Algorithm For Finding The Proper Continuous Distribution Function To A Unimodal Empirical Distribution Function

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Abstract- **We present an algorithm containing the first step towards solving one of the Fundamental Problems of Non-parametric Mathematical Statistics: determining the distribution of an unknown unimodal continuous population from which we have a large random sample of discrete observations. We will present the "algorithm of non-fitting" with the help of which, by using the so-called relative increment functions as auxiliary functions, one can eliminate a large class of classical continuous unimodal distributions which our population does not belong to. In the remaining class of unimodal and smooth distributions one can approximate the distribution of the population in question. The algorithm is illustrated in three numerical examples.**

Keywords- unimodal smooth distribution function,relative increment function, large sample, curve fitting.

I. Introduction

By using only the information obtained from a large random sample of an unknown unimodal continuous population, we can, in some cases, determine the distribution of this population.

Assume UCP denotes an unknown Unimodal Continuous Population with the unknown probability density function f whose open support is the open real interval $I \subseteq (-\infty,+\infty)$. Assume that $f \in C^{\infty}(r,s)$.

Suppose that, from UCP, we have a large random sample of size N spanning the closed real interval J $(\subseteq closure(I))$. Assume we have no other information on our particular UCP. We form an equidistant partition of the sample interval J into n subintervals $I_k = [x_{k-1}, x_k)$ of equal length $x_k - x_{k-1} \equiv a(>0)$, for $k = 1,2,...,n$.

The frequency distribution of the sample will be v_k, $k = 1,2,...,n$ (v_k is the number of sample values belonging to the interval I_k ,for all k).Assume $v_n \geq 1$.

One can form both the relative frequencies $r_k (= \frac{v_k}{N})$ and the cumulative relative frequencies $y_k (= \sum_{j \leq k} r_j)$ which induce the empirical cumulative distribution F_{emp} whose points of discontinuity are at equidistant places x_k , and we have $\boldsymbol{F_{emp}(x_k) = y_k}$ ***for*** $\boldsymbol{k = 0,1,...,n}$ .In addition, we have $F_{emp}(x_k) = y_k < 1$ (for $\boldsymbol{k = 0,1,2,...,n-1}$), since $v_n \geq 1$.

Our aim is to determine the unimodal and infinitely smooth (in I) distribution of our continuous population.

By an infinitely smooth function, we mean a real function which is infinitely many times differentiable in its open support $I \subseteq (r,s) \subseteq (-\infty,+\infty)$.

By a unimodal distribution function, we mean a distribution function whose probability density function is unimodal.

II. Algorithm

Step 1-.[containing "the method of non-fitting" which will allow us to eliminate a relatively large class of unimodal and infinitely smooth distributions which our continuous population does not belong to.]

*W*e consider an arbitrarily chosen "classical" distribution function $F(x) = \int_{-\infty}^{x} f(t)dt$ from the class of Unimodal and infinitely Smooth Distribution Functions (USDF) with at most one point of inflection and $\boldsymbol{F(x) < 1}$ for $x \in I$. We assign our auxiliary functions, the so-called Relative Increment Functions (RIF) to both $\boldsymbol{F_{emp}}$ and $\boldsymbol{F}$, and then we compare the monotonic properties of the RIFs. If they do not match, then we drop $\boldsymbol{F}$, otherwise we keep it (and put it in S).

More precisely, we form the empirical relative *Increment Function-*

$$h_{emp}(x_k) := \frac{y_{k+1} - y_k}{1 - y_k} \quad \text{for} \quad \boldsymbol{k = 0,1,2,...,n-1},$$

and then we check the monotonic behavior of $\boldsymbol{h_{emp}}$.

We keep (and put it in S) the continuous cumulative distribution function $\boldsymbol{F}$ whose relative increment function

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(abbreviated as RIF) $h_F(x) := \frac{F(x+a)-F(x)}{1-F(x)}$ (for $a = x_{k+1} - x_k > 0$) has the same monotonic behavior as h_{emp}

(like, e.g., both relative increment functions h_{emp} and h_F increase,

or both decrease,

or both have a $\cap$ -shape: first increase and then decrease,

or both have a $\cup$ -shape: first decrease and then increase, etc.),

otherwise we drop F .

This way we can eliminate a large part of all known types of unimodal and infinitely smooth classical distribution functions, and we will keep and use a relatively small part S of the USDF functions.

Step 2- We will work with the remaining family S of unimodal and smooth cumulative distribution functions and, by using e.g. the Least Square's Method, we select the unimodal and infinitely smooth distribution function F $(\in S)$ providing the best fit to the cumulative relative frequencies y_k : $\sum_{k=1}^{n}[F(x_k)-y_k]^2 \to \min!$

[or, if F does not have an explicit form, then we choose F whose density function f provides the best fit to the relative frequencies r_k : $\sum_{k=1}^{n}[f(x_k)-r_k]^2 \to \min!$].

If there are more best-fit candidates for F , then Chi-Squer Test (or Kolmogorov's Test) applies for testing goodness of fit ; furthermore, we will compare h_{emp} and h_F , and choose the "best F " in the sense of $\min_F \sum_{k=1}^{n}\left[h_F(x_k)-h_{emp}(x_k)\right]^2$.

Remark 1- Our method is completely different from Density Estimation Methods like Average Shifted Histograms, Smoothing Splines etc.(Venables and Ripley 1994), because the result of our algorithm is an infinitely smooth (in the support I) distribution function with at most one point of inflection, one of the "classical" continuous distribution functions mentioned by Feller (1971), Johnson and Kotz (1970), Stuart and Ord (1987).

Remark 2-Dental researchers have discovered the importance of relative increment of decay or RID Index long time ago. RID Index was first defined by Porter and Dudman (1960), and then used intensively by the author of this paper and co-authors (Sobkowiak, Szabo, Radtke and Adler 1973; Adler and Szabo 1974a,b, 1975, 1976, 1979a,b, 1984; Szabo 1974, 1976, 1989, 1994, 1996, 1999, 2004, 2006; Szondy and Szabo 1976).

Remark 3-. The hazard rate of a continuous distribution function F , being defined to be $\frac{f}{1-F} = \lim_{a \to 0} \frac{h_F(x)}{a}$ (Barlow and Proschan 1967) , is closely related to the RIF of F . Clear that the hazard rate and the relative increment function of F have the same monotonic behavior.

Remark 4.-This algorithm has a disadvantage. It can handle only samples whose empirical RIF has a relatively simple shape that is either increasing, or decreasing, or it has only two monotonic phases

[increases first and, having culminated, it decreases; or decreases first and, having reached its minimum, it increases].

The main research problem was to find sufficient conditions with the help of which one can easily check the monotonic behavior of the relative increment function h_F of a distribution function F [without using the inconvenient term $1-F$ $(= \int_x^\infty f(t)dt)$ appearing in h_F]. These sufficient conditions have been worked out by Adler and Szabo (1974a), and Szabo (1976, 1989, 1994, 1996, 1999, 2006). These conditions contain relatively simple assumptions under which the RIF h_F of the unimodal and infinitely smooth distribution function F increases or decreases, or has two monotonic phases. These assumptions deal with the simple expression f / f' .

These results have been applied to logistic, extreme value, z- , Pareto of the third kind, Weibull, trigonometric, normal, gamma, beta of the first kind, Pareto of the second kind, chi-square and many other distributions [See the Propositions below.]

Monotonic behavior of the relative increment functions of some unimodal and infinitely smooth distribution functions from Feller, W. (1971), Gradshteyn and Ryzhik (1980), Johnson and Kotz (1970), Stuart and Ord (1987) have been checked, and some of the results in Szabo (1976, 1996, 1999, 2006) are summed up as follows.

Examples of Unimodal and infinitely Smooth Distribution Functions (USDF), and monotonic properties of the corresponding Relative Increment Functions (RIF) are given in the following five Propositions.

Prop.1. The following USDFs have increasing RIFs

1. $F(x) = 1 - x^2, \quad I = (-1,0)$;
2. $F(x) = 1-(-x)^k, k > 1$ integer, $I = (-1,0)$;
3. $F(x) = \sin x$; $\quad I = (0, \pi/2)$;
4. $F(x) = 1 + \tan x, I = (-\pi/4, 0)$;
5. $F(x) = 1 + sh(x), I = (\ln(\sqrt{2}-1), 0)$;
6. $F(x) = 2 - ch(x); \quad I = (\ln(2-3^{1/2}), 0)$;
7. $F(x) = \ln x, \quad I = (1, e)$;

8. Uniform distribution;
9. Any convex (below) USDF.
10. $F(x)=(1-\exp(-\lambda x))^k, \lambda>0, k>1, I=(0,\infty).$
11. $F(x)=1-\exp(-\lambda\cdot e^x), \lambda>0,\ I=(-\infty,\infty).$
12. $F(x)=(1+e^{-x})^{-k}, k>0,\ I=(-\infty,\infty).$
13. $F(x)=2^{-k}\cdot(1+th(x))^k, k>0,\ I=(-\infty,\infty).$
14. Logistic Distribution $F(x)=(1+e^{-\lambda x})^{-1}, \lambda>0, I=(-\infty,\infty).$
15. Fisher's z-distribution

$$F(x)=C\cdot\int_{-\infty}^{x} e^{nt}\cdot(1+k\cdot e^{2t})^{-\alpha}dt,$$

$$k=\frac{n}{n'} \qquad \alpha=\frac{n+n'}{2}(>0) \quad ,$$

$$(0<)C=2\cdot k^{n/2}\cdot\Gamma(\alpha)\cdot\left[\Gamma\left(\frac{n}{2}\right)\cdot\Gamma\left(\frac{n'}{2}\right)\right]^{-1}$$

and n, n' are positive integers, $I=(-\infty,\infty).$

16. Weibull distribution when $\alpha>1, F(x)=1-\exp\left(-\lambda\cdot x^{\alpha}\right), \lambda>0, \quad I=(0,\infty).$
17. Extreme value distribution

$$F(x)=\int_{-\infty}^{x}\exp(-t-e^{-t})dt, \qquad I=(-\infty,\infty).$$

18. $F(x)=1-2[c\cdot(1+e^x)^k-c+2]^{-1}, c>0, k=1,2;$ $I=(-\infty,\infty).$
19. Normal distribution

$$F(x)=K\cdot\int_{-\infty}^{x}\exp(-1/2\cdot\sigma^{-2}\cdot(t-m)^2)dt, K=\sigma^{-1}\cdot(2\pi)^{-1/2}, \sigma>0, I=(-\infty,\infty).$$

20. (Special) Gamma distribution

$$F(x)=K\cdot\int_{0}^{x}t^{\alpha-1}\cdot\exp(-\lambda t)dt, K=\lambda^{\alpha}/\Gamma(\alpha), \lambda>0, \alpha>1, I=(0,\infty).$$

21. Beta distribution of the first kind

$$F(x)=C\cdot\int_{0}^{x}t^{\alpha}\cdot(1-t)^{\beta}dt, C=\Gamma(\alpha+\beta+2)/[\Gamma(\alpha+1)\cdot\Gamma(\beta+1)], \alpha,\beta>-1, I=(0,1).$$

22. $F(x)=C\cdot\int_{-s}^{x}(1-t^2/s^2)^n dt, C=[s\cdot B(1/2, n+1)]^{-1}, s>0,$

n is a positive integer, $I=(-s,s)$.

Prop.2. - The following USDFs have DECREASING RIFs:

23. $F(x)=1-x^{-\lambda}, \lambda>0, I=(1,\infty)$;
24. $F(x)=1-(\ln x)^{-\lambda}, \lambda>0, I=(e,\infty)$;
25. $F(x)=1-(\ln\ln x)^{-\lambda}, \lambda>0, I=(e^e,\infty)$;
26. Weibull distribution when $0<\alpha<1$,

$F(x)=1-\exp(-\lambda\cdot x^{\alpha}), \lambda>0, \quad I=(0,\infty)$;

27. $F(x)=1-a\cdot\exp(-bx)-c\cdot\exp(-dx), a,b,c,d>0, a+c=1,$ $I=(0,\infty)$;
28. $F(x)=1-\sum_{j=1}^{N}a_j\cdot\exp(-b_j x), \qquad a_j,b_j>0, \qquad \sum_{j=1}^{N}a_j=1,$ $I=(0,\infty)$;
29. Pareto distribution of the third kind

$F(x)=1-k\cdot\exp(-bx)\cdot x^{-a}, a,b,k>0,$ $I=(k,\infty)$;

30. Chi-Square distribution

$$F(x)=K\cdot\int_{0}^{x}t^{n/2-1}\cdot\exp(-t/2)dt, K=2^{-n/2}/\Gamma(n/2),$$

n is a positive integer, $I=(0,\infty)$;

31. Pareto distribution of the second kind

$F(x)=1-x^{-k}, k>0, I=(1,\infty).$

32. (Special) Gamma distribution

$$F(x)=K\cdot\int_{0}^{x}t^{\alpha-1}\cdot\exp(-\lambda t)dt, K=\lambda^{\alpha}/\Gamma(\alpha), \lambda>0, \alpha<1, I=(0,\infty).$$

Prop.3. - The Exponential Distribution Function $F(x)=1-\exp[-\lambda(x-a)], \lambda>0, \quad I=(a,\infty)$ has a *CONSTANT* Relative Increment Function, and this property of RIF is characteristic for the Exponential Distribution.

Prop.4- The following USDFs have RIFs that INCREASE first and, having culminated, they DECREASE:

34. Cauchy distribution $F(x)=\frac{1}{2}+\frac{1}{\pi}\cdot\mathbf{arctx}$, $I=(-\infty,\infty)$.
35. Inverse Gaussian distribution

$$F(x)=\int_{0}^{x}\left(2\pi t^3/\lambda\right)^{-1/2}\cdot\exp(-\lambda\cdot(t-m)^2/(2m^2\cdot t))dt, \lambda,m>0, I=(0,\infty).$$

36. Lognormal distribution

$$F(x)=K\cdot\int_{0}^{x}\exp(-1/2\cdot\sigma^{-2}\cdot(\ln t)^2)/t dt, K=\sigma^{-1}\cdot(2\pi)^{-1/2}, K=1/\left[\sigma\cdot\sqrt{2\pi}\right], \sigma>0, I=(0,\infty).$$

Prop.5.- The following USDFs have RIFs that DECREASE first and, having reached their minima, they INCREASE:

37. $F(x)=1+\frac{2}{\pi}\cdot\arcsin x, \quad I=(-1,0)$;

38. $F(x)=\sqrt{x},\ I=(0,1)$;
39. $F(x)=(1-x^2)^{1/2},\ I=(-1,0)$.

Remark 4- By using the transformation $x'=Ax+B,\ (for\ A>0)$, the interval I can be transformed, but the monotonic behavior of RIF over the transformed interval I' will NOT change.

Numerical Example 1.

Time points x_k	Relative frequencies r_k
1	0.7059
2	0.1612
3	0.0621
4	0.0324
5	0.0154
6	0.0092
7	0.0054

Table 1. Data

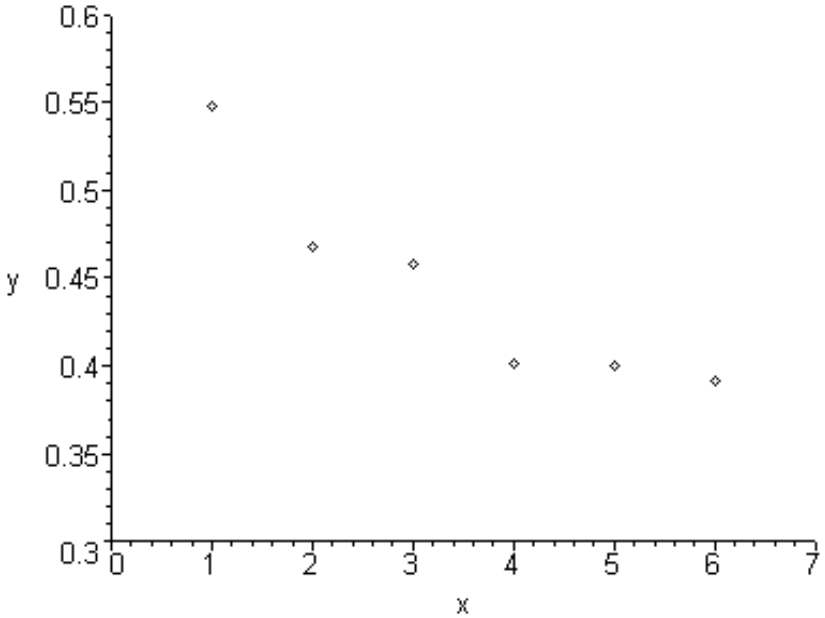

Figure 1. Graph of the corresponding finite sequence of the empirical relative increments $\{(x_k, h_{emp}(x_k))\}$ has a decreasing shape.

This monotonic behavior of the RIF h_F can be found only for USDFs No. 23-32 in Proposition 2. Among these functions, the Weibull distribution function $F(x)=1-\exp[-1.2141(x+0.0771)^{0.6667}]$ provided the best least squares fit $\sum_{k=1}^{7}[F(x_k)-y_k]^2\approx 0.00037$.

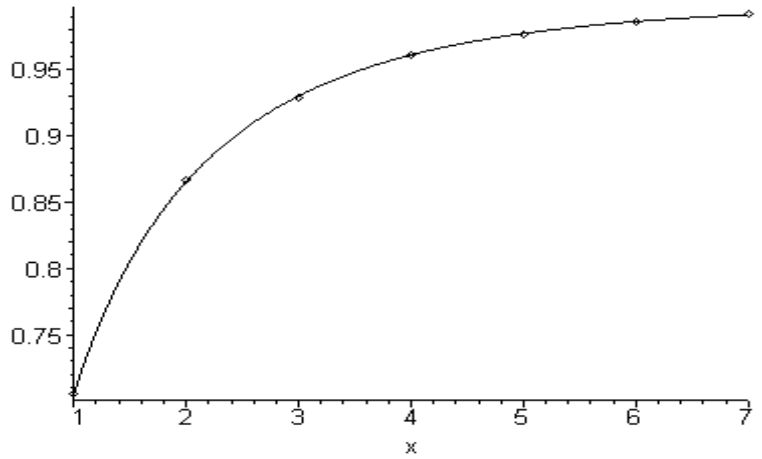

Figure 2. The graphs of F and $\{(x_k, y_k)\}$.

Numerical Example 2. (From Dental Science).

Age x_k	**Cumulative caries prevalence value** y_k
7	0.050
12	0.116
17	0.195
22	0.282
27	0.372
32	0.460
37	0.543
42	0.618
47	0.684
52	0.732
57	0.761
62	0.781

Table 2. Longitudinal data $\{x_k, y_k\}$ collected by Prof.P.Adler, Late Director of Dental Clinics of Debrecen University, Debrecen, Hungary (European Union).

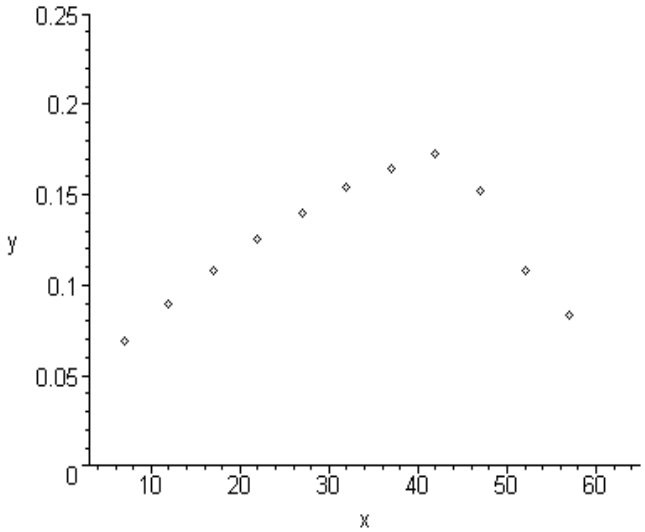

Figure 3. Graph of the corresponding empirical relative increment sequence $\{(x_k, h_{emp}(x_k))\}$.

According to this graph, and the overall clinical experience in Dental Science (Porter and Dudman 1960; Sobkowiak, Szabo, Radtke and Adler 1973; Adler and Szabo 1974a,b, 1975, 1976, 1979a,b, 1984), the relative increments of decay [i.e. the RIF of cumulative caries prevalence curve] increases up to a certain age, and decreases thereafter. This monotonic behavior of the RIF h_F can be found only for USDFs No. 34-36 in Proposition 4. Among these three functions, the truncated Cauchy distribution function $F(x) = \max\{1 + 1.48939\left(-0.5 + \frac{1}{\pi}\arctan(0.042887x - 0.84988)\right), 0\}$ provided the best least squares fit $\sum_{k=1}^{12}[F(x_k) - y_k]^2 \approx 0.0053478$

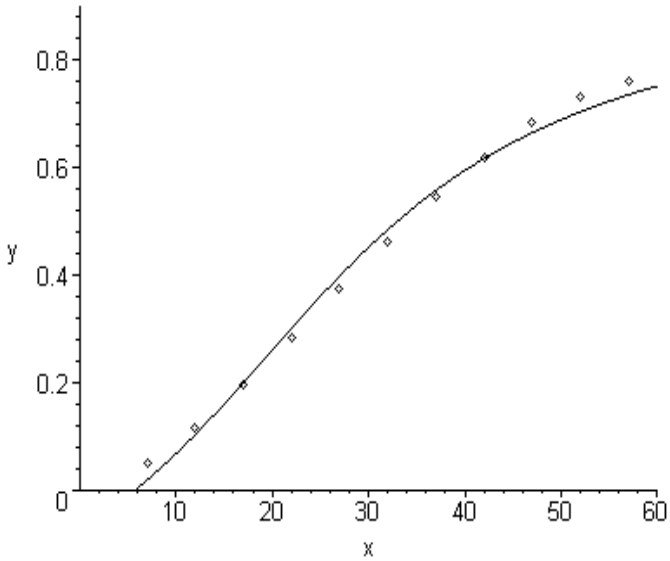

Figure 4. The graphs of F and $\{(x_k, y_k)\}$.

Our method can be used for modeling bounded growth processes as well:

Assume we have the following information on a bounded growth process

$$\boldsymbol{C = \{(x_k, g_k) \quad for \quad k = 1,2,...,n\}}:$$

1.A finite and increasing sequence of values of observations $g_k\,(>0)$ measured at some equidistant points x_k [$x_{k+1}-x_k=a(>0)$, for all k] . Suppose our measurements are free from error.

A reasonable a'priori strict upper bound $B(>0)$ given for the process C beforehand: $g_k<B$ for all values of k ,

(B is greater than any possible value of g_k),

and no other information is given on C .

We want to model the growth process C by using a properly chosen, bounded, strictly increasing and infinitely smooth real function G whose values lie between 0 and B : $0\le G(x)<B$ (for all x) , $G(-\infty)=0$ and $G(\infty)=B$. We use strict mathematical principles, some auxiliary functions and the above mentioned information only ,and no specific information on the growth process C will be used.

Our modeling procedure is as follows.

First, we consider our transformed (normed) data $y_k=\frac{g_k}{B}\ (<1)$ to be the values of an empirical cumulative distribution function F_{em} at the points x_k :

$F_{emp}(x_k)=y_k$ *for* $k=0,1,...,n$.

Then we use our method described above and obtain the USDF F .Eventually, the increasing function $G(x)=B\cdot F(x)$ will serve as a good smooth model for the growth process C in question (Szabo 2004)

Let me illustrate this method by showing

Numerical Example 3:

Modeling and forecasting the average temperature of changing climate.

	x_k	y_k
1926-35	1	13.470
1936-45	2	14.290
1946-55	3	14.883
1956-65	4	15.310
1966-75	5	15.624
1976-85	6	15.927
1986-95	7	16.273
1996-2005	8	16.758

Table 3. For the growth process of average temperatures at a given geographic place of the Earth, the data of ten-year average temperatures (in Celcius) in the past 80 years.

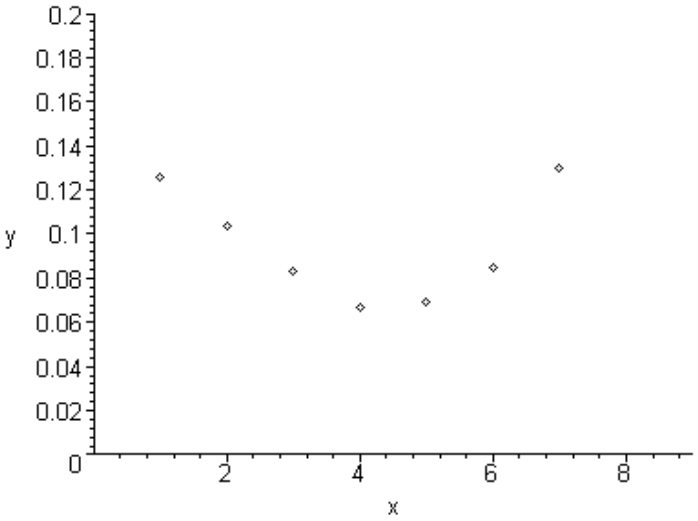

Figure 5. Graph of the corresponding finite sequence of the empirical relative increments $\{(x_k, h_{emp}(x_k))\}$.

According to this graph, the empirical relative increments decrease up to a certain time, and increase thereafter. This monotonic behavior of the RIF h_F can be found only for USDFs No. 37-39 in Proposition 5. Among these three types of functions, the transformed function

$G(x)=B\cdot F(x)=B\cdot\sqrt{1-[a(x-b)]^2}$ for $x\in\left(b-\frac{1}{a},b\right)\approx(-8.226,23.621)$,

$G(x)=0$ for $x\le b-\frac{1}{a}\approx -8.226$, and $G(x)=B$ for $x\ge b\approx 23.621$

provided the best least squares fit $\sum_{k=1}^{12}[G(x_k)-y_k]^2\approx 0.13187935$ with $a=0.0314, b=23.621, B=19.295$

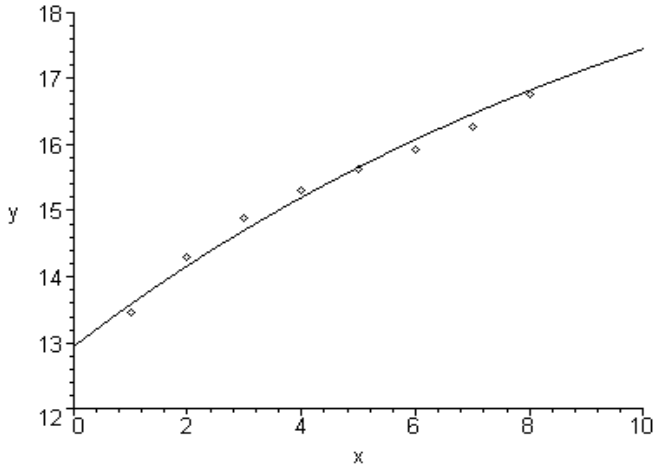

Figure 6. The graphs of G **and the data** $\{(x_k, y_k)\}$.

According to the value of the limit at infinity

$\lim_{x\to\infty} G(x) = B \approx 19.295$, the expected average temperature after a long period of time can be predicted to be $19.295^{o}C$ provided the climatic conditions at that very place will not change dramatically and significantly.

III. References

1) Adler, P., and Szabo, Z. (1974a), "Die Analyse von drei Verteilungsfunktionen zur Beschreibung des kumulativen Kariesbefalles im Spiegel des relativen Karieszuwachses" ("Analysis of Three Distribution Functions for Describing the Cumulative Caries Prevalence in the Light of Relative Growth of Caries"), *Biometrische Zeitschrift [Biometrics Journal]* 16,217-232(In German).
2) Adler, P., and Szabo, Z. (1974b), "The Suitability of Different Mathematical Models in Describing Cumulative Caries Prevalence Curves of Individual Teeth", European Meeting of Statisticians, Budapest,1972. Proceedings in *Colloquia Mathematica Societatis Janos Bolyai - Progress in Statistics*, Vol.I, 43-49, North Holland Publ.Co.1974.
3) Adler, P., and Szabo, Z. (1975), "A Mathematical Description of Cumulative Caries Prevalence Curves of Individual Teeth with Special Regard to the Relative Growth of Caries", *Fogorvosi Szemle* [*Dental Review*] 68,8-9 (In Hungarian).
4) Adler, P., and Szabo, Z. (1976), "Mathematics in Caries Prognosis", *Caries Research* 10,154-155.
5) Adler, P., and Szabo, Z. (1979a), "Interpretation of a Cauchy Distribution Function Describing Cumulative Caries Prevalence Curves", *Community Dentistry and Oral Epidemiology* 7,82-86.
6) Adler, P., and Szabo, Z. (1979b), "Comparison of Two Mathematical Descriptions of the Cumulative Caries Prevalence", *Acta Medica Academiae Scientiarum Hungaricae* 36,211-222.
7) Adler, P., and Szabo, Z. (1984), "Mathematical Description and Use in Forecasting of the Caries Curve", *Community Dentistry and Oral Epidemiology* 12,96-102.
8) Barlow, R. E., and Proschan, F. (1967), "Exponential Life Test Procedures When the Distribution Has Monotone Failure Rate", *Journal of the American Statistical Association* 62,548-560.
9) Feller, W. (1971), "An Introduction to Probability Theory and its Applications"(Vol. I-II), John Wiley, New York.
10) Gradshteyn, I. S., and Ryzhik, I. M. (1980), "Table of Integrals, Series and Products", Academic Press, San Diego.
11) Johnson, N. L., and Kotz, S. (1970), "Distributions in Statistics. Continuous Univariate Distributions" (Vols.1-2), Houghton-Mifflin, Boston,New York.
12) Porter, D. R., and Dudman, J. A. (1960), "Assessment of Caries Increments I. Constitution of the R.I.D. Index", *Journal of Dental Research*. 39,1056 - 1061.
13) Sobkowiak, E. M., Szabo, Z., Radtke, G., and Adler, P. (1973), "Analysis of Cumulative Caries Prevalence in Primary Molars", *Community Dentistry and Oral Epidemiology* 1,127-131.
14) Stuart, A., and Ord, J. K. (1987), "Kendall's Advanced Theory of Statistics. Distribution Theory"(Vol.1.), Griffin, London.
15) Szabo, Z. (1974), "The Relative Growth of Distribution Functions and Its Applications in Medicine", 5th Colloquium of the J. von Neumann Society for Computer Science, Szeged, Hungary. Printed in the Proceedings of the Colloquium, 217-229.(In Hungarian)
16) Szabo, Z. (1976), "Über den relativen Zuwachs von Verteilungsfunktionen" ("On the Relative Growth of Distribution Functions"), Biometrische Zeitschrift [Biometrics Journal] 18,33-40 (In German).
17) Szabo, Z. (1989), "Relative Increments of Distribution Functions and Their Use in Biometrics", Joint Meeting of the American Statistical Association at Washington, D.C. Printed in the Proceedings, Biopharmaceutical Section, 212-216.
18) Szabo, Z. (1994), "Monotone Properties of Hazard Rates and Relative Increments of Distribution Functions", International Congress of Mathematicians ICM'94 held in Zurich, Switzerland in Aug 1994.
19) Szabo, Z. (1996), "Investigation of Relative Increments of Distribution Functions", Publicationes Mathematicae Debrecen 49, 99-112.
20) Szabo, Z. (1999), "Relative Increments of Pearson Distributions", Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis 15,45-54 www.emis.de/journals
21) Szabo, Z. (2004), "Remarks on Modeling Bounded Growth Processes", Hawaii International Conference on Statistics, Mathematics and Related Fields, Honolulu, Hawaii, USA, June 7-13, 2004.
22) Szabo, Z. (2006), "On Monotonic Behavior of Relative Increments of Unimodal Distributions", Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis 22,113-119 www.emis.de/journals
23) Szondy, E., and Szabo, Z. (1976), "A Mathematical Description of Intrauterine Growth of Fetuses", 7th Colloquium of the J. von Neumann Society for Computer Science in Szeged, Hungary, 1976. Printed in the Proceedings of the Colloquium, 591-604 (1977) (In Hungarian).
24) Venables, W. N., and Ripley, B. D. (1994), "Modern Applied Statistics with S-Plus", Springer Verlag, New York,Inc.