Multidimensional Fractional Integral Operators Involving General Class Of Polynomial And $\overline{H}$-Function

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Abstract-In the present paper, we first define a pair of multidimensional fractional integral operators whose kernels involve the product of multivariable polynomial $S_{U_1,\ldots, U_k}^{V_1,\ldots, V_k}(x_1,\ldots, x_k)$ and $\overline{H}$-function. First we obtain images of two useful functions in our operator of study. Next, we establish two theorems giving the multidimensional generalized Stieltjes transform of fractional integral operators and conversely, the fractional integrals of multidimensional generalized Stieltjes transform. Finally, we present results concerning Mellin transform, Mellin convolutions and inversion formulae for these operators. The fractional integral operators studied by us are quite general in nature and may be considered as extensions and unifications of a number of (known or new) results for simpler fractional integral operators.

Keywords-Fractional integral operator, $\overline{H}$-function, Mellin transform, Stieltjes transform, General class of multivariable polynomials.

I. INTRODUCTION

The multivariable polynomial $S_{U_1,\ldots, U_k}^{V_1,\ldots, V_k}(x_1,\ldots, x_k)$ introduced by Srivastava and Garg (1987) [6, p. 686, eq. (1.4)] is defined in the following manner:

\[ S_{U_1,\ldots, U_k}^{V_1,\ldots, V_k}[x_1,\ldots, x_k] = \sum_{-V_{i=1}}^{R_{i=1}} (-V)^{U_{i=1}} A(V, R_1,\ldots, R_k) \frac{X_i}{R_i !} \]

\[ V = 0,1,2,\ldots \]

(1.1)

Where $U_1,\ldots, U_k$ are arbitrary positive integers and the coefficients $A(V, R_1,\ldots, R_k)$ are arbitrary constants (real or complex).

The $\overline{H}$-function will be defined and represented in the following manner

\[ \overline{H}_{M,N}^{M,N}[z] = \overline{H}_{M,N}^{M,N}(z) \]

\[ = \frac{1}{2\pi i} \oint_{L} \phi(\xi) z^{\xi} d\xi \quad (z \neq 0) \]

(1.2)

where

\[ \phi(\xi) = \prod_{j=1}^{M} \Gamma(\beta_j - \alpha_j \xi) \prod_{j=1}^{N} \Gamma(1-\alpha_j + \alpha_j \xi) \]

(1.3)

The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}$-Function given by (1.2) have been recently given by Gupta, Jain and Agrawal (2007) [3].

(i) $|\arg(z)| < 1/2 \Omega \pi$ and $\Omega > 0$

(ii) $|\arg(z)| = 1/2 \Omega \pi$ and $\Omega \geq 0$

(1.4)

and (a) $\mu \neq 0$ and the contour $L$ is so chosen that

(c $\mu + \lambda + 1$) $< 0$

(b) $\mu = 0$ and $(\lambda + 1) < 0$

where

\[ \Omega = \sum_{M=1}^{M} \beta_j + \sum_{N=1}^{N} \alpha_j A_j - \sum_{M+1}^{M+P} \beta_j B_j - \sum_{N+1}^{N} \alpha_j \]

(1.5)

\[ \mu = \sum_{N=1}^{N} \alpha_j A_j + \sum_{M=1}^{M} \alpha_j - \sum_{M+1}^{M+Q} \beta_j B_j - \sum_{N+1}^{N} \alpha_j \]

(1.6)

\[ \lambda = \text{Re} \left( \sum_{M=1}^{M} b_j + \sum_{M=1}^{M} b_j B_j - \sum_{N=1}^{N} a_j A_j - \sum_{N+1}^{N} a_j \right) + \frac{1}{2} \left( -M - \sum_{M=1}^{M} B_j + \sum_{N=1}^{N} A_j + P - N \right) \]

(1.7)

It may be noted that the conditions of validity given above are more general than those given earlier by Buschman (1990)[1].

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The following series representation of the $H$-function was given by Rathie (1997) [4].

$$
\overline{H}^{M,N}_{P,Q} \left( \begin{array}{c}
(a_j, \alpha_j; A_j)_{1,N} \left( a_j, \alpha_j \right)_{N+1,P} \\
(b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}
\end{array} \right)
= \sum_{i=0}^{M} \sum_{j=0}^{N} \overline{g}(s_{j},i) z^{s_{j},i} \tag{1.8}
$$

where

$$
\overline{g}(s_{j},i) = \frac{\prod_{l=1}^{M} \Gamma(b_j - \beta_s, S_{p,u}) \prod_{j=1}^{N} \Gamma(1-a_j + \alpha, S_{p,u})/(1-z)^i}{\prod_{j=1}^{N} \Gamma(b_j + \beta_s, S_{p,u}) \prod_{j=1}^{N} \Gamma(1-a_j - \alpha, S_{p,u})/(1-z)^i}. \quad S_{p,u} = \frac{b_j + P}{\beta_s}
$$

To the sequel, we shall also make use of the following behavior of the $H$-function for small and large value of $z$ as recorded by Saxena (2002) [5, p.112, eqs. (2.3) and (2.4)].

$$
\overline{H}^{M,N}_{P,Q} \left( z \right) = O \left( \left| z \right|^{\alpha} \right) \text{ for small } z, \quad \text{where}
\alpha = \min_{1 \leq j \leq M} \left[ \Re \left( \frac{b_j}{\beta_s} \right) \right] \tag{1.10}
$$

$$
\overline{H}^{M,N}_{P,Q} \left( z \right) = O \left( \left| z \right|^{\beta} \right) \text{ for large } z, \quad \text{where}
\beta = \max_{1 \leq j \leq N} \left[ \Re \left( \frac{a_j - 1}{\alpha_j} \right) \right] \tag{1.11}
$$

and the conditions (1.4) are satisfied.

II. Multidimensional Fractional Integral Operators.

In the present paper we study the following fractional integral operators.

$$
I_{x} \left[ f \left( t_1, \ldots, t_s \right) \right] = I_{x^{\rho} \sigma \epsilon f g Y \lambda} \left[ f \left( t_1, \ldots, t_s ; x_1, \ldots, x_s \right) \right]
= \left[ \prod_{j=1}^{s} \frac{x_j}{t_j} \right] \left[ \prod_{j=1}^{s} \frac{x_j}{t_j} \right] \left[ \prod_{j=1}^{s} \frac{x_j}{t_j} \right] \left[ \prod_{j=1}^{s} \frac{x_j}{t_j} \right] f \left( t_1, \ldots, t_s ; dt_1, \ldots, dt_s \right)
$$

$$
J_{x} \left[ f \left( t_1, \ldots, t_s \right) \right] = J_{x^{\rho} \sigma \epsilon f g Y \lambda} \left[ f \left( t_1, \ldots, t_s ; x_1, \ldots, x_s \right) \right]
= \left[ \prod_{j=1}^{s} \frac{x_j}{t_j} \right] \left[ \prod_{j=1}^{s} \frac{x_j}{t_j} \right] \left[ \prod_{j=1}^{s} \frac{x_j}{t_j} \right] \left[ \prod_{j=1}^{s} \frac{x_j}{t_j} \right] f \left( t_1, \ldots, t_s ; dt_1, \ldots, dt_s \right)
$$

(1.12)

$$
\overline{H}^{M,N}_{P,Q} \left[ z \right] = O \left( \left| z \right|^{\beta} \right) \text{ for large } z, \quad \text{where}
\beta = \max_{1 \leq j \leq N} \left[ \Re \left( \frac{a_j - 1}{\alpha_j} \right) \right] \tag{1.11}
$$

and the conditions (1.4) are satisfied.

II. Some Useful Images

Now we shall obtain the images of some useful functions in our operators of study.

$$
I_{x} \left[ \prod_{j=1}^{s} x_{j}^{\gamma_j} \left( h_j + t_j \right)^{-\gamma_j} \right] = \left[ \prod_{j=1}^{s} x_{j}^{\gamma_j} \left( h_j + t_j \right)^{-\gamma_j} \right]
$$

$$
\sum_{r_1, \ldots, r_s \leq 0} \sum_{r_1, \ldots, r_s \leq 0} \left( -V \right)^{r_1, \ldots, r_s} A \left( V, R_1, \ldots, R_s \right) \frac{E_{r_1}}{R_1} \ldots \frac{E_{r_s}}{R_s}
$$

$$
\sum_{j=1}^{s} \prod_{j=1}^{s} x_{j}^{\gamma_j} \left( h_j + t_j \right)^{-\gamma_j}
$$

(1.13)

Throughout the paper we assume that

$$
f \left( t_1, \ldots, t_s \right) = \begin{cases} \max \left\{ \left| t_j \right| \right\} & \rightarrow 0 \quad j = 1, \ldots, s \\
\min \left\{ \left| t_j \right| \right\} & \rightarrow \infty \quad j = 1, \ldots, s
\end{cases}
$$

and the conditions (1.4) are satisfied.

$$
\overline{H}^{M,N}_{P,Q} \left[ z \right] = O \left( \left| z \right|^{\beta} \right) \text{ for large } z, \quad \text{where}
\beta = \max_{1 \leq j \leq N} \left[ \Re \left( \frac{a_j - 1}{\alpha_j} \right) \right] \tag{1.11}
$$

and the conditions (1.4) are satisfied.
\[
\begin{align*}
&\min_{1 \leq k \leq M} \text{Re}\left[1 + \rho_j + \gamma_j + \eta_j \frac{b_j}{\beta_k}\right] > 0; \\
&\min_{1 \leq k \leq M} \text{Re}\left[\sigma_j + \lambda_j \frac{b_j}{\beta_k}\right] > 0 \\
&J_x \left[\prod_{j=1}^{s} x_j^{t_j}(h_j + t_j)^{-\delta_j}\right] = \left(\prod_{j=1}^{s} x_j^{t_j}(x_j + h_j)^{-\delta_j}\right) \\
&\sum_{j=1}^{s} \sum_{U, R_j \in U, R_j \subseteq V} A(V, R_1, \ldots, R_j) \frac{E_{R_1}}{R_1!} \ldots \frac{E_{R_j}}{R_j!} \\
&\sum_{l=0}^{\infty} \prod_{j=1}^{s} x_j^{t_j} h_j^{t_h}(x_j + h_j)^{-l}(\delta_j)_j \\
&\frac{H_{M,N+2s}}{H_{p+2s,Q+2s}} \left[ (a_j, \alpha_j; A_j)_{1,N} \right] \left[ (a_j, \alpha_j; A_j)_{N+1,P} \right] \\
&\prod_{j=1}^{s} \left[ 1 - \rho_j + \gamma_j - \delta_j - e_j R_j ; \eta_j; 1 \right] \left[ 1 - \sigma_j - l - f_j R_j ; \lambda_j; 1 \right] \\
&\prod_{j=1}^{s} \left[ 1 - \rho_j - \delta_j - \sigma_j - l - \gamma_j - (e_j + f_j) R_j ; (\eta_j + \lambda_j); 1 \right] \\
&\prod_{j=1}^{s} \left[ 1 - \rho_j + \gamma_j - \delta_j - e_j R_j ; \eta_j; 1 \right] \left[ 1 - \sigma_j - l - f_j R_j ; \lambda_j; 1 \right] \\
&\prod_{j=1}^{s} \left[ 1 - \rho_j - \delta_j - \sigma_j - l - \gamma_j - (e_j + f_j) R_j ; (\eta_j + \lambda_j); 1 \right] \\
&\min \text{Re}(e_j, f_j, \eta_j, \lambda_j) \geq 0 \quad (j = 1, \ldots, s) \\
&\text{where} \quad \min \text{Re}(e_j, f_j, \eta_j, \lambda_j) > 0; \min \text{Re}(\sigma_j, \lambda_j, b_j, \beta_k) > 0 \\
&I_x \prod_{j=1}^{s} t_j^{r_j} H \left[ \prod_{j=1}^{s} x_j^{t_j}(1 + (h_j t_j)^{K_j})^{-l_j} \right] \\
&= \left(\prod_{j=1}^{s} x_j^{t_j}\right) \sum_{R_1, \ldots, R_j = 0} (-V) \prod_{j=1}^{s} A(V, R_1, \ldots, R_s) \\
&\frac{E_{R_1}}{R_1!} \ldots \frac{E_{R_j}}{R_j!} \sum_{l=0}^{\infty} \prod_{j=1}^{s} x_j^{t_j} h_j^{t_h}(x_j + h_j)^{-l}(\delta_j)_j \\
&\frac{H_{0,M+2s}}{H_{p+2s,Q+2s}} \left[ (h_1 x_1)^{k_1} \right] \left[ (h_s x_s)^{k_s} \right]
\end{align*}
\]

where
\[
A^* = \left( a_{k_1}; a_{k_2}; a_{k_3}; \ldots, a_{k_r}; 0, \ldots, 0 \right)_{l,p} \left( 1; v_1^{(r)}; v_1^{(r)}; 1, 0, \ldots, 0 \right)_{r-1} \\
\vdots \\
\frac{(-1)^{r-1} \prod_{j=1}^{s} e_j R_j ; u_j^{(r)} ; k_1 ; 0, \ldots, 0 ; \ldots, (-1)^{r-1} \prod_{j=1}^{s} e_j R_j ; u_j^{(r)} ; k_1 ; 0, \ldots, 0 ; k_s}{s-1} \\
(2.2)
\]

and
\[
B^* = \left( b_k ; 0, \ldots, 0 ; b_k^{(r)} ; \ldots, b_k^{(r)} ; 0, \ldots, 0 \right)_{l,q} \left( 1; 0, \ldots, 0 ; v_1^{(r)} ; \ldots, v_1^{(r)} \right)_{r-1} \\
\vdots \\
\frac{(-1)^{r-1} \prod_{j=1}^{s} (e_j + f_j) R_j ; u_j^{(r)} ; k_1 ; 0, \ldots, 0 ; \ldots, (-1)^{r-1} \prod_{j=1}^{s} (e_j + f_j) R_j ; u_j^{(r)} ; k_1 ; 0, \ldots, 0 ; k_s}{s-1} \\
(2.5)
\]

and
\[
C^* = \left( c_{k_1}^{(r)} ; c_{k_2}^{(r)} ; \ldots, c_{k_r}^{(r)} ; 0, \ldots, 0 \right)_{l,p} \\
D^* = (0.1; \ldots; (0.1 ; (d_{k_1}^{(r)} ; \delta_{k_1}^{(r)} ; \ldots, (d_{k_r}^{(r)} ; \delta_{k_r}^{(r)})_{l,q} \\
(2.3)
\]
where \( \bar{\theta}(S_{p,v}) \), \( S_{p,v} \) shall be obtained from (1.9) and occurring on the right hand side of equation (2.4) and (2.5) would mean \( s \) zeros and so on. Provided that not all zero simultaneously,

\[
\min \Re \left[ 1 + \rho_j + \gamma_j + \eta_j \frac{b_k}{\beta_k} + \sum_{i=1}^r u^{(i)}(t) \frac{d_k^{(i)}}{\sigma_k^{(i)}} \right] > 0
\]

Proof: To prove (2.1), first of all we express the \( I \)-operator involved in its left hand side in the integral form with the help of equation (1.12). Next, we express \( S_{V_{1},...,V_k}(x_1,...,x_k) \) polynomials occurring therein in the series form using (1.1). Then, we change the order of the series and \( \int_j \) -integrals and express the \( H \) -function in terms of Mellin Barnes type contour integrals with the help of (1.2). Now we change the order of \( \tilde{z} \) and \( \int_j \) -integrals (\( j = 1,...,s \)) (which is permissible under the conditions stated). Finally, evaluating the \( \int_j \) -integrals with the help of known result (Gradshteyn and Ryzhik (1980)) [2, p. 287, eq. 3.197(8)] we get

\[
3 \cdot 197(8) \text{ we get}
\]

\[
\sum_{j=1}^r U_j R_j \leq V \sum_{j=1}^r U_j R_j A(V, R_1,...,R_s) \frac{E_{R_1}...E_{R_s}}{R_1!...R_s!}
\]

\[
\frac{1}{2\pi i} \int_{\tilde{\theta}(\xi)} Z^\xi \prod_{j=1}^s x_j^{\gamma_j} h_j^{-\delta_j}
\]

\[
B(\sigma_j + f_j R_j + \lambda_j \xi, \rho_j + \gamma_j + e_j R_j + \eta_j \xi + 1)
\]

\[
F_1 \left( \sigma_j + f_j R_j + \lambda_j \xi, \rho_j + \gamma_j + e_j R_j + \eta_j \xi + 1 \right)
\]

\[
\arg \left( \frac{x_j}{h_j} \right) < \pi, \Re \left( \sigma_j + f_j R_j + \lambda_j \xi \right) > 0,
\]

\[
\Re \left( \rho_j + \gamma_j + e_j R_j + \eta_j \xi + 1 \right)
\]

Now reinterpreting the result thus obtained in terms of the \( H \) -function, we easily arrive at the desired result after a little simplification.

Again the proof of result (2.2) can be developed similarly using the formula [Gradshteyn and Ryzhik (1980) 2, p.286, 3.197(2)]. Result (2.6) can be similarly established on expressing the \( H \) -function involved in the operator in its series form using (1.8).
\[
\prod_{j=1}^{s}\left\{ (1-\rho_j-w_j-e_jR_j;\eta_j;1)(1-\sigma_j-l-f_jR_j;\lambda_j;1) \right\} \\
\prod_{j=1}^{s}\left\{ (1-\rho_j-w_j-\sigma_j-l-\left(e_j+f_j\right)R_j;\eta_j+\lambda_j;1) \right\} \\
\psi_2(x_1,\ldots,x_s;h_1,\ldots,h_s) = I_x\left[ \prod_{j=1}^{s}(h_j+t_j)^{-w_j} \right] \\
= \left( \prod_{j=1}^{s}(x_j+h_j)^{-w_j} \right) \sum_{R_1=0}^{\infty} (-V) \sum_{j=1}^{s} A(V,R_1,\ldots,R_s) \\
E_1^{R_1} E_s^{R_s} R_1! \ldots R_s! \sum_{l=0}^{\infty} \left[ \prod_{j=1}^{s} x_j^l(x_j+h_j)^{-l}(w_j)_l \right] \\
\bar{H}_{M,N+1}^{N+2s,Q+1}\left[ \left( x_j+\alpha_j;\lambda_j,\gamma_j;1 \right)_{1,N},(a_j,\alpha_j,\gamma_j)_{N+1,P},(b_j,\beta_j,\gamma_j)_{M+1,Q} \right] \\
\sum_{j=1}^{s}\left\{ (1-\rho_j-w_j-e_jR_j;\eta_j;1)(1-\sigma_j-l-f_jR_j;\lambda_j;1) \right\} \\
\sum_{j=1}^{s}\left\{ (1-\rho_j-w_j-\sigma_j-l-\left(e_j+f_j\right)R_j;\eta_j+\lambda_j;1) \right\} \\
\text{(3.5)}
\]

It is assumed that the integrals on the right hand side of equations (3.2) and (3.3) exist.

Proof: To prove first part of theorem 1, we express the left hand side of (3.2) with the help of (1.12) and (3.1), then we interchange the order of \( t_j \) and \( x_j \) integrals (which is permissible under the conditions stated with the theorem). Finally evaluating the inner \( t_j \)-integrals with the help of result (2.1) (taking \( \gamma_j = 0 \) therein), we easily arrive at desired result after a little simplification.

Similarly the second result (3.3) of theorem 1 can be established on using (2.2).

The following theorem gives the fractional integrals of generalized Stieljes transform given by (3.1)

Theorem 2. Let \( \phi(t_1,\ldots,t_s) \in A \),
\[
\min(e_j, f_j, \eta_j, \lambda_j) \geq 0 \text{ for } (j = 1,\ldots,s) \text{ not all zero simultaneously} \\
\text{Re}(W_j) > 0; \text{Re}(\sigma_j + \lambda_j \frac{b_j}{\beta_k}) > 0; \\
\text{Re}(W_j) = 0; \text{Re}(V_j + w_j) > 0.
\]

then for \( \min_{1 \leq k \leq M} \text{Re}\left[ 1 + \rho_j + \eta_j \frac{b_j}{\beta_k} \right] > 0 \), \( (j = 1,\ldots,s) \)

\[
I_y(S_{w_1,\ldots,w_s}(\phi))(x_1,\ldots,x_s)
\]

and for \( \min_{1 \leq k \leq M} \text{Re}\left[ \rho_j + w_j + \eta_j \frac{b_j}{\beta_k} \right] > 0 \), \( (j = 1,\ldots,s) \)

\[
J_y(S_{w_1,\ldots,w_s}(\phi))(x_1,\ldots,x_s)
\]

where \( \psi_1(t_1,\ldots,t_s; x_1,\ldots,x_s) \) and \( \psi_2(t_1,\ldots,t_s; x_1,\ldots,x_s) \) are as given in (3.4) and (3.5) respectively, provided that the integrals in the right hand side of equations (3.6) and (3.7) exists.

Proof: Results (3.6) and (3.7) of Theorem 2 can be obtained on the similar lines to the proof of Theorem 1.

We can easily obtain the one dimensional analogues of the theorem 1 and 2, however, we omit the details here.

IV Mellin Transforms, Inversion Formulas and convolutions

The multidimensional Mellin Transform of the function \( f(t_1,\ldots,t_s) \in A \) is defined by the following equation
[Saxena and Panda (1978) 7, part I, p. 125, eq. (3.5)].

\[
M[f(t_1,\ldots,t_s);\gamma_1,\ldots,\gamma_s] = \int_0^\infty \cdots \int_0^\infty \left( \prod_{j=1}^{s} t_j^{\gamma_j-1} \right) f(t_1,\ldots,t_s) dt_1 \cdots dt_s
\]

(4.1)

where
\[
\text{Re}(\gamma_j + U_j) > 0, \text{Re}(W_j) > 0 \quad \text{or} \quad \text{Re}(V_j - \gamma_j) > 0 \quad (j = 1,\ldots,s)
\]

Now we shall establish the following results

Result 1

If \( M\left[I_x\{f(t_1,\ldots,t_s);\gamma_1,\ldots,\gamma_s\}\right] \text{ exists} \), then
\[
M\left[I_x\{f(t_1,\ldots,t_s);\gamma_1,\ldots,\gamma_s\}\right] = M\left[f(t_1,\ldots,t_s);\gamma_1,\ldots,\gamma_s\right] \chi(1-\gamma_1,\ldots,1-\gamma_s)
\]

(4.2)

Result 2

If \( M\left[J_x\{f(t_1,\ldots,t_s);\gamma_1,\ldots,\gamma_s\}\right] \text{ exists} \), then
\[
M\left[J_x\{f(t_1,\ldots,t_s);\gamma_1,\ldots,\gamma_s\}\right] = M\left[f(t_1,\ldots,t_s);\gamma_1,\ldots,\gamma_s\right] \chi(1-\gamma_1,\ldots,1-\gamma_s)
\]

(4.3)

where
\[
\chi(\gamma_1,\ldots,\gamma_s) = \sum_{R_1=0}^{\sum_{U,R_j}\in V} (-V) \sum_{R_1=0}^{\sum_{U,R_j}} A(V,R_1,\ldots,R_s)
\]
If \( f(t_1, \ldots, t_s) \in A \), then the fractional integral operators defined by (1.12) and (1.13) can readily be expressed as multidimensional Mellin convolutions in the following form.

**Result 5.**

\[
I_{\rho, \sigma, \varepsilon, f; \eta, \lambda; x; U, V, Z} f(t_1, \ldots, t_s) = (I_{\rho, \sigma, \varepsilon, f; \eta, \lambda; x; U, V, Z} \ast f)(x_1, \ldots, x_s)
\]  

(4.8)

where

\[
I_{\rho, \sigma, \varepsilon, f; \eta, \lambda; x; U, V, Z} = \left( \prod_{j=1}^s x_j^{-\rho_j - \sigma_j} \left( x_j - 1 \right)^{\sigma_j - 1} U \left( x_j - 1 \right) \right)
\]

(4.9)

**Proof:** To prove result 1, first of all we write the multidimensional Mellin Transform of the I-operator with the help of equation (4.1), then we change the order of \( t_j \) and \( x_j \)-integrals. Next, with the help of (2.1) and (4.1) we easily arrive at the desired result (4.2) after a little simplification.

The proof of result 2 can be developed by proceeding on the lines similar to those indicated above.

**Inversion Formulas**

On making use of the inversion theorems for the multidimensional Mellin Transform (4.1), given by Srivastava and Panda (1978) [7, part I,p.125, Lemma 2], we easily get from (4.2) and (4.3) the following inversion formula for the fractional integral operators defined by (1.12) and (1.13).

**Result 3.**

\[
f(t_1, \ldots, t_s) = \frac{1}{(2\pi i)^s} \int_{\gamma_1-i \infty}^{\gamma_1+i \infty} \cdots \int_{\gamma_s-i \infty}^{\gamma_s+i \infty} \prod_{j=1}^s t_j^{\gamma_j} \chi(\gamma_1, \ldots, \gamma_s)
\]

(4.5)

\[
M \left[ I_x \left\{ f(t_1, \ldots, t_s); \gamma_1, \ldots, \gamma_s \right\} \right] d\gamma_1 \ldots d\gamma_s
\]

**Result 4.**

\[
f(t_1, \ldots, t_s) = \frac{1}{(2\pi i)^s} \int_{\gamma_1-i \infty}^{\gamma_1+i \infty} \cdots \int_{\gamma_s-i \infty}^{\gamma_s+i \infty} \prod_{j=1}^s t_j^{\gamma_j} \chi(1-\gamma_1, \ldots, 1-\gamma_s)
\]

(4.6)

\[
M \left[ J_x \left\{ f(t_1, \ldots, t_s); \gamma_1, \ldots, \gamma_s \right\} \right] d\gamma_1 \ldots d\gamma_s
\]

The precise conditions under which the inversion formulas (4.5) and (4.6) are valid can be deduced from the existence condition of the various fractional integral operators and their multidimensional Mellin transforms stated earlier.

**IV. MELLIN CONVOLUTIONS**

The multidimensional Mellin convolutions of two functions \( f(t_1, \ldots, t_s) \) and \( g(t_1, \ldots, t_s) \) will be defined by

\[
(f \ast g)(t_1, \ldots, t_s) = (g \ast f)(t_1, \ldots, t_s)
\]

(4.7)

\[
= \int_0^\infty \int_0^\infty \left( \prod_{j=1}^s x_j^{-1} \right) f \left( \frac{t_1}{x_1}, \ldots, \frac{t_s}{x_s} \right) g(x_1, \ldots, x_s) \, dx_1 \ldots dx_s
\]

provided the multiple integrals exist.
\[ I_{x, i, y, z}^{\rho, \sigma, \eta, \lambda} f(t_1, \ldots, t_s) \]
\[ = \int_0^\infty \left[ \prod_{j=1}^s \left( \frac{x_j}{t_j} - 1 \right)^{\rho_j - \sigma_j} U \left( \frac{x_j}{t_j} - 1 \right) \right] dt_1 \ldots dt_s \]
\[ S_{y, i, j, l}^{x, u, v} \left[ E_{\lambda_i} \left( \frac{x_i}{t_i} - 1 \right)^{\lambda_i} \right] \ldots E_{\lambda_s} \left( \frac{x_s}{t_s} - 1 \right)^{\lambda_s} \]
\[ \overline{H}_{p,q}^{M,N} \left[ Z \prod_{j=1}^s \left( \frac{x_j}{t_j} - 1 \right)^{-\eta_j - \lambda_j} \right] f(t_1, \ldots, t_s) dt_1 \ldots dt_s \]

(4.12)

Now making use of the equation (4.9) and the definition of the Mellin convolutions given by (4.7) in the above equation, we easily arrive at the desired result. The proof of the result 6 can be developed on the same lines.

V. REFERENCES