendicular^2 (5)

When eliminating I*w^2=Ct, that means

1/2*m*Vr1^2+1/2*m*Vp1^2+m*g1*r1= 1/2*m*Vr2^2+1/2*m*Vp2^2+m*g2*r2 (6)

Where,

g is variable

But, we know from physics,

(In an attraction field the work in the perpendicular direction to the direction of the field equal ZERO.)

Then, the equation of energy conservation (6) is written as:

1/2*m*Vr^2+m*g*r+1/2*I*w^2=m*r*dv (7)

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Consequently,

$$Vp1=Vp2=....=Vpn=Ct \quad (8)$$

And these mean:

Kepler's law about "swept out areas equality in equal time" is wrong. Because Vp is constant.
It is said that Newton also confirmed this "area law". And mathematicians attribute to Newton the expression (r*Vp=Ct).

This I don't believe. Vp is constant and not r*Vp. (9)

When continuing, and writing the differentials, the expression (7)
1/2*m*Vr^2+m*g*r+1/2*I*w^2=m*r*dv is written as
1/2*m*(dr/dt)^2+K/dt^2=m*r*d(dr/dt)/dt (10)
Where ,
(K/dt^2=m*g*r+1/2*I*w^2) and
r'^2+2*K/m/dt^2=2*r*r'' the differential equation is found. (11)

The solution of this differential equation is
r=-a*t^2+a*t*tmax+K
Where, (12)
K=-a*tmax^2/4=Constant
tmax is the total life time of the planet.

The expression (12) r=-a*t^2+a*t*tmax+K does not show any sign of ellipse. This is a parabola on Cartesian or a cardioidal looking spiral on Polar plane.
This expression (12) is the equation of the planetary motion, mathematically.

II. PHYSICALLY

Planets do not orbit the Sun on elliptical shapes , but on spiraled shapes: with the SUN at the barycenter of the spiral and having only one maximum point all along the life time (tmax) of the planet.Fig.1

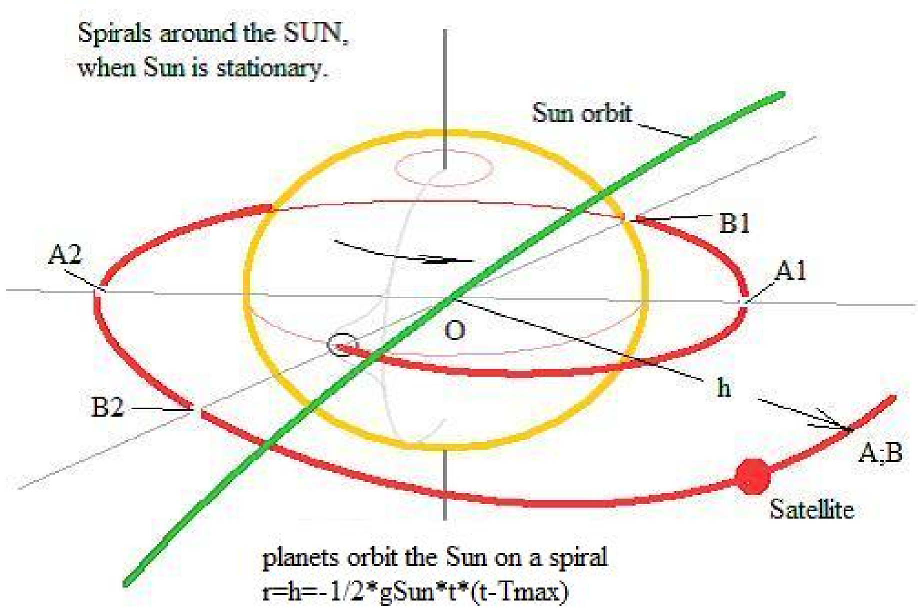

Fig.1-As the SUN is orbiting its galaxy, our Milky-Way, this planetary motion is on a paraboloid.

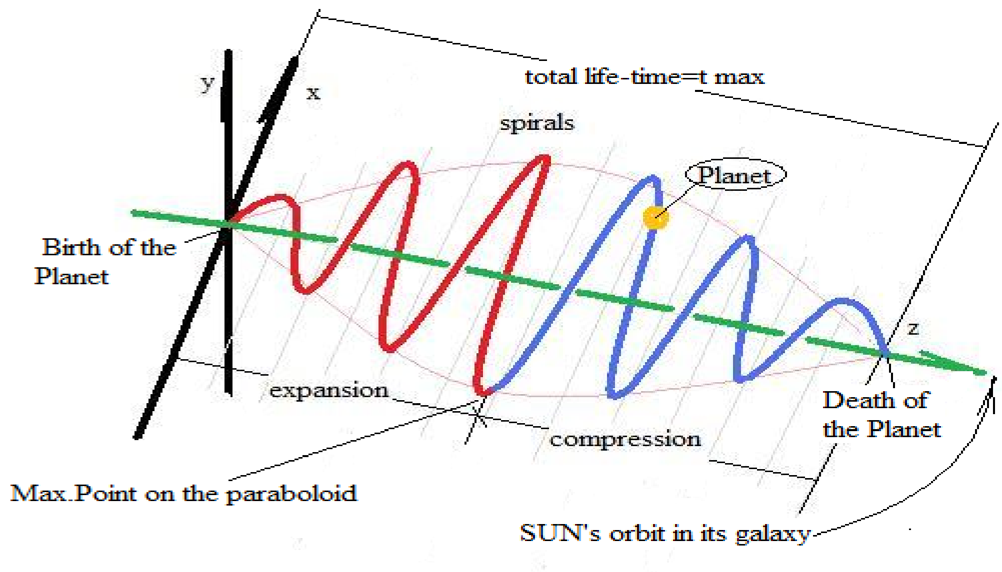

Fig.2

A simulation of the spirals for t=tmax=3 cycles is shown on Fig.3. On a plane figure the paraboloid will be represented by a sinus (left side) with variable amplitude and on the right side the view will be a cardioidal looking spiral. The velocity of the planet along the orbit of the SUN will be its translation velocity by the SUN. This velocity is constant like the orbital, innate, velocity Vp of the planet around the SUN.

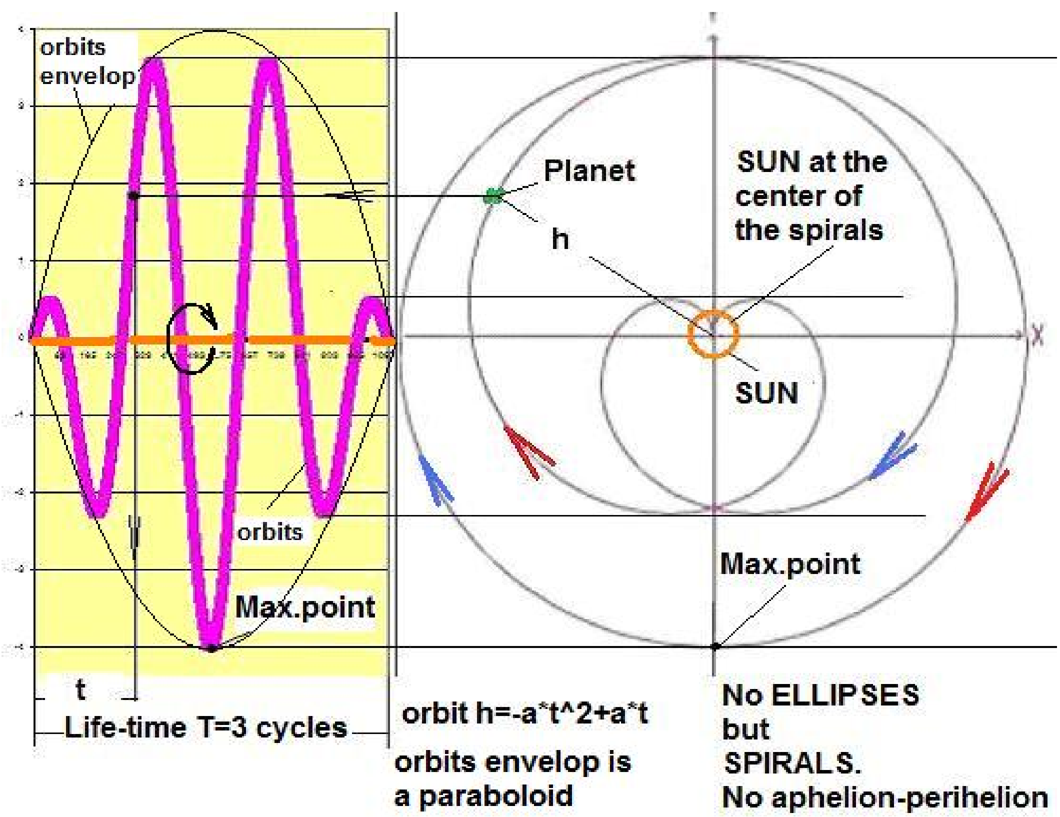

Fig.3

Variable amplitude creates variable eccentricity, in fact eccentricity do not exist: ellipse do not exist. Amplitudes have gone from zero to a max. value (expansion half life-time) and then will go to zero (compression half life-time). This theory accept that the planets have been created from a “small bang” of the SUN, billions years ago. They were ejected from the mass of the SUN. Kepler says the planets were at their actual position since the beginning and they will stay on their elliptical rigid orbit for eternity. No.
The Universe is not a dead media but living, changing. Elliptical orbits are mathematically blocked, rigid, invariable, dead orbits. Spiraled orbits are living. They have a life-time: billions of cycles around the SUN.

III. VARIABLE AMPLITUDES

Mean variable distances of the planet to the SUN. Then aphelion, perihelion reasoning is not valid. There is not a constant distance, repeating each cycle of the planet around the SUN. All distances are variable. An aphelion distance will be the perihelion distance after half cycle of the planet around the SUN, when on expansion mode.
And vice-versa, when on the compression mode.

Actually, for our EARTH, the distances to the SUN are so close in each cycle that anyone may comments the shape of the orbits in a wrong way, and claim for elliptical orbits as done Kepler. But Kepler corrected his wrong claim by the law of the periods which Newton confirmed ONLY for circular orbits. Probably Kepler discovered that the elliptical orbits will be transformed to circular orbits, with time. Then the orbits are not elliptical if their shape tends to be circular. An ellipse is an ellipse. A circular is a circular. A spiraled orbit, enveloped by a paraboloid, will reach to a circular form at its max.point; an instant circular shape and then the compression mode will start.

Table. I. resumes and compares Keplerian orbits with spiraled orbits for our EARTH.
The equation of the planetary motion (12) r=-a*t^2+a*t*tmax+K is written as y=c*x^2+d*x+0
For y=1 ; x=0,5 (unit equation) and (dx steps)=0,5/k are considered for k=631257 excel working lines.(arbitrary)
Then the angle step=0,2085607 is found to fit the actual Keplerian eccentricity e=0,016710218…

For Earth's orbit assumed to be a spiral			Keplerian	
aEarth=	1		definitions	Evaluations
eccentr=e Earth	0,016710219		a2=aphelion	1,000000000
bEarth=	0,999860375		b2	0,999953463
We choose k=	63257	working lines	a1=perihelion	0,999813850
Projectile trajectory on Cartesian.		y=r=c*x^2+d*x	b1	0,999581163
c=	-4		a=(a2+a1)/2	0,999906925
d=	4		b=(b2+b1)/2	0,999767313
(For x=0,5 ;y=1) dx=	7,90426E-06		eccen. e=	0,016710218
d angle=	0,2085607	approx.evalua.		
k=counter	**x=k*dx**	**y=r**	**Angle o**	
0	0	0	0	
1	7,90426E-06	3,16168E-05	0,2085607	
2	1,58085E-05	6,32331E-05	0,4171214	
3	2,37128E-05	9,48489E-05	0,6256821	
4	3,16171E-05	0,000126464	0,8342428	
5	3,95213E-05	0,000158079	1,0428035	
63252	0,499960479	0,999999994	13191,8814	
63253	0,499968383	0,999999996	13192,08996	
63254	0,499976287	0,999999998	13192,29852	
63255	0,499984191	0,999999999	13192,50708	
63256	0,499992096	1	13192,71564	
63257	0,5	1	13192,9242	

Table. I. (Ref: 2010.06.30-1201)

The Keplerian evaluations (a2,b2,a1,b1) are realized with Table II. Just go 90 degrees back from a2, evaluate (k), find (x), find (y=r=a2;b2;a1;b1).

Points	angle values o	k=counter	x	y=r
b1	12922,9242	61962,41286	0,489767242	0,999581163
a1	13012,9242	62393,94191	0,493178161	0,99981385
b2	13102,9242	62825,47095	0,496589081	0,999953463
a2	13192,9242	63257	0,5	1

Table.II.

Time equation: Ref: 2010.04.01-1321

From the expression (12) r=-a*t^2+a*t*t.max+K, with reference K=0, we solve (t)
t=t.max/2 (+,-) (t.max^2/4-r/a)^(1/2) (13)

When r=r.max; t=t.max/2 and then
(t.max^2/4=r.max/a) and (14)

a=4*r.max/t.max^2 (15)

With unit values (that is [r.max=1 and t.max=1])
When r=1, t=0,5 and a=4 (16)

IV. Scientists

Have calculated the age of the Earth to be 4 600 000 000 years at the date 2009
Then, Table III is prepared. That is:
When r=a1 ; time for a1=4600000000 years and x=fictive unit time=0,493178161 units
Then time for xi =xi*4600000000/0, 493178161
For x=1, tmax=1*4600000000/0, 493178161=9 327 258 099 years (our time's year)

x	**t=Total life-time=2*t for a2**	9327258099	
0,489767242	t for b1	4568185475	
0,493178161	t for a1	4600000000	Scientists evaluation
0,496589081	t for b2	4631814525	
0,5	t for a2	4663629050	

Table. III.

V. Scientists

Have also **measured** the distance of the Earth to the Sun=149 597 890 km.in 2009. Then we may prepare the Table IV: the Points column indicate some special dates. Period's column is about the comparison of the length of a year when 2009 is 1 year.(365 days).

It is evident, year's days are variable: for example,

When the real time is 100 000 000 years, (one cycle around the Sun)=0,0424331744 years=15,488 days
Accurate distances should be known to confirm the following table.

date relative	real time=t	Point	r formular	r corresponding=km	periods
-4599997991	0		0,0000E+00	0,000000	0,000000000000
-4599997990	1		3,7309E+10	0,064167	0,000000000429
-4599997891	100		3,7309E+12	6,416709	0,000000042893
-4599996991	1000		3,7309E+13	64,167079	0,000000428930
-4599987991	10000		3,7309E+14	641,670171	0,000004289300
-4599897991	100000		3,7309E+15	6416,639790	0,000042892582
-4598997991	1000000		3,7305E+16	64160,206331	0,000428884434
-4589997991	10000000		3,7269E+17	640982,906250	0,004284705528
-4499997991	100000000		3,6909E+18	6347913,356248	0,042433174400
-3599997991	1000000000		3,3309E+19	57287562,936941	0,382943656070
-2599997991	2000000000		5,8618E+19	100816080,039350	0,673913783405
-1000	4599996991		8,6982E+19	149597887,365633	0,999999982390
-500	4599997491	Thales	8,6982E+19	149597887,803390	0,999999985317
0	4599997991	Christ	8,6982E+19	149597888,241143	0,999999988243
850	4599998841	El-Harezmi	8,6982E+19	149597888,985315	0,999999993217
1299	4599999290	Osman Bey	8,6982E+19	149597889,378409	0,999999995845
1453	4599999444	Fatih	8,6982E+19	149597889,513234	0,999999996746
1609	4599999600	Kepler	8,6982E+19	149597889,649809	0,999999997659
1881	4599999872	Atatürk	8,6982E+19	149597889,887939	0,999999999251
2009	4600000000	This year	8,6982E+19	149597890,000000	1,000000000000
2010	4600000001	Coming year	8,6982E+19	149597890,000875	1,000000000006
	4663629050	Max.Point	8,6998E+19	149625742,821763	1,000186184590
	6300000000		7,6287E+19	131204376,295209	0,877046971018
	6400000000		7,4938E+19	128884090,775701	0,861536822315
	6500000000		7,3509E+19	126426214,797849	0,845106938325
	6600000000		7,2000E+19	123830748,361650	0,827757319048
	6900000000		6,6992E+19	115218806,302984	0,770190049492
	7200000000		6,1265E+19	105368550,119210	0,704345162350
	8000000000		4,2472E+19	73047220,128618	0,488290443994
	9327258099		3,7309E+10	0,064167	0,000000000429
	9327258100		0,0000E+00	0,000000	0,000000000000

Table. IV.

VI. OLD TESTAMENT

Affirms Adam has lived 930 years and we understand 930*365=339 450 days. No!
When Adam does have lived? We don't know. But we can estimate that he lived long time. How long? Compared to our era, we may say he may have lived 100 years in the condition of date 2009.
100 years=365*100=36500 days
When 930 years=36500 days?
When the period was=100/930=0,107526881
At what real time we got this period?
4 342 185 804 years ago, at real time 257 812 187, when 1 year was =39,2473 days,
930 years=930*39,2473=36500 days
Then we may deduct that "homo sapiens" existence started at the real time 257 814 196=257812187+2009.
This may be against "habitable zone" theory.
Also we may calculate how many days existed in Kepler time? And more! Table V.

In the time of Kepler there were 365*0,999999997659=364,999999145578 days /year.
For the last 10 000 000 years, year's length did not changed too much =364,9744858 days
For the coming 63 629 050 years, year's length will reach to=365,0679574 days.(ref days =365 in 2009)

date	real time	Point	r Formular	r real=km	period
t1	4600000000,00	r1,0	8,6982E+19	149597890,000000	1,000000000000
3 months later	4600000000,25	r1,25	8,6982E+19	149597890,000219	1,000000000001
6 months later	4600000000,50	r1,5	8,6982E+19	149597890,000438	1,000000000003
9 months later	4600000000,75	r1,75	8,6982E+19	149597890,000657	1,000000000004
t2	4600000001,00	r2,0	8,6982E+19	149597890,000875	1,000000000006
3 months later	4600000001,25	r2,25	8,6982E+19	149597890,001094	1,000000000007
6 months later	4600000001,50	r2,5	8,6982E+19	149597890,001313	1,000000000009
9 months later	4600000001,75	r2,75	8,6982E+19	149597890,001532	1,000000000010
t Kepler	4599999600,00	r.Kepler	8,6982E+19	149597889,649809	0,999999997659
					364,999999145578

Table .V.

VII. WHAT ABOUT AREAS

Swept out in equal interval of time?

Considering Kepler's elliptical orbits, the area swept out in 2009 should be equal to the area swept out in 2010.Table VI shows that this is not correct.

	r formular	corresponding real (km;km^2)
a1	8,69815E+19	149597890,000219
a2	8,69815E+19	149597890,001094
b1	8,69815E+19	149597890,000438
b2	8,69815E+19	149597890,001313
Area1; in 2009	pi*a1*b1	70307362931318100,00
Area2; in 2010	pi*a2*b2	70307362932141000,00
Area difference		822888,00
Turkey's area	814578	1,010201601
Area1	pi*r1^2 approx.	70307362931318100,00
Area2	pi*r2^2 approx.	70307362932141000,00
Area difference		822896,00

Table.VI.

VIII. THE MOST

Trendy question, since Kepler era.
What is the length of the orbit for 1 cycle of the Earth around the Sun.
According Kepler and other mathematician this is the perimeter of an ellipse. No! Table VII gives the evaluation steps of these lengths; length adds due to translation along the orbit of the Sun is neglected.

Length of the spiral,on plane		
r=-a*time*(time-Tmax)		
Tmax=	9327258100	
real time in 2009 t1	4600000000	
real time in 2010 t2	4600000001	
r real in 2009 (km)	149597890	
we write	r=149597890=-a*4600000000*(4600000000-Tmax)	

acceleration factor a=	6,879522917266270E-12	is found
let the years be	2*pi*n	
d year=	2*pi*dn	is written;then we write
	r=-a*2*pi*n*(2*pi*n-2*pi*nmax)	
Tmax=	2*pi*nmax	
r=-A*(n^2-n*nmax)	and	dL=r*dn
	L=Integral[n1 to n2]B*(n^3/3-nmax*n^2/2)	
L/B=	2,17454E+19	
L1=2*Pi*r (in 2009)	939951264,433068000	approximately
L2=2*Pi*r (in 2010)	939951264,438568000	approximately
L=(L1+L2)/2	939951264,435818000	
B=	-4,32253E-11	is found
Example		
L=Integral [t1 to t2] B*[t^3/3-T*t^2/2]		
n1=t1	4663629049	
n2=t2	4663629050	Max.point
L real=	940126162,9	
Lapproximate=	940126268,9	km

Table.VII.

IX. What means light-year?

It means nothing, it has no sense; years length (evaluated as number of days) is not constant. Days are constant Because , (I*w^2=Constant, innate). Big distances should be evaluated with “light-day” or 1000 LD.

X. Conclusion

The Orbit of the planets is not elliptical but spiraled; expanding and then after compressing.

There is no aphelion, no perihelion, nor equality of swept out areas in equal interval of time. The Sun is not at a focus of an ellipse but at the barycenter of the solar system. There are no ellipses. There is only one max.point, for the total life-time of the planet.

And the trendy problem of orbit’s perimeter evaluation is closed. Mathematical ellipse’s perimeter evaluation is a different problem: there, the arc length on positive Cartesian L1^s=a^s+b^s and is due to Thales theorem.

XI. Reference

No reference as this is an original work; will be used as reference for researchers.

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Final Points:

A purpose of organizing a research paper is to let people to interpret your effort selectively. The journal requires the following sections, submitted in the order listed, each section to start on a new page.

The introduction will be compiled from reference matter and will reflect the design processes or outline of basis that direct you to make study. As you will carry out the process of study, the method and process section will be constructed as like that. The result segment will show related statistics in nearly sequential order and will direct the reviewers next to the similar intellectual paths throughout the data that you took to carry out your study. The discussion section will provide understanding of the data and projections as to the implication of the results. The use of good quality references all through the paper will give the effort trustworthiness by representing an alertness of prior workings.

Writing a research paper is not an easy job no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record keeping are the only means to make straightforward the progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear

· Adhere to recommended page limits

Mistakes to evade

- Insertion a title at the foot of a page with the subsequent text on the next page
- Separating a table/chart or figure - impound each figure/table to a single page
- Submitting a manuscript with pages out of sequence

In every sections of your document

· Use standard writing style including articles ("a", "the," etc.)

· Keep on paying attention on the research topic of the paper

· Use paragraphs to split each significant point (excluding for the abstract)

· Align the primary line of each section

· Present your points in sound order

· Use present tense to report well accepted

· Use past tense to describe specific results

· Shun familiar wording, don't address the reviewer directly, and don't use slang, slang language, or superlatives

· Shun use of extra pictures - include only those figures essential to presenting results

Title Page:

Choose a revealing title. It should be short. It should not have non-standard acronyms or abbreviations. It should not exceed two printed lines. It should include the name(s) and address (es) of all authors.

Abstract:

The summary should be two hundred words or less. It should briefly and clearly explain the key findings reported in the manuscript--must have precise statistics. It should not have abnormal acronyms or abbreviations. It should be logical in itself. Shun citing references at this point.

An abstract is a brief distinct paragraph summary of finished work or work in development. In a minute or less a reviewer can be taught the foundation behind the study, common approach to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Yet, use comprehensive sentences and do not let go readability for briefness. You can maintain it succinct by phrasing sentences so that they provide more than lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study, with the subsequent elements in any summary. Try to maintain the initial two items to no more than one ruling each.

- Reason of the study - theory, overall issue, purpose
- Fundamental goal
- To the point depiction of the research
- Consequences, including definite statistics - if the consequences are quantitative in nature, account quantitative data; results of any numerical analysis should be reported
- Significant conclusions or questions that track from the research(es)

Approach:

- Single section, and succinct
- As a outline of job done, it is always written in past tense
- A conceptual should situate on its own, and not submit to any other part of the paper such as a form or table
- Center on shortening results - bound background information to a verdict or two, if completely necessary
- What you account in an conceptual must be regular with what you reported in the manuscript
- Exact spelling, clearness of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else

Introduction:

The **Introduction** should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable to comprehend and calculate the purpose of your study without having to submit to other works. The basis for the study should be offered. Give most important references but shun difficult to make a comprehensive appraisal of the topic. In the introduction, describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will have no attention in your result. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here. Following approach can create a valuable beginning:

- Explain the value (significance) of the study
- Shield the model - why did you employ this particular system or method? What is its compensation? You strength remark on its appropriateness from a abstract point of vision as well as point out sensible reasons for using it.
- Present a justification. Status your particular theory (es) or aim(s), and describe the logic that led you to choose them.
- Very for a short time explain the tentative propose and how it skilled the declared objectives.

Approach:

- Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done.
- Sort out your thoughts; manufacture one key point with every section. If you make the four points listed above, you will need a least of four paragraphs.
- Present surroundings information only as desirable in order hold up a situation. The reviewer does not desire to read the whole thing you know about a topic.
- Shape the theory/purpose specifically - do not take a broad view.

- As always, give awareness to spelling, simplicity and correctness of sentences and phrases.

Procedures (Methods and Materials):

This part is supposed to be the easiest to carve if you have good skills. A sound written Procedures segment allows a capable scientist to replacement your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt for the least amount of information that would permit another capable scientist to spare your outcome but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section. When a technique is used that has been well described in another object, mention the specific item describing a way but draw the basic principle while stating the situation. The purpose is to text all particular resources and broad procedures, so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step by step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

- Explain materials individually only if the study is so complex that it saves liberty this way.
- Embrace particular materials, and any tools or provisions that are not frequently found in laboratories.
- Do not take in frequently found.
- If use of a definite type of tools.
- Materials may be reported in a part section or else they may be recognized along with your measures.

Methods:

- Report the method (not particulars of each process that engaged the same methodology)
- Describe the method entirely
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures
- Simplify - details how procedures were completed not how they were exclusively performed on a particular day.
- If well known procedures were used, account the procedure by name, possibly with reference, and that's all.

Approach:

- It is embarrassed or not possible to use vigorous voice when documenting methods with no using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result when script up the methods most authors use third person passive voice.
- Use standard style in this and in every other part of the paper - avoid familiar lists, and use full sentences.

What to keep away from

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings - save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part a entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Carry on to be to the point, by means of statistics and tables, if suitable, to present consequences most efficiently.

You must obviously differentiate material that would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matter should not be submitted at all except requested by the instructor.

Content

- Sum up your conclusion in text and demonstrate them, if suitable, with figures and tables.
- In manuscript, explain each of your consequences, point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation an exacting study.
- Explain results of control experiments and comprise remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or in manuscript form.

What to stay away from

- Do not discuss or infer your outcome, report surroundings information, or try to explain anything.
- Not at all take in raw data or intermediate calculations in a research manuscript.
- Do not present the similar data more than once.
- Manuscript should complement any figures or tables, not duplicate the identical information.
- Never confuse figures with tables - there is a difference.

Approach

- As forever, use past tense when you submit to your results, and put the whole thing in a reasonable order.
- Put figures and tables, appropriately numbered, in order at the end of the report
- If you desire, you may place your figures and tables properly within the text of your results part.

Figures and tables

- If you put figures and tables at the end of the details, make certain that they are visibly distinguished from any attach appendix materials, such as raw facts
- Despite of position, each figure must be numbered one after the other and complete with subtitle
- In spite of position, each table must be titled, numbered one after the other and complete with heading
- All figure and table must be adequately complete that it could situate on its own, divide from text

Discussion:

The Discussion is expected the trickiest segment to write and describe. A lot of papers submitted for journal are discarded based on problems with the Discussion. There is no head of state for how long a argument should be. Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implication of the study. The purpose here is to offer an understanding of your results and hold up for all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of result should be visibly described.
Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved with prospect, and let it drop at that.

- Make a decision if each premise is supported, discarded, or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

- Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work
- You may propose future guidelines, such as how the experiment might be personalized to accomplish a new idea.
- Give details all of your remarks as much as possible, focus on mechanisms.
- Make a decision if the tentative design sufficiently addressed the theory, and whether or not it was correctly restricted.
- Try to present substitute explanations if sensible alternatives be present.
- One research will not counter an overall question, so maintain the large picture in mind, where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

- When you refer to information, differentiate data generated by your own studies from available information
- Submit to work done by specific persons (including you) in past tense.
- Submit to generally acknowledged facts and main beliefs in present tense.

Administration Rules Listed Before Submitting Your Research Paper to Global Journals

Please carefully note down following rules and regulation before submitting your Research Paper to Global Journals:

Segment Draft and Final Research Paper: You have to strictly follow the template of research paper. If it is not done your paper may get rejected.

- The **major constraint** is that you must independently make all content, tables, graphs, and facts that are offered in the paper. You must write each part of the paper wholly on your own. The Peer-reviewers need to identify your own perceptive of the concepts in your own terms. NEVER extract straight from any foundation, and never rephrase someone else's analysis.
- Do not give permission to anyone else to "PROOFREAD" your manuscript.

Written Material: You may discuss with your guides and key sources.

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- To guard yourself and others from possible illegal use please do not permit anyone right to use to your paper and files.

Criterion for Grading a Research Paper *(Compilation)*
by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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ISSN 9755896