Generalized Fractional Calculus Of Certain Product Of Special Functions

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***Abstract*- The paper is devoted to study the generalized fractional calculus of arbitrary complex order for the $\overline{\mathrm{H}}$ -function defined by Inayat Hussain [8]. The classical fractional integrals and derivatives of Riemann-Liouville type are treated. The considered generalized fractional integration and differentiation operators contain the Gauss Hypergeometric function as a kernel and generalize classical Riemann-Liouville, Erdelyi-Kober types ones. It is proved that the generalized fractional integrals and derivatives of $\overline{\mathrm{H}}$ -function turn also out $\overline{\mathrm{H}}$ -functions but of greater order. Especially, the obtained results define more precise and general ones than known. Corresponding assertion for Riemann-Liouville and Erdelyi-Kober fractional integral operators are also presented.**

Keywords- Fractional integral operator, $\overline{H}$ - function, Riemann Zeta-function, General class of polynomials.

I. Introduction

By evaluating certain Feynman integrals which arise naturally in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions, Inayat-Hussain [7] derived a number of interesting properties and characteristics of hypergeometric functions of one and more variables. While presenting some further examples of the use of these Feynman integrals, Inayat-Hussain [8] was led to a novel generalization of the familiar H-function of Charles Fox [4]. This function is popularly known as $\overline{H}$ -function and contains some new special cases, such as the polylogarithm of a complex order and the exact partition function of the Gaussian model in Statistical mechanics. In terms of Mellin-Barnes contour integral, it is defined as follows:

$$\overline{H}^{M,N}_{P,Q}\left(z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right)$$

$$=\frac{1}{2\pi i}\int_L\overline{\phi}(\xi)z^{\xi}d\xi, \qquad (z\neq 0) \tag{1.1}$$

Where

$$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.2}$$

Here $a_j\ (j=1,...,p)$ and $b_j\ (j=1,...,q)$ are complex parameters, $A_j>0\ (j=1,...,p)$ and the exponents $a_j\ (j=1,...,p)$ can take noninteger values. The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}$ - Function given by (1.1) have been recently given by Gupta, Jain and Agrawal [5]:

(i) $|\arg(z)|<1/2\,\Omega\pi$ *and* $\Omega>0$,

(ii) $|\arg(z)|=1/2\,\Omega\pi$ *and* $\Omega\geq 0$,

and (a) $\mu\neq 0$ and the contour L is so chosen that $(c\mu+\lambda+1)<0$,

(b) $\mu=0$ *and* $(\lambda+1)<0$,

where

$$\Omega=\sum_1^M\beta_j+\sum_1^N\alpha_jA_j-\sum_{M+1}^{Q}\beta_jB_j-\sum_{N+1}^{P}\alpha_j, \tag{1.4}$$

$$\mu=\sum_1^N\alpha_jA_j+\sum_{N+1}^{P}\alpha_j-\sum_1^M\beta_j-\sum_{M+1}^{Q}\beta_jB_j, \tag{1.5}$$

$$\lambda=\mathrm{Re}\left(\sum_1^M b_j+\sum_{M+1}^{Q}b_jB_j-\sum_1^N a_jA_j-\sum_{N+1}^{P}a_j\right)+\frac{1}{2}\left(-M-\sum_{M+1}^{Q}B_j+\sum_1^N A_j+P-N\right) \tag{1.6}$$

It may be noted that the conditions of validity given above are more general than those given earlier [1].

The following series representation of the $\overline{H}$ -function was given by Rathie [12].

$$\overline{H}^{M,N}_{P,Q}\left(z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right)=\sum_{\upsilon=1}^{M}\sum_{P=0}^{\infty}\overline{\theta}(S_{P,\upsilon})z^{S_{P,\upsilon}} \tag{1.7}$$

where

$$\overline{\theta}(S_{P,\upsilon})=\frac{\prod_{j=1,j\neq\upsilon}^{M}\Gamma(b_j-\beta_jS_{P,\upsilon})\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_jS_{P,\upsilon})\}^{A_j}(-1)^P}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_jS_{P,\upsilon})\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_jS_{P,\upsilon})P!\beta_\upsilon} \tag{1.8}$$

The following behavior of $\overline{H}^{M,N}_{P,Q}[z]$ for small and large value of z as recorded by Saxena [16, p.112, eqs. (2.3) and (2.4)] will be required in the sequel

$\overline{H}^{M,N}_{P,Q}[z]=O\left[|z|^{\alpha}\right]$ for small z, where

$$\alpha=\min_{1\leq j\leq M}\left[\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)\right] \tag{1.9}$$

$\overline{H}_{P,Q}^{M,N}[z]=O\left[|z|^{\beta}\right]$ for large z, where

$$\beta=\max_{1\le j\le N}\left[\operatorname{Re}\left(\frac{a_j-1}{\alpha_j}\right)\right] \quad (1.0)$$

and the conditions (1.3) are satisfied. Also $S_V^U[x]$ occurring in the sequel denotes the general class of polynomials [17, p.1, Eq. (1)]

$$S_V^U[x]=\sum_{R=0}^{[V/U]}(-V)_{UR}A(V,R)\frac{x^R}{R!} \quad (1.11)$$

where U is an arbitrary positive integer, V=0,1,2,… and the coefficients A(V,R) $A_{V,R}\ (V,R\ge 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{V,R}$ yields a number of known polynomials as its special cases.

II. Classical and Generalized Fractional Calculas Operators

For α ∈ C (Re (α) > 0), the Riemann-Liouville left and right-sided fractional calculus operators are defined as the following ([15, sections 2.3 and 2.4)]:

$$(I_{0+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad (x>0) \quad (1)$$

$$(I_{0-}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}\frac{f(t)}{(t-x)^{1-\alpha}}dt, \quad (x>0) \quad (2)$$

and

$$(D_{0+}^{\alpha}f)(x)=\left(\frac{d}{dx}\right)^{[\operatorname{Re}(\alpha)+1]}\left(I_{0+}^{1-\alpha+[\operatorname{Re}(\alpha)]}f\right)(x)$$

$$=\left(\frac{d}{dx}\right)^{[\operatorname{Re}(\alpha)]+1}\frac{1}{\Gamma(1-\alpha+[\operatorname{Re}(\alpha)])}$$

$$\int_0^x\frac{f(t)}{(x-t)^{\alpha-[\operatorname{Re}(\alpha)]}}dt, \quad (x>0) \quad (3)$$

$$(D_{-}^{\alpha}f)(x)=\left(-\frac{d}{dx}\right)^{[\operatorname{Re}(\alpha)]+1}\left(I_{-}^{1-\alpha+[\operatorname{Re}(\alpha)]}f\right)(x)$$

$$=\left(-\frac{d}{dx}\right)^{[\operatorname{Re}(\alpha)]+1}\frac{1}{\Gamma 1-\alpha+\operatorname{Re}(\alpha)])}$$

$$\int_x^{\infty}\frac{f(t)}{(t-x)^{\alpha-[\operatorname{Re}(\alpha)]}}dt, \quad (x>0) \quad (4)$$

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respectively, where [Re(α)] is the integral part of Re (α). In particular, for real α > 0, the operators D_{0+}^{α} and D_{-}^{α} take more simple forms.

$$(D_{0+}^{\alpha}f)(x)=\left(\frac{d}{dx}\right)^{[\alpha]+1}(I_{0+}^{1-\{\alpha\}}f)(x)$$

$$=\left(\frac{d}{dx}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\int_0^x\frac{f(t)}{(x-t)^{\{\alpha\}}}dt, \quad (x>0) \quad (2.5)$$

$$(D_{-}^{\alpha}f)(x)=\left(-\frac{d}{dx}\right)^{[\alpha]+1}(I_{-}^{1-\{\alpha\}}f)(x)$$

$$=\left(-\frac{d}{dx}\right)^{[\alpha]+1}\frac{1}{\Gamma(1-\{\alpha\})}\int_x^{\infty}\frac{f(t)}{(t-x)^{\{\alpha\}}}dt, \quad (x>0) \quad (2.6)$$

where [α] and {α} are integral and fractional parts of α. For α, β, η ∈ C and x > 0 the generalized fractional calculus operators [13] are defined by

$$(I_{0+}^{\alpha,\beta,\eta}f)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}$$

$$ {}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})\,f(t)\,dt, \quad [\operatorname{Re}(\alpha)>0] \quad (2.7)$$

$$(I_{0+}^{\alpha,\beta,\eta}f)(x)=\left(\frac{d}{dx}\right)^{n}(I_{0+}^{\alpha+n,\beta-n,\eta-n}f)(x)$$

$$(\operatorname{Re}(\alpha)<0;\eta=[\operatorname{Re}(-\alpha)]+1); \quad (2.8)$$

$$(I_{-}^{\alpha,\beta,\eta}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\beta}$$

$$ {}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t})f(t)\,dt \quad (\operatorname{Re}(\alpha)>0); \quad (2.9)$$

$$(I_{-}^{\alpha,\beta,\eta}f)(x)=\left(-\frac{d}{dx}\right)^{n}(I_{-}^{\alpha+n,\beta-n,\eta}f)(x)$$

$$(\operatorname{Re}(\alpha)<0;\eta=[\operatorname{Re}(-\alpha)]+1); \quad (2.10)$$

and

$$(D_{0+}^{\alpha,\beta,\eta}f)(x)\equiv(I_{0+}^{-\alpha,-\beta,\alpha+\eta})f(x)$$

$$=\left(\frac{d}{dx}\right)^{n}(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n}f)(x)$$

$$(\operatorname{Re}(\alpha)>0;n=[\operatorname{Re}(\alpha)]+1); \quad (2.11)$$

$$(D_{-}^{\alpha,\beta,\eta}f)(x)\equiv(I_{-}^{-\alpha,-\beta,\alpha+\eta})f(x)$$

$$=\left(-\frac{d}{dx}\right)^{n}(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta}f)(x)$$

$$(\operatorname{Re}(\alpha)>0;n=[\operatorname{Re}(\alpha)]+1) \quad (2\ 12)$$

Here ${}_2F_1(a,b,c;z)\ (a,b,c.z\in C)$ is the Gauss hypergeometric function of the series form

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \quad (2.13)$$

With

$$(a)_0 = 1, (a)_k = a(a+1)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} \left(k \in N\right) \quad (2.14)$$

where $\Gamma(z)$ is the gamma function [3, Chap.I] and **N** denotes the set of positive integers.

The series in (2.13) is convergent for $|z| < 1$ and $|z| = 1$ with Re (c-a-b) > 0 and can be analytically continued into $\{z \in C : |\arg(1-z)| < \pi\}$ [3, Chapter II].

Since

$$ {}_2F_1(0,b;c;z) = 1 , \quad (2.15)$$

the generalized fractional calculus operators (2.7), (2.9), (2.11) and (2.12) coincide, if $\beta = -\alpha$ with the Riemann-Liouville operators (2.1) – (2.4) for Re (α) > 0:

$$(I_{0+}^{\alpha,-\alpha,\eta} f)(x) = (I_{0+}^{\alpha} f)(x),$$

$$(I_{-}^{\alpha,-\alpha,\eta} f)(x) = (I_{-}^{\alpha} f)(x) \quad (2.16)$$

$$(D_{0+}^{\alpha,-\alpha,\eta} f)(x) = (D_{0+}^{\alpha} f)(x),$$

$$(D_{-}^{\alpha,-\alpha,\eta} f)(x) = (D_{-}^{\alpha} f)(x) \quad (2.17)$$

According to the relation [3, 2.8 (4)]

$$ {}_2F_1(a,b;a;z) = (1-z)^{-b} , \quad (2.18)$$

the operators (2.7) and (2.9) coincide with the Erdelyi-Kober fractional integrals [15, § 18.1] when β = 0:

$$(I_{0+}^{\alpha,0,\eta} f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt$$

$$= (I_{\eta,\alpha}^{+} f)(x), \quad (\alpha, \eta \in \text{ Re}(\alpha) > 0), \quad (2.19)$$

$$(I_{-}^{\alpha,0,\eta} f)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\eta-\alpha} f(t)\, dt$$

$$\equiv (K_{\eta,\alpha}^{-} f)(x) \quad (\alpha, \eta \in \mathbf{C}, \text{Re}(\alpha) > 0). \quad (2.20)$$

Therefore the operators (2.7), (2.9) and (2.11), (2.12) are called 'generalized' fractional integrals and derivatives, respectively. Moreover, the operators (2.11) and (2.12) are inverse to (2.7) and (2.9):

$$D_{0+}^{\alpha,\beta,\eta} = (I_{0+}^{\alpha,\beta,\eta})^{-1}, \; D_{-}^{\alpha,\beta,\eta} = (I_{-}^{\alpha,\beta,\eta})^{-1} \quad (2.21)$$

We also need following asymptotic behaviour of ${}_2F_1(a, b ; c ; z)$ at the point z = 1.

Lemma 1. For a, b, c ∈ **C** with Re (c) > 0 and z ∈ C, there hold the following asymptotic relations near z = 1:

$$ {}_2F_1(a,b;c;z) = 0(1) \quad (z \to 1-) \quad (2.22)$$

for Re (c – a – b) > 0 ;

$$ {}_2F_1(a,b;c;z) = 0\left((1-z)^{c-a-b}\right) \quad (z \to 1-)$$

Re (c – a – b) < 0 ; (2.23)

And

$$ {}_2F_1(a,b;c;z) = 0(\log(1-z)) \quad (z \to 1-);$$ for c – a – b = 0, a, b ≠ 0, -1, -2,… and $|\arg(z)| < \pi$. (2.24)

III. *LEFT-SIDED GENERALIZED FRACTIONAL INTEGRATION OF THE* $\overline{\mathbf{H}}$ *-FUNCTION*

Here we consider the left sided generalized fractional integration $I_{0+}^{\alpha,\beta,\eta}$ defined by (2.7).

THEOREM 1- Let α, β, η ∈ C with Re (α) > 0, Re (β) ≠ Re (η). Let the constants aj, bj ∈ C, αj, βj > 0 (j = 1,…, P; j = 1,…,Q) and λ ∈ C, $\sigma_1, \sigma_2 > 0$ and

$$\sigma_1 \min_{1\le j\le M_1} \text{Re}\left(\frac{f_j}{F_j}\right) + \sigma_2 \min_{1\le j\le M} \text{Re}\left(\frac{b_j}{\beta_j}\right)$$

$$+ \text{Re}(\lambda) + \min[0, \text{Re}(\eta-\beta)] > 0 \quad (3.1)$$

Then the generalized fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the product of $\overline{H}$ -functions with $S_V^U[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1} \overline{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1} \left| \begin{matrix} (e_j,E_j;\varepsilon_j)_{1,N_1},(e_j,E_j)_{N_1+1,P_1} \\ (f_j,F_j)_{1,M_1},(f_j,F_j;\mathfrak{I}_j)_{M_1+1,Q_1} \end{matrix} \right. \right] \right.$$

$$\left. \overline{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2} \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right] S_V^U[\delta t^{\rho}] \right)(x)$$

$$= x^{\lambda-\beta-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A(V,R)\delta^R x^{\rho R} \sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty} \bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{p,\upsilon}} \overline{H}_{P+2,Q+2}^{M,N+2}\left[w_2 x^{\sigma_2} \left| \begin{matrix} \left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right.$$

$$\left. \begin{matrix} \left(1-\lambda-\eta+\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left(1-\lambda+\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda-\alpha-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1\right) \end{matrix} \right] \quad (3.2)$$

Proof-By (2.7), we have

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1} \overline{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1} \right] \overline{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2} \right] S_V^U[\delta t^{\rho}] \right)(x)$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\varpi} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)$$

$$\overline{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1} \right] \overline{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2} \right] S_V^U[\delta t^{\rho}]\, dt$$

According to (2.22), (2.23) the integrand in (3.3) for any x > 0 has the asymptotic estimate at zero

$$(x-t)^{\alpha-1} t^{\omega} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;\frac{t}{x}\right)$$

$$\bar{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\bar{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S^U_V[\delta t^{\rho}]$$

$$= O\left(t^{\lambda+\sigma_1 e^*+\sigma_2 f^*+\min[0,\,\mathrm{Re}(\eta+\beta)]-1}\right)(t\to+0)$$

$$= O\left(t^{\lambda+\sigma_1 e^*+\sigma_2 f^*+\min[0,\,\mathrm{Re}(\eta+\beta)]-1}[\log(t)]^{N^*}\right),(t\to+0)$$

Here

$$e^* = \min_{1\le j\le M_1}\left[\frac{\mathrm{Re}\,(f_j)}{F_j}\right],\ f^* = \min_{1\le j\le M}\left[\frac{\mathrm{Re}\,(b_j)}{\beta_j}\right]$$

and N^* is the order of one of the poles $b_{j\ell} = \dfrac{-b_j-\ell}{\beta_j}$ (j = 1,…,M; ℓ = 0,1,2,…) to which some other poles of $\Gamma(b_j+\beta_j s)$ (j = 1,…,M) coincide. Therefore the condition (3.1) ensures the existence of the integral (3.3).

Applying (1.1), (1.7), (1.11), making the change of variable t = x τ, changing the order of summation and integration and taking into account the formula [11, § 2.21.1.11]

$$\int_0^x t^{\alpha-1}(x-t)^{c-1}\,{}_2F_1(a,b;c;1-\frac{t}{x})dt$$

$$= \frac{\Gamma(c)\Gamma(\alpha)\Gamma(\alpha+c-a-b)}{\Gamma(\alpha+c-a)\Gamma(\alpha+c-b)}x^{\alpha+c-1} (a, b, c, \alpha \in$$

C, Re (α) > 0, Re (c) > 0,

Re (α+c – a – b > 0). (3.4)

We obtain

$$\left(I^{\alpha,\beta,\eta}_{0+} t^{\lambda-1}\bar{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\bar{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S^U_V[\delta t^{\rho}]\right)(x)$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_L \bar{\phi}(\xi)(w_2)^{\xi}d\xi\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\delta^R$$

$$\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}(S_{P,\upsilon})(w_1)^{S_{P,\upsilon}}\int_0^x t^{\lambda+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi-1}$$

$$(x-t)^{\alpha-1}\,{}_2F_1(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})\,dt$$

$$= x^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{V,R}\left(\delta t^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}(S_{P,\upsilon})\left(w_1 t^{\sigma_1}\right)^{S_{P,\upsilon}}$$

$$.\frac{1}{2\pi i}\int_{\alpha}\bar{\phi}(\xi)\frac{\Gamma(\lambda+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi)}{\Gamma(\lambda-\beta+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi)}$$

$$\frac{\Gamma(\lambda+\eta-\beta+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi)}{\Gamma(\lambda+\alpha+\eta+\rho R+\sigma_1 S_{p,\upsilon}+\sigma_2\xi\eta)}x^{\sigma_2\xi}\,d\xi$$

and in accordance with (1.1), we obtain (3.2) which completes the proof of Theorem 1.

Corollary 1.1- Let α ∈ **C** with Re (α) > 0 and let the constants λ ∈ **C**, $\sigma_1,\sigma_2 > 0$ and

$$\sigma_1\min_{1\le j\le M_1}\mathrm{Re}\left(\frac{f_j}{F_j}\right)+\sigma_2\min_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right) + \mathrm{Re}\,(\lambda) > 0 \quad (3.5)$$

then the Riemann-Liouville fractional integral I^{α}_{0+} of the product of $\bar{H}$ -functions with $S^U_V[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(I^{\alpha}_{0+} t^{\lambda-1}\bar{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\bar{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S^U_V[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda+\alpha-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}(S_{P,\upsilon})$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}}\bar{H}^{M,N+1}_{P+1,Q+1}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\lambda-\alpha-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1)\end{matrix}\right] \quad (3.6)$$

Corollary 1.2- Let α, η, ∈ **C** with Re (α) > 0 and let the constants λ ∈ **C**, $\sigma_1,\sigma_2 > 0$ satisfy and

$$\sigma_1\min_{1\le j\le M_1}\mathrm{Re}\left(\frac{f_j}{F_j}\right)+\sigma_2\min_{1\le j\le M}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right)$$

$$+\mathrm{Re}\,(\lambda)+\min[0,\mathrm{Re}\,(\eta)]>0 \quad (3.7)$$

then the Erdélyi-kober fractional integral $I^{+}_{\eta,\alpha}$ of the product of $\bar{H}$ -functions with $S^U_V[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(I^{+}_{\eta,\alpha} t^{\lambda-1}\bar{H}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\bar{H}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S^U_V[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}(S_{P,\upsilon})$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{P,\upsilon}}\bar{H}^{M,N+1}_{P+1,Q+1}\left[w_2 x^{\sigma_2}\left|\begin{matrix}(1-\lambda-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\lambda-\alpha-\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2;1)\end{matrix}\right] \quad (3.8)$$

IV. RIGHT-SIDED FRACTIONAL INTEGRATION OF THE $\bar{H}$ FUNCTION

In this section we consider the right sided generalized fractional integration $I^{\alpha,\beta,\eta}_{-}$ defined by (2.9).

Theorem 2.- Let α, β, η ∈ **C** with Re (α) > 0, Re (β) ≠ Re (η). Let the constants λ ∈ **C**, $\sigma_1,\sigma_2 > 0$ and

$$\sigma_1\max_{1\le j\le N_1}\left[\frac{\mathrm{Re}(e_j)-1}{E_j}\right]+\sigma_2\max_{1\le j\le N}\left[\frac{\mathrm{Re}(a_j)-1}{\alpha_j}\right]$$

$$+\operatorname{Re}(\lambda) < \min[\operatorname{Re}(\beta), \operatorname{Re}(\eta)] \qquad (4.1)$$

then the generalized fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the product of $\bar{H}$ -functions with $S_V^U[\delta t^\rho]$ exists and the following relation holds:

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1} \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^\rho]\right)(x)$$

$$= x^{\lambda-\beta-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \left(\delta x^\rho\right)^R \sum_{\upsilon=1}^{M_1} \sum_{p=0}^{\infty} \bar{\theta}\left(S_{p,\upsilon}\right)\left(w_1 x^{\sigma_1}\right)^{S_{p,\upsilon}}$$

$$\bar{H}_{P+2,Q+2}^{M+2,N}\left[w_2 x^{\sigma_2} \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left(1-\lambda+\beta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2\right) \end{matrix} \right.\right.$$

$$\left.\left. \begin{matrix} \left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2\right)\left(1-\lambda+\alpha+\beta+\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2\right) \\ \left(1-\lambda+\eta-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2\right)(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right]$$

(4.2)

Proof-By (2.9), we have

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1} \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^\rho]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{\lambda-\alpha-\beta-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)$$

$$\bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^\rho] \, dt \qquad (4.3)$$

Due to (2.22) and (2.23) the integrand in (4.3) for any x > 0 has the asymptotic estimate at infinity

$$(t-x)^{\alpha-1} t^{\lambda-\alpha-\beta-1} {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)$$

$$\bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^\rho]$$

$$= O\left(t^{\lambda+\sigma_1 e_1^*+\sigma_2 f_1^*-\min[\operatorname{Re}(\beta),\operatorname{Re}(\eta)]-1}\right), \ (t \to +\infty)$$

$$= O\left(t^{\lambda+\sigma_1 e_1^*+\sigma_2 f_1^*-\min[\operatorname{Re}(\beta),\operatorname{Re}(\eta)]-1} [\log(t)]^{N^*}\right), \ (t \to +\infty)$$

Here

$$e_1^* = \max_{1\le j\le N_1}\left[\frac{\operatorname{Re}(e_j)-1}{E_j}\right], \quad f_1^* = \max_{1\le j\le N}\left[\frac{\operatorname{Re}(a_j)-1}{\alpha_j}\right]$$

and N^* is the order of one of the poles $a_i k = \frac{1-a_i+k}{\alpha_i}$ (i = 1,…,N ; k = 0,1,2,…) to which some other poles of Γ(1- a_i- α_is) (i = 1,2,…,N) coincide. Therefore the condition (4.1) ensures the existence of the integral (4.3).

Applying (1.2) making the change $t = \frac{1}{\tau}$ and using (3.4), we obtain

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1} \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^\rho]\right)(x)$$

$$= \frac{1}{\Gamma(\alpha)} \int_{1/x}^\infty \left(t-\frac{1}{x}\right)^{\alpha-1} t^{\lambda-\alpha-\beta-1}$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{1}{tx}\right) \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \delta^R$$

$$\sum_{\upsilon=1}^{M_1} \sum_{p=0}^{\infty} \bar{\theta}\left(S_{p,\upsilon}\right)\left(w_1\right)^{S_{p,\upsilon}} \frac{1}{2\pi i} \int_L \bar{\phi}(\xi)\left(w_2 t^{\sigma_2}\right)^{\xi} d\xi \, dt$$

$$= \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \tau^{\alpha+\beta-\lambda-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi-1}$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{x}\right) d\tau \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \delta^R$$

$$\sum_{\upsilon=1}^{M_1} \sum_{p=0}^{\infty} \bar{\theta}\left(S_{p,\upsilon}\right)\left(w_1\right)^{S_{p,\upsilon}} \frac{1}{2\pi i} \int_L \bar{\phi}(\xi)\left(w_2\right)^{\xi} d\xi$$

$$= x^{\lambda-\beta-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \left(\delta t^\rho\right)^R \sum_{\upsilon=1}^{M_1} \sum_{p=0}^{\infty} \bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1 t^{\sigma_1}\right)^{S_{p,\upsilon}} \frac{1}{2\pi i} \int_L \bar{\phi}(\xi)\left(w_2\right)^{\xi} \frac{\Gamma(1-\lambda+\beta-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi)}{\Gamma(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi)}$$

$$\frac{\Gamma(1-\lambda+\eta-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi)}{\Gamma(1-\lambda+\alpha+\beta+\eta-\rho R-\sigma_1 S_{p,\upsilon}-\sigma_2\xi)} x^{\sigma_2\xi} d\xi$$

and in accordance with (1.1), we obtain (4.2) which completes the proof of Theorem 2.

Corollary 2.1 -Let α ∈ **C**, with Re (α) > 0 and let the constants λ ∈ **C**, $\sigma_1,\sigma_2 > 0$ satisfy and

$$\sigma_1 \max_{1\le j\le N_1}\left[\frac{\operatorname{Re}(e_j)-1}{E_j}\right] + \sigma_2 \max_{1\le j\le N}\left[\frac{\operatorname{Re}(a_j)-1}{\alpha_j}\right]$$

$$+\operatorname{Re}(\lambda) + \operatorname{Re}(\alpha) < 0 \qquad (4.5)$$

Then the Riemann-Liouville fractional integral I_{-}^{α} of the product of $\bar{H}$ -functions with $S_V^U[\delta t^\rho]$ exist and the following relation hold:

$$\left(I_{-}^{\alpha} t^{\lambda-1} \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[w_1 t^{\sigma_1}\right] \bar{H}_{P,Q}^{M,N}\left[w_2 t^{\sigma_2}\right] S_V^U[\delta t^\rho]\right)(x)$$

$$= x^{\lambda+\alpha-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{(V,R)} \left(\delta x^\rho\right)^R \sum_{\upsilon=1}^{M_1} \sum_{p=0}^{\infty} \bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1 x^{\sigma_1}\right)^{S_{p,\upsilon}} \bar{H}_{P+1,Q+1}^{M+1,N}\left[w_2 x^{\sigma_2} \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ \left(1-\lambda-\alpha-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2\right) \end{matrix} \right.\right.$$

$$\left.\left. \begin{matrix} \left(1-\lambda-\rho R-\sigma_1 S_{p,\upsilon},\sigma_2\right) \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right] \qquad (4.6)$$

Corollary 2.2- Let α ∈ **C**, with Re (α) > 0 and let the constants λ ∈ **C,** $\sigma_1,\sigma_2>0$ satisfy and

$$\sigma_1 \max_{1\le j\le N_1}\left[\frac{\operatorname{Re}\left(e_j\right)-1}{E_j}\right]+\sigma_2 \max_{1\le j\le N}\left[\frac{\operatorname{Re}\left(a_j\right)-1}{\alpha_j}\right]$$

$$+\operatorname{Re}(\lambda)\;+\operatorname{Re}(\eta)<0 \qquad (4.7)$$

Then the Erdelyi-Kober fractional integral $K^{-}_{\eta,\alpha}$ of the product of $\overline{\mathrm{H}}$ -functions with $S_V^U[\delta t^{\rho}]$ exist and the following relation hold:

$$\left(K^{-}_{\eta,\alpha}\, t^{\lambda-1}\,\overline{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{\mathrm{H}}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)\left(w_1x^{\sigma_1}\right)^{S_{p,\upsilon}}$$

$$\overline{H}^{M+1,N}_{P+1,Q+1}\left[w_2x^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ \left(1-\lambda+\eta-\alpha-\rho R-\sigma_1S_{p,\upsilon},\sigma_2\right)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda+\alpha+\beta+\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2\right)\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right] \qquad (4.8)$$

V. Left- sided generalized fractional differentition of $\overline{\mathrm{H}}$ -function

Now we treat the left-sided generalized fractional derivative $D^{\alpha,\beta,\eta}_{0+}$ given by (2.11).

Theorem 3-Let α, β, η ∈ **C** with Re (α) > 0, Re (α+β+η) ≠ 0. Let the constants a_j, b_j ∈ **C**, α_j, β_j > 0 (j = 1,…, P; j = 1,…,Q) and λ ∈ **C,** $\sigma_1,\sigma_2>0$ and

$$\sigma_1 \min_{1\le j\le M_1}\operatorname{Re}\left(\frac{f_j}{F_j}\right)+\sigma_2 \min_{1\le j\le M}\operatorname{Re}\left(\frac{b_j}{\beta_j}\right)$$

$$+\operatorname{Re}(\lambda)\;+\min\,[0,\operatorname{Re}(\alpha+\beta+\eta)]>0 \qquad (5.1)$$

Then the generalized fractional derivative $D^{\alpha,\beta,\eta}_{0+}$ of the product of $\overline{\mathrm{H}}$ -functions with $S_V^U[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(D^{\alpha,\beta,\eta}_{0+}\, t^{\lambda-1}\,\overline{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{\mathrm{H}}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda+\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1x^{\sigma_1}\right)^{S_{p,\upsilon}}\overline{H}^{M,N+2}_{P+2,Q+2}\left[w_2x^{\sigma_2}\left|\begin{matrix}\left(1-\lambda-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda-\alpha-\beta-\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ \left(1-\lambda-\beta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda-\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right]$$

(5.2)

Proof.- Let n = [Re (α)] +1. From (2.11) we have

$$\left(D^{\alpha,\beta,\eta}_{0+}\, t^{\lambda-1}\,\overline{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{\mathrm{H}}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$=\left(\frac{d}{dx}\right)^n\left(I^{-\alpha+n,-\beta-n,\alpha+\eta-n}_{0+}\, t^{\lambda-1}\,\overline{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{\mathrm{H}}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)$$

(5.3)

which exists according to Theorem 1 with α, β and η being replaced by - α + n, - β - n and α + η - n respectively. Then we obtain

$$\left(D^{\alpha,\beta,\eta}_{0+}\, t^{\lambda-1}\,\overline{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{\mathrm{H}}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$=\left(\frac{d}{dx}\right)^n x^{\lambda+\beta+n-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1x^{\sigma_1}\right)^{S_{p,\upsilon}}\overline{H}^{M,N+2}_{P+2,Q+2}\left[w_2x^{\sigma_2}\left|\begin{matrix}\left(1-\lambda-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}\left(1-\lambda-\alpha-\beta-\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ \left(1-\lambda-\beta-n-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda-\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right]$$

(5.4)

on differentiating n times and the relation $n\Gamma(n)=\Gamma(n+1)$ imply (5.2) which completes the proof of the theorem.

Corollary 3.1- Let α ∈ C with Re (α) > 0 and let the constants and λ ∈ C, $\sigma_1,\sigma_2>0$ satisfy the conditions in (3.8). Then the Riemann-Liouville fractional derivative D^{α}_{0+} of the product of $\overline{\mathrm{H}}$ -functions with $S_V^U[\delta t^{\rho}]$ exists and the following relation holds:

$$\left(D^{\alpha}_{0+}\, t^{\lambda-1}\,\overline{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[w_1 t^{\sigma_1}\right]\overline{\mathrm{H}}^{M,N}_{P,Q}\left[w_2 t^{\sigma_2}\right]S_V^U[\delta t^{\rho}]\right)(x)$$

$$= x^{\lambda-\alpha-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^R\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$

$$\left(w_1x^{\sigma_1}\right)^{S_{p,\upsilon}}\overline{H}^{M,N+1}_{P+1,Q+1}\left[w_2x^{\sigma_2}\left|\begin{matrix}\left(1-\lambda-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ \left(1-\lambda+\alpha-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right] \qquad (5.2)$$

VI. Right-sided fractional differentiation of $\overline{\mathrm{H}}$ -function

Theorem 4-Let α, β, η ∈ **C** with Re (α) > 0, Re (α + β + η) + Re(α) + 1 ≠ 0. Let the constants λ ∈ **C**, $\sigma_1, \sigma_2 > 0$ satisfy and

$$\sigma_1 \max_{1\le j\le N_1}\left[\frac{\mathrm{Re}\left(e_j\right)-1}{E_j}\right]+\sigma_2 \max_{1\le j\le N}\left[\frac{\mathrm{Re}\left(a_j\right)-1}{\alpha_j}\right]$$
$$+\,\mathrm{Re}\,(\lambda)\;-1+\max\left[\mathrm{Re}\,(\beta),\left\{\mathrm{Re}\,(\alpha)+1\right\}-\mathrm{Re}\,(\alpha+\eta)\right]<0 \qquad (6.1)$$

Then the generalized fractional derivative $\mathrm{D}_{-}^{\alpha,\beta,\eta}$ of the product of $\bar{\mathrm{H}}$ -functions with $\mathrm{S}_V^U[\delta\, \mathrm{t}^{\rho}]$ exists and the following relation holds:

$$\left(D_{-}^{\alpha,\beta,\eta}\mathrm{t}^{\lambda-1}\,\bar{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[\mathrm{w}_1\mathrm{t}^{\sigma_1}\right]\bar{\mathrm{H}}^{M,N}_{P,Q}\left[\mathrm{w}_2\mathrm{t}^{\sigma_2}\right]\mathrm{S}_V^U[\delta\,\mathrm{t}^{\rho}]\right)(\mathrm{x})$$
$$=\left(-1\right)^{\left[\mathrm{Re}(\alpha)+1\right]}\mathrm{x}^{\lambda+\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^{R}\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$
$$\left(w_1x^{\sigma_1}\right)^{S_{p,\upsilon}}\bar{H}^{M+2,N}_{P+2,Q+2}\left[w_2x^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\lambda+\alpha+\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1)\end{matrix}\right.\right.$$
$$\left.\begin{matrix}\left(1-\lambda-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda-\beta+\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\\\left(1-\lambda-\beta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right] \qquad (6.2)$$

Proof.- Let n = [Re (α)] + 1. From (2.12) we have

$$\left(D_{-}^{\alpha,\beta,\eta}\mathrm{t}^{\lambda-1}\,\bar{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[\mathrm{w}_1\mathrm{t}^{\sigma_1}\right]\bar{\mathrm{H}}^{M,N}_{P,Q}\left[\mathrm{w}_2\mathrm{t}^{\sigma_2}\right]\mathrm{S}_V^U[\delta\,\mathrm{t}^{\rho}]\right)(\mathrm{x})$$
$$=\left(-\frac{\mathrm{d}}{\mathrm{dx}}\right)^{\mathrm{n}}\left(\mathrm{I}_{-}^{-\alpha+\mathrm{n},-\beta-\mathrm{n},\,\alpha+\eta}\,\mathrm{t}^{\lambda-1}\right.$$
$$\left.\bar{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[\mathrm{w}_1\mathrm{t}^{\sigma_1}\right]\bar{\mathrm{H}}^{M,N}_{P,Q}\left[\mathrm{w}_2\mathrm{t}^{\sigma_2}\right]\mathrm{S}_V^U[\delta\,\mathrm{t}^{\rho}]\right) \qquad (6.3)$$

which exists according to Theorem 2 with α, β and η being replaced by - α + n, - β - n and α + η respectively.
Then we obtain

$$\left(D_{-}^{\alpha,\beta,\eta}\mathrm{t}^{\lambda-1}\,\bar{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[\mathrm{w}_1\mathrm{t}^{\sigma_1}\right]\bar{\mathrm{H}}^{M,N}_{P,Q}\left[\mathrm{w}_2\mathrm{t}^{\sigma_2}\right]\mathrm{S}_V^U[\delta\,\mathrm{t}^{\rho}]\right)(\mathrm{x})$$
$$=\left(-\frac{\mathrm{d}}{\mathrm{dx}}\right)^{\mathrm{n}}\mathrm{x}^{\lambda+\beta+n-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^{R}\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$
$$\left(w_1\,x^{\sigma_1}\right)^{S_{p,\upsilon}}\bar{H}^{M+2,N}_{P+2,Q+2}\left[w_2x^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\lambda-\beta+\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1)\end{matrix}\right.\right.$$
$$\left.\begin{matrix}\left(1-\lambda-\beta-n-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\left(1-\lambda+\alpha+\eta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\left(1-\lambda-\beta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right] \qquad (6.4)$$

which implies the formula (6.2) in view of n times differentiation and the relation $n\Gamma\left(n\right)=\Gamma\left(n+1\right)$.

Corollary 4.1-Let α ∈ **C** with Re (α) > 0 and let the constants λ ∈ **C**, $\sigma_1, \sigma_2 > 0$ satisfy and

$$\sigma_1 \max_{1\le j\le N_1}\left[\frac{\mathrm{Re}\left(e_j\right)-1}{E_j}\right]+\sigma_2 \max_{1\le j\le N}\left[\frac{\mathrm{Re}\left(a_j\right)-1}{\alpha_j}\right]$$
$$+\,\mathrm{Re}\,(\lambda)+\mathrm{Re}\,(\alpha)<0 \qquad (6.5)$$

Then the Riemann-Liouville fractional derivative D_{-}^{α} of the product of $\bar{\mathrm{H}}$ -functions with $\mathrm{S}_V^U[\delta\, \mathrm{t}^{\rho}]$ exists and the following relation holds:

$$\left(D_{-}^{\alpha}\mathrm{t}^{\lambda-1}\,\bar{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}\left[\mathrm{w}_1\mathrm{t}^{\sigma_1}\right]\bar{\mathrm{H}}^{M,N}_{P,Q}\left[\mathrm{w}_2\mathrm{t}^{\sigma_2}\right]\mathrm{S}_V^U[\delta\,\mathrm{t}^{\rho}]\right)(\mathrm{x})$$
$$=\left(-1\right)^{\left[\mathrm{Re}(\alpha)+1\right]}\mathrm{x}^{\lambda+\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{(V,R)}\left(\delta x^{\rho}\right)^{R}\sum_{\upsilon=1}^{M_1}\sum_{p=0}^{\infty}\bar{\theta}\left(S_{p,\upsilon}\right)$$
$$\left(w_1x^{\sigma_1}\right)^{S_{p,\upsilon}}\bar{H}^{M+1,N}_{P+1,Q+1}\left[w_2x^{\sigma_2}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(1-\lambda+\alpha-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1)\end{matrix}\right.\right.$$
$$\left.\begin{matrix}\left(1-\lambda-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\\\left(1-\lambda-\beta-\rho R-\sigma_1S_{p,\upsilon},\sigma_2;1\right)\end{matrix}\right] \qquad (6.6)$$

Special cases

i. If in theorem 1, we reduce S_V^U to $L_V^{\alpha}\left(x\right)$ Laguerre polynomial, $\bar{\mathrm{H}}^{M_1,N_1}_{P_1,Q_1}$ to generalized Wright hypergeometric function ${}_{P_1}\bar{\psi}_{Q_1}$ and $\bar{\mathrm{H}}^{M,N}_{P,Q}$ to generalized Wright Bessel function with the help of results [20, p.101, Eq. (5.1.6)], [6, p.271, Eq. (7)], [18, p.271, Eq. (9)] respectively, we get the following theorem:

$$\left(\mathrm{I}_{0+}^{\alpha,\beta,\eta}\,\mathrm{t}^{\lambda-1}\;{}_{P_1}\bar{\psi}_{Q_1}\left[\mathrm{w}_1\mathrm{t}^{\sigma_1}\left|\begin{matrix}(1-e_j,E_j;\varepsilon_j)_{1,P_1}\\(1-f_j,F_j;\Im_j)_{M_1+1,Q_1}\end{matrix}\right.\right]\right.$$
$$\left.\bar{J}_{\zeta}^{\,\upsilon,\mu}\left[\mathrm{w}_2\mathrm{t}^{\sigma_2}\right]L_V^{(\alpha')}[\delta\,\mathrm{t}^{\rho}]\right)(x)$$
$$=\mathrm{x}^{\lambda-\beta-1}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}\binom{V+\alpha}{V}\frac{\left(\delta x^{\rho}\right)^{R}}{(1+\alpha)_R}$$
$$\sum_{p=0}^{\infty}\frac{\prod_{j=1}^{P_1}\left\{\Gamma\left(1-e_j+E_j\xi\right)\right\}^{\varepsilon_j}}{\prod_{j=1}^{Q_1}\left\{\Gamma\left(1-f_j+F_j\xi\right)\right\}^{\Im_j}}\left(w_1x^{\sigma_1}\right)^{p}\bar{H}^{1,2}_{2,4}\left[w_2x^{\sigma_2}\left|\begin{matrix}(1-\lambda-\rho R-\sigma_1p,\sigma_2;1)\\(0,1),(-\zeta,\upsilon;\mu)(1-\lambda+\beta-\rho R-\sigma_1p,\sigma_2;1)\end{matrix}\right.\right.$$
$$\left.\begin{matrix}\left(1-\lambda-\eta+\beta-\rho R-\sigma_1p,\sigma_2;1\right)\\\left(1-\lambda-\alpha-\eta-\rho R-\sigma_1p,\sigma_2;1\right)\end{matrix}\right]$$

ii. Once again in theorem 1, if we reduce S_V^U polynomial to $y_V(-\beta' x, \alpha', \beta')$ Bessel polynomial, $\bar{H}^{M_1,N_1}_{P_1,Q_1}$ to generalized Riemann Zeta function $\phi(x)$ and $\bar{H}^{M,N}_{P,Q}$ to generalized hypergeometric function ${}_P\bar{F}_Q$ using results [9, p.108, Eq. (34)], [3, p.271, Eq. (1)], [6, p.471, Eq. (9)] respectively, it takes the following interesting form:

$$\Big(I_{0+}^{\alpha,\beta,\eta} t^{\lambda-1} \phi(w_1 t^{\sigma_1}, k, r)\, {}_P\bar{F}_Q \left[w_2 t^{\sigma_2} \left| \begin{matrix} (1-a_j, A_j)_{1,P} \\ (1-b_j, B_j)_{1,Q} \end{matrix} \right. \right]$$

$$y_V[-\beta'\delta t^\rho, \alpha', \beta']\Big)(x)$$

$$= x^{\lambda-\beta-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} (\alpha'+V-1)_R (\delta x^\rho)^R$$

$$\sum_{p=0}^{\infty} \frac{\prod_{j=1}^{Q} \{\Gamma(1-b_j)\}^{B_j}}{\prod_{j=1}^{P} \{\Gamma(1-a_j)\}^{A_j}} \frac{(w_1 x^{\sigma_1})^p}{(p+r)^k}$$

$$\bar{H}^{1,P+2}_{P+2,Q+3} \left[w_2 x^{\sigma_2} \left| \begin{matrix} (1-a_j, A_j)_{1,P} \\ (0,1)_{1,M}, (1-b_j, B_j)_{1,Q} \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (1-\lambda-\rho R-\sigma_1 p, \sigma_2; 1)(1-\lambda-\eta+\beta-\rho R-\sigma_1 p, \sigma_2; 1) \\ (1-\lambda+\beta-\rho R-\sigma_1 p, \sigma_2; 1)(1-\lambda-\alpha-\eta-\rho R-\sigma_1 p, \sigma_2; 1) \end{matrix} \right]$$

iii. If in Theorem 2, we reduce S_V^U polynomial to Hermite polynomial $H_V(x)$, $\bar{H}^{M_1,N_1}_{P_1,Q_1}$ to H-function and $\bar{H}^{M,N}_{P,Q}$ to g_1 function with the help of [20, p.106, Eq. (5.5.4)], [8, p.4125, Eq. (20)] we arrive at the following result after a little simplification:

$$\Big(I_{-}^{\alpha,\beta,\eta} t^{\lambda-1} \bar{H}^{M_1,N_1}_{P_1,Q_1} \left[w_1 t^{\sigma_1} \right] g\left[r, \mu, \tau, m, w_2 t^{\sigma_2} \right]$$

$$[\delta t^\rho]^{V/2} H_V \left[\frac{1}{2\sqrt{\delta t^\rho}} \right] \Big)(x)$$

$$= x^{\lambda-\beta-1} \sum_{R=0}^{[V/U]} \frac{(-V)_{2R}}{R!} (-1)^R (\delta x^\rho)^R$$

$$\sum_{\upsilon=1}^{M_1} \sum_{p=0}^{\infty} \bar{\theta}(S_{p,\upsilon}) (w_1 x^{\sigma_1})^{S_{p,\upsilon}} \frac{\Gamma(m+1)\Gamma\left(\frac{1+\tau}{2}\right)}{\pi^{d/2} 2^{m+d} \Gamma\left(\frac{d-1}{2}\right) \Gamma(r) \Gamma\left(r-\frac{\tau}{2}\right)}$$

$$\bar{H}^{3,3}_{5,5} \left[w_2 x^{\sigma_2} \left| \begin{matrix} (1-r, 1; 1), \left(1-r+\frac{\tau}{2}, 1; 1\right) \\ (0,1), \left(-\frac{\tau}{2}, 1; 1\right) (-\mu, 1; 1+m) \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (1-\lambda-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2)(1-\lambda+\alpha+\beta+\eta-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2) \\ (1-\lambda+\beta-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2)(1-\lambda+\eta-\rho R-\sigma_1 S_{p,\upsilon}, \sigma_2) \end{matrix} \right]$$

where

$$\theta(S_{P,\upsilon}) = \frac{\prod_{j=1, j\neq\upsilon}^{M_1} \Gamma(b_j - \beta_j S_{p,\upsilon}) \prod_{j=1}^{N_1} \Gamma(1-a_j+\alpha_j S_{p,\upsilon})(-1)^p}{\prod_{j=M+1}^{Q_1} \Gamma(1-b_j+\beta_j S_{p,\upsilon})\} \prod_{j=N+1}^{P_1} \Gamma(a_j-\alpha_j S_{p,\upsilon}) p! \beta_\upsilon}$$

$$S_{p,\upsilon} = \frac{b_\upsilon + p}{\beta_\upsilon}$$

The results obtained by M. Saigo and A.A. Kilbas in [14] can be easily deduced from our results. If in theorem 1 and 2 we put $w=1$, reduce the polynomial S_V^U and $\bar{H}^{M_1,N_1}_{P_1,Q_1}$ to unity and $\bar{H}^{M,N}_{P,Q}$ to familiar H-function we arrive at known results recorded in [10, pp. 109-110, Eqs. (3.130), (3.131)]. Further, if in corollary (1.1) and (1.2) we reduce S_V^U and $\bar{H}^{M_1,N_1}_{P_1,Q_1}$ to unity, we get the results given by Srivastava H.M. [19, p. 97, Eqs. (2.4), (2.5)]. Also by reducing $\bar{H}^{M_1,N_1}_{P_1,Q_1}$ to unity, we at once get the results obtained by Chaurasia et al. [2].

VII. REFERENCES

1) Buschman R.G. and Srivastava H.M., The $\bar{H}$ -Function associated with a certain class of Feynman integrals, J. Phys. A: Math. Gen., 23 (1990), 4707-4710.
2) Erdélyi A., Magnus W., Oberhettinger F. and Tricomi F.G., Higher Transcendental Functions, Vol. I, Mc-Graw Hill, New York, 1953.
3) Chaurasia V. B. L. and Jain M., Genaralized Fractional calculas of the $\bar{H}$ -function, Acta Cien. Indica, V0l. XXXIII M, (4), (2007) 1653-1666.
4) Fox C, The G- and H- Functions as Symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98 (1961) 395-429.
5) Gupta KC, Jain R and Agrawal R, On Existence Conditions for Generalized Mellin-Barnes Type Integral, Nat. Acad. Sci. Lett. 30(5&6) (2007) 169-172.
6) Gupta KC, Jain R and Sharma A, A study of Unified Finite Integral Transforms with

Applications, J. Raj. Aca. Phy. Sci. 2(4) (2003) 269-282.

7) Inayat Hussain A.A., New properties of hypergeometric series derivable from Feynman integrals: I – Transformation and reduction formula, J. Phys. A: Math. Gen., 20 (1987), 4109-4117.
8) Inayat Hussain A.A., New properties of hypergeometric series derivable from Feynman integrals: II. A generalization of the H-function, J. Phys. A: Math. Gen., 20 (1987), 4119-4128.
9) Krall H.L. and Frink O., A new class of orthogonal polynomials: The Bessel polynomials, Trans. Amer. Math. Soc., 65 (1949), 100-115.
10) Mathai AM and Saxena RK and Haubold H.J., The H-function, Theory and Application, Springer, 2010.
11) Prudnikov A.P., Brychkov A. Yu and Marichev O.I., Integrals and series, Vol.3, More Special Functions, Gordon and Breach, New York, 1990.
12) Rathie AK, A New Generalization of Generalized Hypergeometric Functions, Le Matematiche Fasc.II, 52 (1997) 297-310.
13) Saigo M., A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. Kyushu Univ. 11 (1978), 135-143.
14) Saigo M.and Kilbas A.A., Generalized Fractional Calculus of the H-function, Fukuoka Univ. Sci. Reports, 29(1), 31-45 (1999).
15) Samko S.G., Kilbas A.A. and Marichev O.I., Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, New York, 1993.
16) Saxena RK, Ram C and Kalla SL, Applications of Generalized H-Function in Bivariate Distributions, Rev. Acad. Canar. Cienc., 14(2002)111-120.
17) Srivastava H.M., A contour integral involving Fox's H-function, Indian J. Math., 14 (1972).1-6
18) Srivastava HM, Gupta KC and Goyal SP, The H-functions of One and Two Variables with Applications, South Asian Publishers, New Delhi and Madras, 1982.
19) Srivastava H.M., Lim S. and Wang P., Some Fractional-calculas results for the $\overline{H}$ -function associated with a class of Feynman integrals, Russ. J. math. Phy., 13 (1) (2007) 94-100.
20) Szegö G., Orthogonal polynomials, Amer. math. soc. colloq. publ., Vol. 23, Fourth edition, Amer. math. soc., Providence, Rhode Island.