

Graph Transformations, Interpolation And Extremal Theorems For Graph Parameters

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Abstract—Let $\mathcal{G}$ be the class of all graphs and $\mathcal{J} \subseteq \mathcal{G}$. A graph parameter π is called an interpolation graph parameter over $\mathcal{J}$ if there exist integers a and b such that

$$\{\pi(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : a \leq k \leq b\}.$$

Thus if π is an interpolation graph parameter over $\mathcal{J}$ then $\{\pi(G) : G \in \mathcal{J}\}$ is uniquely determined by

$$\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\} \text{ and } \max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in \mathcal{J}\}.$$

The problem of finding $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$ is called the extremal problem in graph theory. We will discuss our results which have been done, in this direction, in the past ten years. Some open problems are also reviewed.

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I. INTRODUCTION

Only finite simple graphs are considered in this paper. For the most part, our notation and terminology follows that of Bondy and Murty [1].

Let $\mathcal{J}$ be a class of non-isomorphic graphs. A graph transformation on $\mathcal{J}$ is a subset of $\mathcal{J} \times \mathcal{J}$. Let ρ be a graph transformation on $\mathcal{J}$. We can define the p -graph having $\mathcal{J}$ as its vertex set and there is a directed edge from G to H if and only if $(G, H) \in \rho$. If ρ is symmetric, it yields an undirected graph and otherwise a directed graph.

Harary [10] used a graph transformation called a fundamental exchange or an edge exchange as follows: Let G be a connected graph of order $n > 3$. The tree graph, $T(G)$, of G is defined by specifying $V(T(G))$ as the set of all spanning trees of G , and two vertices $T_1, T_2 \in V(T(G))$ are adjacent in $T(G)$ if and only if T_1 and T_2 have exactly $n - 2$ edges in common. This is an example of an undirected p -graph. It was proved by Harary [10] that the tree graph $T(G)$ is connected.

A non-increasing sequence $d = (d_1; d_2; \dots; d_n)$ of non-negative integers is a graphic degree sequence if it is a degree sequence of some graph G . In this case, G is called a realization of d : A degree sequence of an r -regular graph of order n is denoted by r^n .

Let G be a graph. For the distinct vertices a, b, c , and d in $V(G)$ such that ab and cd are edges in G while ac and bd are not edges in G . Define $G^{\sigma(a,b;c,d)}$, simply written G^{σ} to be the graph obtained from G by deleting the edges ab and cd and adding the edges ac and bd .

The operation $\sigma(a; b; c; d)$ is called a switching operation. For a graphic degree sequence d , let $R(d)$ and $CR(d)$ be the

sets of non-isomorphic realizations and connected realizations of d , respectively. The $\Sigma(d)$ is defined as a relation on $R(d)$ as $(G, H) \in \Sigma(d)$ if $G \not\cong H$ and there is a switching σ on G such that $H = G^{\sigma}$. Thus the $\Sigma(d)$ -graph is simple. The concept of $\Sigma(d)$ -graph was introduced and developed in a joint paper by Eggleton and Holton [7]. It provides a structured way to examine all the graphs which "realize" a given degree sequence. The $\Sigma(d)$ -graph and the subgraph induced by $CR(d)$ are connected as a consequence of Taylor [22, 23]. For positive integers m and n with $0 \leq m \leq \binom{n}{2}$, let $G(m; n)$ and $CG(m; n)$ be the sets of all non-isomorphic graphs and the set of connected graphs of order n and size m , respectively. Let

and $f \notin E(G)$. Define $G^{t(e,f)}$ to be a graph with $V(G^{t(e,f)}) = V(G)$ and $E(G^{t(e,f)}) = E(G - e + f)$. A transformation $t(e, f)$ is called an edge jump. Now let $T(m, n)$ be a relation on $G(m; n)$ defined by $(G; H) \in T(m, n)$ if $G \not\cong H$ and H can be obtained from G by an edge jump. Since $T(m; n)$ is symmetric, it follows that the $T(m; n)$ -graph is simple.

II. THE INTERPOLATION GRAPH PARAMETERS

Let $\mathcal{G}$ be the class of all graphs. A function $\pi : \mathcal{G} \rightarrow \mathbb{Z}$ is called a graph parameter if $\pi(G) = \pi(H)$ for all isomorphic graphs G and H . A graph parameter π is called an interpolation graph parameter over $\mathcal{J} \subseteq \mathcal{G}$ if there exist integers x and y such that

$$\{\pi(G) : G \in \mathcal{J}\} = \{k \in \mathbb{Z} : x \leq k \leq y\}.$$

If π is an interpolation graph parameter over $\mathcal{J}$ then $\{\pi(G) : G \in \mathcal{J}\}$ is uniquely determined by $\min(\pi, \mathcal{J}) := \min\{\pi(G) : G \in \mathcal{J}\}$ and $\max(\pi, \mathcal{J}) := \max\{\pi(G) : G \in \mathcal{J}\}$.

In 1964, Erdős and Gallai [8] proved that any regular graph on n vertices has chromatic number $k \leq \frac{3n}{5}$ unless the graph is complete. Commenting on their result in a personal communication, Erdős wrote to Pullman "probably such a graph exists for every $k \leq \frac{3n}{5}$ except possibly for trivial exceptional cases."

Caccetta and Pullman [3] confirmed and strengthened Erdős' conjecture by showing that if $k > 1$, then for every $n \geq \frac{5k}{3}$, there exists a connected, regular, k -chromatic graph of order n . This is an example of interpolation graph parameter X over the class of all connected regular graphs of order n .

A. The $\Sigma(d)$ -graphs

We will review in this subsection the interpolation properties of various graph parameters over $R(d)$ and $CR(d)$. We first prove a general result as follows:

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Theorem 2.1- Let $\mathcal{J} \subseteq \mathcal{R}(d)$ and the subgraph of $\Sigma(d)$ -graph induced by $\mathcal{J}$ be connected. Let π be a graph parameter. For any graph G of degree sequence d and any switching σ . If $|\pi(G) - \pi(G^\sigma)| \leq 1$, then π : an interpolation graph parameter over $\mathcal{J}$.

Proof- Let $H, K \in \mathcal{J}$ such that

$$\pi(H) = \min\{\pi(G) : G \in \mathcal{J}\} \text{ and } \pi(K) = \max\{\pi(G) : G \in \mathcal{J}\}.$$

Since the subgraph of $\Sigma(d)$ -graph induced by $\mathcal{J}$ is connected, there exists a path $P : H = G_1; G_2; \dots; G_t = K$ in $\mathcal{J}$. Thus there exists a sequence $\sigma_1, \sigma_2, \dots, \sigma_{t-1}$ such that $G_{i+1} = G_i^{\sigma_i}$. Since $|\pi(G_i) - \pi(G_{i+1})| = |\pi(G_i) - \pi(G_i^{\sigma_i})| \leq 1$, it follows that $\{\pi(G_i) : i = 1, 2, \dots, t\} = \{k \in \mathbb{Z} : \pi(H) \leq k \leq \pi(K)\}$. Thus π is an interpolation graph parameter over $\mathcal{J}$.

The following result can be obtained as consequences of Taylor [22, 23].

Corollary 2.2- Let π be a graph parameter. For a graph G of degree sequence d and A switching σ $|\pi(G) - \pi(G^\sigma)| < 1$. then π is an interpolation graph parameter over $\mathcal{R}(d)$ and $\mathcal{CR}(d)$.

We will now review interpolation results on graph parameters over $\mathcal{R}(d)$ and $\mathcal{CR}(d)$. Here we use $\omega(G)$ and $\alpha(G)$ for the clique and independent number of a graph G respectively.

We proved in [13] and [14] the following result.

Theorem 2.3- Let G be a graph and σ be a switching on G .

If $\pi \in \{\chi, \omega\}$, then $|\pi(G) - \pi(G^\sigma)| \leq 1$.

Note that $\alpha(G) = \omega(\overline{G})$ for any graph G and $\overline{G}^{\sigma(a,b,c,d)} = \overline{G^{\sigma(a,b,c,d)}}$.

Thus we have the following corollary.

Corollary 2.4- Let G be a graph and σ be a switching on G . Then $|\alpha(G) - \alpha(G^\sigma)| \leq 1$. For the matching number $\alpha'(G)$ of a graph G we obtained in [18] the following result.

Theorem 2.5- If σ is a switching on G ; then $|\alpha'(G) - \alpha'(G^\sigma)| \leq 1$. The following results were obtained by Gallai [9] showing a relationship between the independence and covering number. Here we use $\beta(G)$ and $\beta'(G)$ for the covering and edge covering number of a graph G respectively.

Theorem 2.6- For a graph G of order n ; $\alpha(G) + \beta(G) = n$.

Theorem 2.7- For a graph G of order n and $\delta \geq 1$, $\alpha'(G) + \beta'(G) = n$.

As a consequence we obtain the following result.

Theorem 2.8- Let G be a graph, $\delta(G) \geq 1$ and σ be a switching on G . If $\pi \in \{\beta, \beta'\}$, then $|\pi(G) - \pi(G^\sigma)| \leq 1$.

Let G be a graph and $F \subseteq V(G)$. Then F is called an induced forest of G if $G[F]$ contains no cycle. For a graph G ; we define, $f(G)$ as:

$$f(G) := \max\{|F| : F \text{ is an induced forest in } G\}.$$

The graph parameter f is called the forest number. The problem of determining the minimum number of vertices whose removal eliminates all cycles in a graph G is known as the decycling number of G , and is denoted by $\phi(G)$. Thus for a graph G of order n , $\phi(G) + f(G) = n$. We proved in [15] the following results on f and ϕ .

Theorem 2.9- If S is any subset of vertices of G such that $G[S]$ is a forest, and ϕ is any switching on G ; then $G^\sigma[S]$

contains at most one cycle.

Proof- Let $S \subseteq V(G)$ and $G[S]$. Let $a; b; c; d \in V(G)$ with $ab; cd \in E(G)$ and $ac; bd \notin E(G)$. Since $G[S]$ contains no cycle, it follows that $G[S] + ac$ and $G[S] + bd$ contains at most one cycle. Thus if $|S \cap \{a, b, c, d\}| \leq 3$, then $G[S]$ contains at most one cycle. Now suppose that $\{a; b; c; d\} \subseteq S$. Since $G[S]$ is a forest, for any two vertices $u, v \in S$ there is at most one $(u; v)$ -path in $G[S]$. In particular, if there is an $(a; c)$ -path in $G[S]$, then there is no $(b; d)$ -path in $G[S]$. Thus $G^\sigma[S]$ contains at most one cycle, where $\sigma = \sigma(a, b; c, d)$.

The following corollary can be obtained as a consequence of above theorem.

Corollary 2.10- Let G be a graph and ϕ be a switching on G . If $\pi \in \{f, \phi\}$, then $|\pi(G) - \pi(G^\sigma)| \leq 1$.

A dominating set of a graph $G = (V; E)$ is a subset D of V such that each vertex of $V - D$ is adjacent to at least one vertex of D : The domination number $\gamma(G)$ of a graph G is the cardinality of a minimal dominating set with the least number of elements. We proved in [17] the following results.

Theorem 2.11- If G is a graph with $\gamma(G) = \gamma$ and ϕ is a switching on G ; then $\gamma(G^\sigma) \leq \gamma + 1$.

Proof- Let D be a minimum dominating set of G . Let $a; b; c; d \in V(G)$ with $ab; cd \in E(G)$ and $ac; bd \in E(G)$. Put $\phi = \phi(a; b; c; d)$. If $\{a; b; c; d\} \cap D = \emptyset$ or $\{a, b, c, d\} \subseteq D$, then D is a dominating set of G^σ . If $a, b \in D$ or $c, d \in D$, then D is a dominating set of G^σ . Finally if $a \in D$ or $c \in D$, then $D \cup \{b\}$ or $D \cup \{d\}$ is a respective dominating set of G^σ . Thus $\gamma(G^\sigma) \leq \gamma + 1$.

By the fact that a switching is symmetric we obtain the following result.

Corollary 2.12- If ϕ is a switching on G ; then $|\gamma(G) - \gamma(G^\sigma)| \leq 1$.

Combining the results in this subsection we can conclude the following theorem.

Theorem 2.13- Let $d = (d_1; d_2; \dots; d_n)$; $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$ be a graphic degree sequence. Then $\chi, \omega, f, \phi, \alpha, \alpha', \beta, \beta'$ and γ are interpolation graph parameters over $\mathcal{R}(d)$ and $\mathcal{CR}(d)$.

B. The $T(m; n)$ -graphs

We recently proved in [21] that the $T(m; n)$ -graph and the subgraph of the $T(m; n)$ -graph induced by $\mathcal{CG}(m; n)$ are connected. We also obtained in the same paper the following results.

Theorem 2.14- $\pi \in \{\chi, \omega, f, \phi, \alpha, \alpha', \beta, \beta', \gamma\}$. Then for any $G \in \mathcal{G}(m, n)$ and an edge jump $t(e; f)$ on G , $|\pi(G) - \pi(G^{t(e,f)})| \leq 1$.

Theorem 2.15- $\pi \in \{\chi, \omega, f, \phi, \alpha, \alpha', \beta, \beta', \gamma\}$. Then π is an interpolation graph parameter over $\mathcal{G}(m; n)$ and $\mathcal{CG}(m; n)$:

III. THE EXTREMAL PROBLEMS

An extremal problem asks for minimum and maximum values of a function π over a class of objects. In our context we consider the problem of determining $\min(\pi, \mathcal{J})$ and $\max(\pi, \mathcal{J})$, where π is a graph parameter and

$\mathcal{J}$ is a class of graphs. We emphasize on the graph parameters as stated in Section 2 and the classes of graphs $\mathcal{J} \in \{\mathcal{R}(r^n), \mathcal{CR}(r^n), \mathcal{G}(m, n), \mathcal{CG}(m, n)\}$. Therefore we use the following notation.

- $\min(\pi, r^n) = \min\{\pi(G) : G \in \mathcal{R}(r^n)\}$,
- $\max(\pi, r^n) = \max\{\pi(G) : G \in \mathcal{R}(r^n)\}$,
- $\text{Min}(\pi, r^n) = \min\{\pi(G) : G \in \mathcal{CR}(r^n)\}$,
- $\text{Max}(\pi, r^n) = \max\{\pi(G) : G \in \mathcal{CR}(r^n)\}$,
- $\min(\pi; m, n) = \min\{\pi(G) : G \in \mathcal{G}(m, n)\}$,
- $\max(\pi; m, n) = \max\{\pi(G) : G \in \mathcal{G}(m, n)\}$,
- $\text{Min}(\pi; m, n) = \min\{\pi(G) : G \in \mathcal{CG}(m, n)\}$, and
- $\text{Max}(\pi; m, n) = \max\{\pi(G) : G \in \mathcal{CG}(m, n)\}$.

A. $\mathcal{R}(r^n)$ and $\mathcal{CR}(r^n)$

We will focus on the extremal problem over the classes of regular graphs and some other related classes in this subsection. A classical result of Erdős and Gallai [8] gives a motivation to the extremal problem.

Theorem 3.1- An r -regular graph G of order $n > r + 1$ has chromatic number $k \leq \frac{3n}{5}$, with equality if and only if the complementary graph $\overline{G}$ of G is the union of disjoint 5-cycles.

We obtained in [13] the extremal values of X

Theorem 3.2- $r \geq 2$ and $n \geq 2r$,

$$\min(\chi, r^n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3.3- If $r \geq 2$, then

- i. $\min(\chi, r^{r+1}) = \max(\chi, r^{r+1}) = r + 1$, and
- ii. $\min(\chi, r^{r+2}) = \max(\chi, r^{r+2}) = (r + 2)/2$.

Theorem 3.4- For any $r \geq 4$ and odd integer s such that $3 \leq s \leq r$, let q and t be integers satisfying Then

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q + 1 & \text{if } 1 \leq t \leq s - 2, \\ q + 2 & \text{if } t = s - 1. \end{cases}$$

Theorem 3.5- For any even integer $r \geq 6$ and any even number s such that $4 \leq s \leq r$, let q and t be integers satisfying $r + s = sq + t$, $0 \leq t < s$, then

$$\min(\chi, r^{r+s}) = \begin{cases} q & \text{if } t = 0, \\ q + 1 & \text{if } t \geq 2. \end{cases}$$

By using Brooks' theorem [2] and some graph construction we obtained the following theorems in [13].

Theorem 3.6- Let $r \geq 2$. Then

- i. $\max(\chi, r^{2r}) = r$,
- ii. $\max(\chi, r^{2r+1}) = \begin{cases} 3 & \text{if } r = 2, \\ r & \text{if } r \geq 4, \end{cases}$
- iii. $\max(\chi, r^n) = r + 1$ for $n \geq 2r + 2$.

Theorem 3.7- For any r and s such that $3 \leq s \leq r - 1$, we have

- i. $\max(\chi, r^{r+s}) \geq (r + s)/2$ if $r + s$ is even, and
- ii. $\max(\chi, r^{r+s}) \geq (r + s - 1)/2$ if $r + s$ is odd.

The exact values of $\max(\chi, r^n)$ are not easy to obtain if $r + 3 \leq n \leq 2r - 1$. Result of Theorem 3.1 gives an upper bound for χ in the class of connected regular graphs of order n but the bound can be very far from the actual value depending on the regularity. We were able to improve the bound in [12] by introducing the definition of $F(j)$ -graph.

Let $\mathcal{J}$ be a positive integer. An $F(j)$ -graph is a $(j - 1)$ -regular graph G of minimum order $f(j)$ with the property that $\chi(\overline{G}) > f(j)/2$. It is easy to see that $F(3)$ -graph is C_5 and $f(3) = 5$.

We found $F(j)$ -graphs for all odd integers j as stated in the following theorems.

Theorem 3.8- For odd integer $j \geq 3$, we have

$$f(j) = \frac{5}{2}(j - 1) \text{ if } j \equiv 3 \pmod{4} \text{ and } f(j) = 1 + \frac{5}{2}(j - 1) \text{ if } j \equiv 1 \pmod{4}.$$

Theorem 3.9- [12]- Any r -regular graph of order n with $n - r = j$ odd and $j \geq 3$ has chromatic number at most

$$\frac{f(j) + 1}{2f(j)} n,$$

and this bound is achieved precisely for those graphs with complement equal to a disjoint union of $F(j)$ -graphs.

Problem 1.- Find an $F(j)$ -graph for even integer $j \geq 4$.

Problem 2.- Find $\max(\chi, r^{r+j})$ if j is even and $4 \leq j \leq r - 2$.

The extremal problem for ω has been completely answered in [14]. Since K_{r+1} is the only r -regular graph of order $r + 1$, it follows that $\min(\omega, r^{r+1}) = \max(\omega, r^{r+1}) = r + 1$. Given positive integers n and k with $k \leq n$, there exists a connected graph G of order n with $\omega(G) = k$. As we shall see in the next theorem that there is no regular graph G of order n having $\omega(G)$ strictly lies between $\frac{n}{2}$ and n .

Theorem 3.10- Let $d = r^n$ be a graphic degree sequence with $r + 2 \leq n \leq 2r + 1$. Then $\max(\omega, r^n) = \lfloor \frac{n}{2} \rfloor$.

The idea of obtaining $\min(\omega, r^n)$ is similar to what we have done for $\min(\chi, r^n)$ and we have $\min(\omega, r^n) = \min(\chi, r^n)$ in all situations.

Problem 3.- We have obtained $\min(\omega, r^n) = \min(\chi, r^n)$ in all situations. It is interesting to find $\min(\omega, r^n) = \min(\chi, r^n)$

Problem 4.- By using the relation $\alpha(G) = \omega(\overline{G})$, can we obtain $\min(\alpha, r^n)$, $\max(\alpha, r^n)$, $\text{Min}(\alpha, r^n)$ and $\text{Max}(\alpha, r^n)$?

For the graph parameter f , we found in [16] a lower bound of $\min(f; d)$ by using the probabilistic method. In particular, we proved the following theorem.

Theorem 3.11- Let G be a graph having degree sequence $d = (d_1, d_2, \dots, d_n)$, $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$. Then

$$f(G) \geq 2 \sum_{i=1}^n \frac{1}{d_i + 1}.$$

The value of $\min(f, r^n)$ is not easy to obtain if we work on r -regular graphs. It is reasonable to extend the class of r -regular graphs of order n to a larger class $\mathcal{G}_{\Delta}(n)$. Let n and positive integers with $n > \Delta$. Let $\mathcal{G}_{\Delta}(n)$ be the class of all graphs G of order n and $\Delta(G) = \Delta$. Let $d = (d_1; d_2; \dots; d_n)$

be a sequence of non-negative integers. Define $\bar{d}$ a degree sequence $(\bar{d}_1, \bar{d}_2, \dots, \bar{d}_n)$, where $\bar{d}_i = n - d_i - 1$, for $i = 1, 2, \dots, n$. It is clear that d is graphic if and only if $\bar{d}$ is. We proved in [16] the following results.

Theorem 3.12-Let $d =$

$d = (d_1, d_2, \dots, d_n), d_1 \geq d_2 \geq \dots \geq d_n \geq 1$ be a graphic degree sequence and $d_1 + 1 \leq n \leq 2d_1 + 1$. Then

- i. $\min(f; d) = 2$ if and only if $d_1 = d_2 = d_3 = \dots = d_n$ and $n = d_1 + 1$ and

if d does not have a complete graph as its realization, then $\min(f; d) = 3$ if and only if $\bar{d}$ has a disjoint union of stars as its realization.

Theorem 3.13-Let $n = (\Delta + 1)q + t, 0 \leq t \leq \Delta$. then

1. $\min(f, \mathcal{G}_\Delta(n)) = 2q$, if $t = 0$,
2. $\min(f, \mathcal{G}_\Delta(n)) = 2q + 1$, if $t = 1$, and
3. $\min(f, \mathcal{G}_\Delta(n)) = 2q + 2$, if $2 \leq t \leq \Delta$.

With some modification of Theorem 3.13 in the class of r -regular graphs of order n and some properties of $F(j)$ -graph, we found $\min(f, r^n)$ in all situations as stated in the following theorems in [19].

Theorem 3.14- For $r \geq 3$, and $n = r + j, 1 \leq j \leq r + 1$

- i. $\min(f; r^n) = 2$, if and only if $n = r + 1$,
- ii. $\min(f; r^n) = 3$, if and only if $n = r + 2$,
- iii. $\min(f; r^n) = 4$, for all even integers $n, r + 3 \leq n$,
- iv. $\min(f; r^n) = 4$, for all odd integers $n, n, r + 3 \leq n$ and $n \geq f(j)$,
- v. $\min(f; r^n) = 5$, for all odd integers $n, r + 3 \leq n$ and $n < f(j)$, where $f(j) = \frac{5}{2}(j - 1)$ if $j \equiv 3 \pmod{4}$, and $f(j) = 1 + \frac{5}{2}(j - 1)$ if $j \equiv 1 \pmod{4}$.

Theorem 3.15-

For $n \geq 2r + 2$ and $r \geq 3$, write $n = (r + 1)q + t, q \geq 2$ and $0 \leq t \leq r$. Then

- i. $\min(f; r^n) = 2q$ if $t = 0$,
- ii. $\min(f; r^n) = 2q + 1$ if $t = 1$,
- iii. $\min(f; r^n) = 2q + 2$ if $2 \leq t \leq r - 1$,
- iv. $\min(f; r^n) = 2q + 3$ if $t = r$.

We obtained in [15] the values of $\max(f; r^n)$, for all r and n as stated in the following theorems.

Theorem 3.16- $\max(f, r^n) = \begin{cases} n - r + 1 & \text{if } r + 1 \leq n \leq 2r - 1, \\ \lfloor \frac{nr-2}{2(r-1)} \rfloor & \text{if } n \geq 2r. \end{cases}$

Note that if $r \geq 2$, then $\max(f; r^n) = \text{Max}(f; r^n)$: The investigation of $\text{Min}(f; r^n)$ was considered in [20] and we settled almost all cases as stated in the following results.

Theorem 3.17-Let n be an even integer $n \geq 12$. Then

$$\text{Min}(f, 3^n) = \begin{cases} \frac{5}{8}n - \frac{1}{4} & \text{if } n \equiv 2 \pmod{8}, \\ \lfloor \frac{5}{8}n \rfloor & \text{otherwise.} \end{cases}$$

Theorem 3.18-Let n and r be integers with $r \geq 4$. Then

$$\text{Min}(f, r^n) \geq \lceil \frac{2n}{r} \rceil.$$

Let $n = rq + t, 0 \leq t \leq r - 1, r \geq 4$. Then $\text{Min}(f, r^n) \geq 2q + \lceil \frac{2t}{r} \rceil$. By construction we have the following results.

$$\text{Min}(f, r^n) = \begin{cases} 2q & \text{if } t = 0, \\ 2q + 1 & \text{if } t = 1, 2, \\ 2q + 2 & \text{if } t > \frac{r}{2}. \end{cases}$$

Problem 5.- Find $\text{Min}(f; r^n)$ if $3 \leq t \leq \frac{r}{2}$.

Let $\mathcal{B}(r^{2n})$ be the class of r -regular bipartite graphs of order $2n$. It was shown in [27], page 53 that the subgraph of the $\Sigma(r^{2n})$ -graph induced by $\mathcal{B}(r^{2n})$ is connected. Therefore f is an interpolation graph parameter over $\mathcal{B}(r^{2n})$. We write $\min(f, \mathcal{B}(r^{2n}))$ for $\min\{f(G) : G \in \mathcal{B}(r^{2n})\}$ and $\max(f, \mathcal{B}(r^{2n}))$ for $\max\{f(G) : G \in \mathcal{B}(r^{2n})\}$. Thus $f(\mathcal{B}(r^{2n}))$ is uniquely determined by $\min(f, \mathcal{B}(r^{2n}))$ and $\max(f, \mathcal{B}(r^{2n}))$. Evidently, $\min(f, \mathcal{B}(r^{2n})) = \max(f, \mathcal{B}(r^{2n})) = 2n$ if $r \in \{0, 1\}$ $\max(f, \mathcal{B}(2^{2n})) = 2n - 1$ and $\min(f, \mathcal{B}(2^{2n})) = \lceil \frac{3n}{2} \rceil$. We proved in [5] the following theorems.

Theorem 3.19- If $r \geq 2$, then $\max(f, \mathcal{B}(r^{2n})) = \max(f, r^{2n}) = \lfloor \frac{nr-1}{r-1} \rfloor$.

Theorem 3.20- $\min(f, \mathcal{B}(3^{2n})) = n + \lceil \frac{n}{4} \rceil$.

Theorem 3.21- $\min(f, \mathcal{B}(4^{2n})) = n + \lceil \frac{n}{7} \rceil$.

The problem of determining $\min(f, \mathcal{B}(r^{2n}))$ is not easy if $r \geq 5$.

Problem 6.-Find $(f, \mathcal{B}(r^{2n}))$ if $r \geq 5$.

Problem 7-Let $\mathcal{CB}(r^{2n})$ be the class of connected r -regular bipartite graphs of order $2n$ and $r \geq 2$. It is clear that $\max(f, \mathcal{CB}(r^{2n})) = \max(f, \mathcal{B}(r^{2n}))$. Find $\min(f, \mathcal{CB}(r^{2n}))$.

Problem 8-The hypercube Q_n is a connected n -regular bipartite graph of order $2n$. The exact values of $f(Q_n)$ have been obtained when n is a power of 2. Details can be found in [6]. Find $f(Q_n)$ for other values of n .

In [18], we determined the values of $\min(\alpha', r^n)$ and $\max(\alpha', r^n)$ for all r and n . Since

$$\min(\alpha', 0^n) = \max(\alpha', 0^n) = 0 \text{ and } \min(\alpha', 1^{2n}) = \max(\alpha', 1^{2n}) = n,$$

we can assume that $r \geq 2$ and $n > r + 1$. An existence of an r -regular Hamiltonian graph of order n implies that $\max$

$(\alpha', r^n) = \lfloor \frac{n}{2} \rfloor$. A component of a graph is odd or even according as it has odd or even number of vertices. We denote by $o(G)$ the number of odd components of G . Tutte [25] proved the following theorem.

Theorem 3.22-The number of edges in a maximum matching of a graph G is $\frac{1}{2}(|V(G)| - d)$, where $d = \max_{S \subseteq V(G)} \{o(G - S) - |S|\}$.

Let $F(r; d)$ be the minimum order of an r -regular graph G with $\alpha'(G) = \frac{1}{2}(|V(G)| - d)$. It is clear that

$$|V(G)| \equiv d \pmod{2}. \text{ Wallis [26] found } F(r, 2) \text{ for all } r \geq 3.$$

More precisely, he proved the following theorem.

Theorem 3.23-Let G be an r -regular graph with no 1-factor and no odd component. Then

$$|V(G)| \geq \begin{cases} 3r + 7 & \text{if } r \text{ is odd, } r \geq 3, \\ 3r + 4 & \text{if } r \text{ is even, } r \geq 6, \\ 22 & \text{if } r = 4. \end{cases}$$

Furthermore, no such graphs exist for $r = 1$ or 2 .

If G is an r -regular graph with $\alpha'(G) = \frac{1}{2}(|V(G)| - d)$, there exists a k -subset K of $V(G)$ such that $o(G - K) = k + d$. If $k = 0$, then r is even, G contains d odd components, and each component of G has order at least $r +$

1. Suppose that $k \geq 1$ and $G - K$ has an odd component with p vertices where $p \leq r$. Thus the number of edges within the component is at most $\frac{1}{2}p(p-1)$. This means that the sum of degrees of these p vertices in $G - K$ is at most $p(p-1)$. But G is an r -regular graph, so the sum of degrees of these p vertices in G is pr . Hence the number of edges joining K to the component must be at least $pr - p(p-1)$. For a fixed integer r and an integer p satisfying $1 \leq p \leq r$, the function $f(p) = pr - p(p-1)$, $1 \leq p \leq r$ has minimum value $f(1) = f(r) = r$. So any odd component with r or less vertices is joined to K by r or more edges. Suppose that there are o_+ odd components of $G - K$ with more than r vertices and o_- odd components with less than or equal to r vertices. Thus

$$o_+ + o_- = k + d \quad (1)$$

$$o_+ + ro_- \leq kr. \quad (2)$$

From these 2 relations, we have

$$o_+ \geq \left\lceil \frac{rd}{r-1} \right\rceil = d + \left\lceil \frac{d}{r-1} \right\rceil \text{ and } k \geq \left\lceil \frac{d}{r-1} \right\rceil.$$

We obtained the following results in [18].

Theorem 3.24- Let r be an even integer, $r \geq 2$. Then $F(r, d) = d(r+1)$. then

$$\min(\alpha', r^n) = \frac{dr}{2} + \left\lfloor \frac{1+e}{2} \right\rfloor.$$

Suppose that r is odd and $r \geq 3$. Let G be an r -regular graph of order n such that $\alpha'(G) = \frac{1}{2}(n-d)$. Then d must be even. Put $d = 2q$. There exists a nonempty subset K of $V(G)$ of cardinality k such that $e(G-K) = k + 2q$. By (1) and (2), we have

$$n \geq k + (r+2)o_+ \geq \left\lceil \frac{2q}{r-1} \right\rceil + (r+2)(2q + \left\lceil \frac{2q}{r-1} \right\rceil) = \left\lceil \frac{2q}{r-1} \right\rceil(r+3) + 2q(r+2).$$

Wallis [26] defined $G(x; y)$ to be a graph with $x+y$ vertices, x and y being of degree $x+y-3$ and $x+y-2$, respectively. Thus $G(x; y)$ exists if and only if y is even and $y \geq 2$. It is noted that for any graph $G(x; y)$, it has y vertices of degree r and x vertices of degree $r-1$. Let $x_i, y_i, i = 1, 2, \dots, m$, be integers such that $G(x_i; y_i)$ exists for all $i = 1; 2; \dots; m$. We then construct a graph $G(x_1, y_1) * G(x_2, y_2) * \dots * G(x_m, y_m)$

from disjoint copies of the graphs by inserting a new vertex, say u , by joining u to all vertices of $G(x_i; y_i)$ which have the smallest degree, for $i = 1; 2; \dots; m$. With this notion we see that for an odd integer $r \geq 3$, $q = 1, 2, \dots, \frac{r-1}{2}$ and for any odd positive integers $a_i, i = 1; 2; \dots; 1 + 2q$ whose sum is r , it follows that

$$G_q = G(a_1, r+2-a_1) * G(a_2, r+2-a_2) * \dots * G(a_{1+2q}, r+2-a_{1+2q})$$

is an r -regular graph on $(r+2)(1+2q) + 1$ vertices with $\alpha'(G_q) = \frac{1}{2}(|V(G_q)| - 2q)$. We have the following results.

Theorem 3.26-For an odd integer $r \geq 3$, then

- $F(r, 2q) = (r+2)(1+2q) + 1$, for $q = 1, 2, \dots, \frac{r-1}{2}$,
- if $q = \frac{r-1}{2}s + t, 0 \leq t < \frac{r-1}{2}$, then $F(r, 2q) = sF(r, r-1) + F(r, 2t)$, where $F(r, 0) = 0$;

Corollary 3.27- Let r be an odd integer, $r \geq 3$. If $F(r, 2q) \leq n < F(r, 2(q+1))$, then $\min(\alpha', r^n) = \frac{1}{2}(n-2q)$.

Problem 9.- It is clear that α' is an interpolation graph parameter over $CR(rn)$ and it is easy to see that

$\text{Max}(\alpha', r^n) = \left\lfloor \frac{n}{2} \right\rfloor$. Find $\text{Min}(\alpha', r^n)$. Find $\text{Min}(\alpha', r^n)$.

B. $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$

We will discuss in this subsection the extremal problem for graph parameters over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$. Mantel's theorem [11] provides the maximum number of edges that a 2-chromatic graph of order n can have. On the other hand the minimum number of edges in a 2-chromatic graph of order $n \geq 2$ is 1 and the minimum number of edges in a 2-chromatic connected graph of order $n \geq 2$ is 1 and the minimum number of edges in a 2-chromatic connected graph of order $n \geq 2$ is $n-1$. Turán [24] extended the result of Mantel by introducing the Turán graph. This result of Turán is viewed as the origin of extremal graph theory. The Turán graph $T_{n,r}$ is the complete r -partite graph of order n whose partite sets differ in cardinality by at most 1.

Theorem 3.28-Among the graphs of order n containing no complete subgraph of order $r+1$, $T_{n,r}$ has the maximum number of edges.

In order to apply Turán theorem in our context, we would like to state the following facts.

- If $n = rq + t$, $0 \leq t < r$, then $T_{n,r}$ then $T_{n,r}$ consists of t partite sets of cardinality $\left\lceil \frac{n}{r} \right\rceil$ and $r-t$ partite sets of cardinality $\left\lfloor \frac{n}{r} \right\rfloor$.
Let $G \in \mathcal{G}(m, n)$. If $\omega(G) \leq r$, then $m \leq \varepsilon(T_{n,r})$.
 $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \left\lfloor \frac{n}{r} \right\rfloor$.
- $\varepsilon(T_{n,r}) = \varepsilon(T_{n,r})$. Then for a fixed n , we get $t(n, r-1) < t(n, r)$ for all $r, 2 \leq r \leq n$. In fact $t(n, r) - t(n, r-1) \geq \binom{a+1}{2}$, where $a = \left\lfloor \frac{n}{r} \right\rfloor$.

We obtained in [21] the following theorems.

Theorem 3.29-Let $m; n$ and k be positive integers with $n \geq k \geq 3$ and $\binom{k}{2} \leq m < \binom{k+1}{2}$. Then $\max(\chi; m, n) = k$.

Theorem 3.30-Let $m; n$ and $k \geq 2$ be positive integers satisfying $t(n, k-1) < m \leq t(n, k)$. Then $\max(\chi; m, n) = k$.

We now conclude the following corollary.

Corollary 3.31-Let $m; n$ and k be positive integers.

- If $n \geq k$ and $\binom{k}{2} \leq m < \binom{k+1}{2}$, then $\max(\omega; m, n) = k$
- If $t(n, k-1) < m \leq t(n, k)$, then $\min(\omega; m, n) = k$.
- If $t(n, k-1) < m \leq t(n, k)$, then $\text{Min}(\chi; m, n) = k$.
- If $k \geq 3$ and $t(n, k-1) < m \leq t(n, k)$, then $\text{Min}(\omega; m, n) = k$.

Results on $\text{Max}(\chi; m, n)$ and $\text{Max}(\omega; m, n)$ be obtained similarly as stated in the following theorems.

Theorem 3.32-Let $n; m$ and k be positive integers with $n \geq k \geq 3$ and $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1$. then $\text{Max}(\chi; m, n) = k$.

Theorem 3.33-Let $n; m$ and k be positive integers with $n \geq k \geq 3$ and $\binom{k}{2} + n - k \leq m < \binom{k+1}{2} + n - k - 1$. then

$\max(\omega; m, n)$

Thus all extreme values of χ and ω over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$ are obtained in all situations. The extremal values of the graph parameter f over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$ were obtained in [4].

Let G be a graph and X, Y be disjoint nonempty subsets of $V(G)$. Denote by $\varepsilon(X)$ the number of edges in $G[X]$ and $\varepsilon(X, Y)$ the number of edges in G connecting vertices in X to vertices in Y :

Let $G \in \mathcal{G}(m, n)$ and F be a maximum induced forest of G . Let $|F| = a$: Therefore $G - F$ has order $n - a$. An upper bound for m can be obtained by the following inequality.

$$m = \varepsilon(G - F) + \varepsilon(G - F, F) + \varepsilon(F) \leq \binom{n-a}{2} + a(n-a) + (a-1).$$

Let $a = n - i$ for any $i \in \{1, 2, \dots, n-2\}$. Then $m \leq (i+1)n - \frac{i^2+3i+2}{2}$.

For an integer $i = 1; 2; \dots; n-2$, let $M_n(n-i) := (i+1)n - \frac{i^2+3i+2}{2}$.

It is clear that $M_n(n-i)$ is an integer. We showed in [4] that $\max\{f; m, n\} = n-i$ if and only if $M_n(n-i+1) < m \leq M_n(n-i)$. And $\max\{f; m, n\} = n-i$ if and only if $m \geq n-1$ and $M_n(n-i+1) < m \leq M_n(n-i)$.

In order to obtain the values of $\min\{f; m; n\}$, we first find the minimum number of edges of a graph of order n having the forest number a . Let $\mathcal{G}(n; f=a)$ be the set of graphs of order n having the forest number a . It is clear that $\mathcal{G}(n; f=a) \neq \emptyset$ if and only if $2 \leq a \leq n$. For integers n and a , let

$$m_n(a) := \min\{\varepsilon(G) : G \in \mathcal{G}(n; f=a)\}.$$

Thus $m_n(n) = 0$, $m_n(n-1) = 3$ and $m_n(2) = \binom{n}{2}$. It is easy to see that for a graph G of order $n \geq 2$, $f(G) = 2$ if and only if $G \cong K_n$. We now find $m_n(a)$ for $2 < a < n$. Theorem 3.12 gives a characterization of graphs having forest number 3. Thus $m_n(3) = \binom{n}{2} - n + 1$, for all $n \geq 4$. We proved in [4] the following lemma.

Lemma 3.35-If G is a graph of order n with $\Delta(G) = \Delta$ and $f(G) = 2q+1$ for some integer q , then $n \leq (\Delta+1)q+1$.

By Lemma 3.35, we have a lower bound for the maximum degree of a given graph in terms of its order and its forest number. In other words, if G is a graph of order n , then

$$\Delta(G) \geq \left\lceil \frac{2n}{f(G)} \right\rceil - 1. \text{ In particular, if } f(G) = 2q \text{ for some integer } q, \text{ then } n \leq (\Delta+1)q+1.$$

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By Lemma 3.35 the lower bound for $\Delta(G)$ can be improved if $f(G)$ is odd. That is, if $f(G) = 2q+1$ for some integer q , then $n \leq (\Delta(G)+1)q+1$ which is equivalent to $\Delta(G) \geq \left\lceil \frac{n-1}{q} \right\rceil - 1$. We have the following corollary.

Corollary 3.36-Let G be a graph of order n and q be a positive integer. If $f(G) = 2q$, then

$$\Delta(G) \geq \left\lceil \frac{n}{q} \right\rceil - 1, \text{ and if } f(G) = 2q+1, \text{ then } \Delta(G) \geq \left\lceil \frac{n-1}{q} \right\rceil - 1.$$

Let $\mathcal{G}^*(n; f=a) = \{G \in \mathcal{G}(n; f=a) : \text{is a union of } \left\lceil \frac{n}{2} \right\rceil \text{ cliques}\}$. It is clear that $\mathcal{G}^*(n; f=a) \subset \mathcal{G}(n; f=a)$. We have the following theorem.

Theorem 3.37-Let G be a graph of order n with $f(G) = a$. Then there exists a graph $H \in \mathcal{G}^*(n; f=a)$ such that $\varepsilon(H) \leq \varepsilon(G)$.

By Theorem 3.37, we know the structure of graphs of order n with prescribed the forest number. In general, for a graph $G \in \mathcal{G}(n; f=a)$, there may be many such graphs $H \in \mathcal{G}^*(n; f=a)$. We now seek for such a graph H with minimum number of edges.

By using the results of Mantel [11] and Turán [24] as mentioned in the previous subsection, we have the following results.

- i. Let $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \dots \cup p_kK_k$. Then the order of G is $n = p_1 + 3p_3 + 4p_4 + \dots + kp_k$ and $f(G) = p_1 + 2(p_3 + p_4 + \dots + p_k)$. Suppose that $p_1 \geq 2$, $p_k \geq 1$ and $k \geq 4$. Then, by replacing $2K_1 \cup K_k$ by $K_3 \cup K_{k-1}$ we obtain a graph H with $\varepsilon(H) \leq \varepsilon(G)$. Further, $\varepsilon(H) = \varepsilon(G)$ if and only if $k = 4$.
- ii. $m_n(n-1) = 3$ if $n \geq 4$. Let $G \in \mathcal{G}(n; f=n-1)$. Then $f(G) = 3$ if and only if $n \geq 4$ and $G = (n-3)K_1 \cup K_3$.
- iii. Let a be an integer with $\frac{2n}{3} \leq a \leq n-1$. If $(p; q)$ is the solution of $p+3q = n$ and $p+2q = a$, then $G = pK_1 \cup qK_3$ satisfies $f(G) = a$.
- iv. Let a be an integer with $\frac{2n}{3} \leq a \leq n-1$ and $G \in \mathcal{G}(n; f=a)$ such that $\varepsilon(G) = m_n(a)$. Then by Theorem 3.37, we can choose $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \dots \cup p_kK_k \in \mathcal{G}^*(n; f=a)$ and $k \leq 4$. If $k = 4$, then $p_1 \geq 2$. Thus, there exists a graph $H = pK_1 \cup qK_3$ such that $p+3q = n$, $p+2q = a$ and $\varepsilon(H) = \varepsilon(G) = m_n(a)$.
- v. Let a be an integer with $a < \frac{2n}{3}$ and $G \in \mathcal{G}(n; f=a)$ and $\Delta(G) \geq 3$. Thus if $G = p_1K_1 \cup p_3K_3 \cup p_4K_4 \cup \dots \cup p_kK_k$, $f(G) = a < \frac{2n}{3}$ and $\varepsilon(G) = m_n(a)$, then $p_1 \leq 1$ and $k \geq 4$.
- vi. If $n = rq + t$, $0 \leq t < r$, then $T_{n,r}$ consists of t partite sets of cardinality $\left\lceil \frac{n}{r} \right\rceil$ and $r-t$ partite sets of cardinality $\left\lfloor \frac{n}{r} \right\rfloor$.
- vii. $\varepsilon(T_{n,r}) = \binom{n-a}{2} + (r-1)\binom{a+1}{2}$, where $a = \left\lfloor \frac{n}{r} \right\rfloor$.
- viii. Let $t(n, r) = \varepsilon(T_{n,r})$. Then for a fixed n , by using elementary arithmetic, we get $t(n; r-1) < t(n, r)$ for all r , $2 \leq r \leq n$. In fact $t(n, r) - t(n, r-1) \geq \binom{a+1}{2}$, where $a = \left\lfloor \frac{n}{r} \right\rfloor$.
- ix. Let $\tilde{t}(n, r) = \binom{n}{2} - \varepsilon(T_{n,r})$. Summarizing the results, we have the following theorems.

Theorem 3.38-Let n and a be integers with $2 \leq a \leq n-1$. then

- i. $m_n(n) = 0$,
- ii. $m_n(n-1) = 3$ if $n \geq 3$ and $G = (n-3)K_1 \cup K_3$ is the only graph of order n satisfying $f(G) = n-1$ and $\varepsilon(G) = 3$,
- iii. $m_n(n-i) = 3i$ if $1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil$,

- iv. Suppose $4 \leq a < \frac{2n}{3}$. Then $m_n(a) = \bar{t}(n, q)$ if $a = 2q$, and $m_n(a) = \bar{t}(n-1, q)$ if $a = 2q+1$, for some integer q , and
- v. $m_n(3) = \binom{n-1}{2}$ if $n \geq 3$, and $m_n(2) = \binom{n}{2}$ if $n \geq 2$.

Theorem 3.39-Let n and m be integers with $2 \leq a \leq n-1$. then

- i. $\min(f; m; n) = \max(f; m; n) = n$ if and only if $\in \{0, 1, 2\}$,
- ii. $\min(f; m; n) = \max(f; m; n) = 2$ if and only if $m = \binom{n}{2}$,
- iii. for $3 \leq a \leq n-1$, $\min(f; m; n) = a$ if and only if $m_n(a) \leq m < m_n(a-1)$.

We now find the minimum number of edges of a connected graph order n having the forest number a . Let $\mathcal{CG}(n; f=a)$ be the set of all connected graphs of order n having the forest number a . For integers n and a , let

$$cm_n(a) = \min\{e(G) : G \in \mathcal{CG}(n; f=a)\}.$$

Further, $cm_n(n) = n-1$, $cm_n(2) = \binom{n}{2}$. We now find $cm_n(a)$ for $2 < a < n$.

Let $\mathcal{CG}^*(n; f=a) = \{G \in \mathcal{CG}(n; f=a) : G \text{ is obtained from } \lceil \frac{a}{2} \rceil \text{ disjoint cliques and } \lceil \frac{a}{2} \rceil - 1 \text{ edges}\}$. We have the following theorem.

Theorem 3.40- Let G be a connected graph of order n with $f(G) = a$. Then there exists A graph $H \in \mathcal{CG}^*(n; f=a)$

By Theorem 3.40 we know that for a graph $G \in \mathcal{CG}(n; f=a)$, there may be many such graphs $H \in \mathcal{CG}^*(n; f=a)$ We now seek for such a graph H with minimum number of edges. By applying Turán Theorem once again, we have the following theorems.

Theorem 3.41- Let n and a be integers with $2 \leq a \leq n-1$. then

- i. $cm_n(n) = n-1$,
Suppose that $4 \leq a \leq n-1$. Then
- ii. $cm_n(a) = \bar{t}(n, q) + q - 1$ if $a = 2q$, and $cm_n(a) = \bar{t}(n-1, q) + q$ if $a = 2q+1$, for some integer q , and
- iii. $cm_n(3) = \binom{n-1}{2} + 1$ if $n \geq 3$, $cm_n(2) = \binom{n}{2}$ if $n \geq 2$.

Theorem 3.42-Let n and m be integers with $n-1 \leq m \leq \binom{n}{2}$. Then

- i. $\min(f; m; n) = \max(f; m; n) = n$ if and only if $m = n-1$,
- ii. $\min(f; m; n) = \max(f; m; n) = 2$ if and only if $m = \binom{n}{2}$
- iii. for $3 \leq a \leq n-1$, $\min(f; m; n) = a$ if and only if $cm_n(a) \leq m < cm_n(a-1)$

Problem 10.- Several graph parameters have been proved to interpolate over $\mathcal{G}(m, n)$ and $\mathcal{CG}(m, n)$ as stated in Theorem 2.15. Find $\min(\pi; m, n)$, $\max(\pi; m, n)$,

$\min(\pi; m, n)$ and $\max(\pi; m, n)$ where $\pi \in \{\alpha, \alpha', \gamma\}$.

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