A Solution of Fractional Volterra Type Integro-Differential Equations Associated With a Generalized Lorenzo and Hartley Function

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Abstract: **In this paper, we applied Sumudu transform to solve a fractional integro-differential equation involving a Lorenzo-Hartley function. A Cauchy-type problem involving the Caputo fractional derivatives and a generalized Volterra integral equation are also considered. Several special cases are also mentioned.**

Keywords: Sumudu transform, fractional differential operator, fractional integral operator, Mittag–Leffler function.

I. Introduction and Preliminaries

The free-electron laser (FEL) high-gain equation (HGE) describes the evolution of the optical field when it is far from the saturation. Denoting with $a(\tau)$ the dimensionless field amplitude, the HGE is written in the form of a Volterra integro-differential equation, namely ([9], [10]):

$$\frac{d}{d\tau}a(\tau) = -i\pi g_0 \int_0^\tau \xi a(\tau - \varsigma) d\xi \qquad (1.1)$$

The Riemann-Liouville operator of fractional integration of order v is defined by

$$D_x^{-v}[h(x)] = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} h(t) dt \qquad (1.2)$$

provided that the integral (1.2) exists. The Riemann-Liouville fractional derivative of order v is defined in the form ([17], [18], [19], [15])

$$D_x^{\alpha}[h(x)] = \frac{1}{\Gamma(v)} \frac{d^n}{dx^n} \int_0^x \frac{h(t)}{(x-t)^{\alpha+n-1}} dt \qquad (n-1 < \alpha < n) \qquad (1.3)$$

Boyadjiev et al. [6] studied the following nonhomogeneous form of fractional integro-differential equation of Volterra type:

$$D_\tau^\alpha a(\tau) = \lambda \int_0^\tau \xi a(\tau - \xi) \exp(iv\xi) d\xi + \beta \exp(iv\tau), \;\; 0 \le \tau \le 1 \qquad (1.4)$$

where $\beta, \lambda \in C$ and $v \in R$.

Al-Shammery et al. [1] considered a generalization of (1.4) in the form

$$D_\tau^\alpha a(\tau) = \lambda \int_0^\tau \xi^\delta a(\tau - \xi) \exp(iv\xi) d\xi + \beta \exp(iv\tau), \quad 0 \le \tau \le 1 \qquad (1.5)$$

where $\beta,\lambda,\delta \in C$, $v \in R$. $\Re(\alpha)>0$, and $\Re(\delta)>-1$. Al-Shammery et al. [2] further studied another generalization of (1.5)
in the form

$$D_\tau^\alpha a(\tau)=\lambda\int_0^\tau \xi^\delta a(\tau-\xi)\Phi(b,\delta+1;iv\xi)d\xi+\beta\Phi(b',1;iv\tau), \quad 0\le\tau\le1$$

……...(1.6)

where $\alpha,\beta,\lambda,\delta \in C$, $v \in R$. $\Re(\alpha)>0$, and $\Re(\delta)>-1$.

Saxena and Kalla [21] derived the solution of a further generalizaion of (1.6) in the form

$$D_\tau^\alpha a(\tau)=\lambda\int_0^\tau \xi^\delta a(\tau-\xi)\Phi(b,\delta+1;iv\xi)d\xi+\mu t^\gamma\Phi(\beta,\gamma+1;iv\tau), \quad 0\le\tau\le1$$

……….... (1.7)

where $\alpha,\beta,\lambda,\delta,\mu \in C$, $v \in R$. $\Re(\alpha)>0, \Re(\gamma)>-1$ and $\Re(\delta)>-1$.

Recently Kilbas et al. [13] systematically studied a generalization of (1.7) in the

followingform

$$D_\tau^\alpha a(\tau)=\lambda\int_a^x (x-t)^{\mu-1}E_{\rho,\mu}^\gamma(w(x-t)^\rho)h(t)d\ \ t+f(x), \quad a\le x\le b \quad (1.8)$$

where $\lambda,\mu,\rho,\gamma \in C$, $w \in R$. $\Re(\alpha)>0, \Re(\mu)>0$
and f is assumed to be Lebesgue integrable over the interval (a,b) and the function

$$E_{\rho,\mu}^\gamma(z)=\sum_{r=0}^\infty \frac{(\gamma)_{tr}}{\Gamma(r\rho+\mu)}.\frac{z^r}{r!}, \qquad \Re(\rho)>0,$$

…………………………………………………...(1.9)

A detailed account of various operators of fractional integration and their applications can be found in a recent survey paper of Srivastava and Saxena [20], Oldham and Spenier [18], and Miller ad Ross [17].
In 90's, Watugala [22] introduced the following integral transform

Sumudu Transform: The Sumudu transform is defined over the set of the functions

$$A=\{f(t)\,|\,\exists M,\tau_1,\tau_2>0,\left|f(t)\right|<Me^{|t|/\tau_j}, \text{if } t\in(-1)^j\times[0,\infty)\},$$

…………………………………(1.10)

by

$$G(s)=S\{f(t)\}:=\int_0^\infty f(st)e^{-t}dt, \qquad s\in(-\tau_1,\tau_2).$$

……………………………………………………….. (1.11)

More detail and properties about this transform, please see ([5],[6]) and others.

The Lorenzo-Hartley function and its relationship with some other functions:-

The Lorenzo-Hartley function $G_{\upsilon,\mu,\delta}(a,c,t)$ is introduced by Lorenzo & Hartley [16] defined as

$$G_{\upsilon,\mu,\delta}(a,c,t)=\sum_{k=0}^\infty \frac{(\delta)_k a^k (t-c)^{(k+\delta)\upsilon-\mu-1}}{k!\Gamma((k+\delta)\upsilon-\mu)}, \qquad …….(1.12)$$

$(\Re \upsilon > 0, \Re \mu > 0, \Re(\upsilon\delta - \mu) > 0)$

where $(\delta)_k$ is Pochhammer's symbol defined by

$$(\delta)_k = \begin{cases} 1, & k=0 \\ \delta(\delta+1)\ldots(\delta+k-1), & \delta \neq 0, k \in \square \end{cases},$$

be the set of natural numbers. Particularly at c=0, the above Lorenzo-Hartley function reduces in to the following form

$$G_{\upsilon,\mu,\delta}(a,t) = \sum_{k=0}^{\infty} \frac{(\delta)_k a^k (t)^{(k+\delta)\upsilon-\mu-1}}{k!\Gamma((k+\delta)\upsilon-\mu)}.$$
$$(\Re \upsilon > 0, \Re \mu > 0, \Re(\upsilon\delta-\mu) > 0)$$

……(1.13)

Lorenzo-Hartley function yields the following relationships with various classical special functions:

Mittag-Leffler function (see [13])

$$G_{\upsilon,\upsilon-1,1}(-a,t) = E_{\upsilon}[-at^{\upsilon}] = \sum_{k=0}^{\infty} \frac{(-a)^k t^{k\upsilon}}{\Gamma(k\upsilon+1)}, \quad (\Re \upsilon > 0). \quad \ldots (1.14)$$

Robtonov&Hartleyfunction(LorenzoandHertley [16])

$$G_{\upsilon,0,1}(-a,t) = F_{\upsilon}[-a,t] = \sum_{k=0}^{\infty} \frac{(-a)^k t^{(k+1)\upsilon-1}}{\Gamma((k+1)\upsilon)}.$$

where $E_{\upsilon,\upsilon-\mu}[at^{\upsilon}]$ is the well known generalized Mittag-Leffler function (see [13]) defined as

$$E_{\upsilon,\upsilon-\mu}[t] = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\upsilon+\mu)}, \quad \Re \upsilon > 0, \quad \Re \mu > 0 .$$

R function (Lorenzo & Hartley [16])

The method followed here in finding the solution of Cauchy-type problems (2.1) and (2.2) and other Cauchy problems is based upon certain properties of fractional calculus and the Sumudu transform. The solutions derived are in closed forms and are suitable for numerical computation.

In order to prove our main results, we shall required the following results

Lemma 1.1. Lorenzo-Hartley function $G_{\upsilon,\mu,\delta}(a,c,t)$ is given by (1.12), the Sumudu transform of $G_{\upsilon,\mu,\delta}(a,c,t)$ is given by .

$$S[G_{\upsilon,\mu,\delta}(a,c,t)] = \frac{s^{\delta\upsilon-\mu-1}}{(1-as^{\upsilon})^{\delta}}$$

,

provided that $\Re \upsilon > 0,\ \Re\mu>0,\ \Re(\upsilon\text{-}\mu)>0.$ **Proof:** By definition of Lorenzo-Hartley function, we may see that

$$G_{\upsilon,\mu,\delta}(a,t) = \sum_{k=0}^{\infty} \frac{(\delta)_k a^k (t)^{(k+\delta)\upsilon-\mu-1}}{k!\Gamma((k+\delta)\upsilon-\mu)},$$

$$\Re \upsilon > 0,\ \Re\mu>0,\ \Re(\upsilon\text{-}\mu)>0.$$

$$S[G_{\upsilon,\mu,\delta}(a,t)] = S\left[\sum_{k=0}^{\infty} \frac{(\delta)_k a^k (t)^{(k+\delta)\upsilon-\mu-1}}{k!\Gamma((k+\delta)\upsilon-\mu)}\right]$$

Taking the Sumudu transform both side

$$= \sum_{k=0}^{\infty} \frac{(\delta)_k a^k}{k!\Gamma((k+\delta)\upsilon-\mu)} \int_0^{\infty} e^{-t}(st)^{(k+\delta)\upsilon-\mu-1} dt = \sum_{k=0}^{\infty} \frac{(\delta)_k a^k s^{k\upsilon}}{k!} s^{\delta\upsilon-\mu-1}$$

which gives

$$S[G_{\upsilon,\mu,\delta}(a,c,t) = \frac{s^{\delta\upsilon-\mu-1}}{(1-as^{\upsilon})^{\delta}},\ \Re \upsilon > 0,\ \Re\mu>0,\ \Re(\upsilon\text{-}\mu)>0.$$

2. Solution of generalized fractional integro-differential equation of Volterra type.

We begin by proving

Theorem 2.1. Consider the following generalized integro-differential equation of Volterra type

$$D_{\tau}^{\alpha} h(\tau) = \kappa \int_0^{\tau} h(\tau-\xi) G_{\upsilon,\mu,\delta}(a,\xi) d\xi + \eta f(\tau), \qquad \text{.....(2.1)}$$

where $0 \le \tau \le 1; \kappa, \mu, \upsilon, \delta, \eta \in C$ and $\Re \upsilon > 0,\ \Re\mu>0,\ \Re(\upsilon\text{-}\mu)>0.$

together with the initial conditions

$$D_{\tau}^{\alpha-k} h(\tau)|_{\tau=0} = a_k \ \ (k=1,2,\ldots N)$$

$$(N := -[-\operatorname{Re}(\alpha)]) : N-1 < \operatorname{Re}(\alpha) \le N : N \in \mathbb{N}), \qquad \text{.................................... (2.2)}$$

where $a_1, \ldots a_N$ are prescribed constant and $f(\tau)$ is assumed to be continuous solution of the Cauchy-type problem (2.1) and (2.2) given by

$$h(\tau)=\sum_{k=1}^{N}a_k\Lambda_k(\tau)+\eta\int_0^{\tau}\Theta(\tau-\xi)f(\xi)d\xi,$$

……………………………………………………(2.3)

where

$$\Lambda_k(\tau)=\sum_{r=0}^{\infty}\kappa^r G_{\upsilon,\mu_{r'},\delta_r}(a,\tau)\ ,$$

where $\mu_{r'}=-\alpha(1+r)+\mu_r+k-1$

and

$$\Theta(\tau)=\sum_{r=0}^{\infty}\kappa^r G_{\upsilon,\mu_{r''},\delta_r}(a,\tau),$$

where $\mu_{r''}=-\alpha(1+r)+\mu_r-1$. (2.4)

Proof: Applying the Sumudu transform to (2.1) and using the Lemma 1 , we find that

$$s^{-\alpha}H(s)-\sum_{k=1}^{N}s^{-k}D_{\tau}^{\alpha-k}h(\tau)|_{\tau=0}=\kappa sH(s)\frac{s^{\delta\upsilon-\mu-1}}{(1-as^{\upsilon})^{\delta}}+\eta F(s)$$

…………………………………………………... (2.5)

where H(s) and F(s) represent, respectively, the Sumudu transform of the function $h(\tau)$ and $f(\tau)$. Solving (3.6) under the initial conditions (2.2) we find that,

$$H(s)=\sum_{k=1}^{N}s^{\alpha-k}a_k\left[1-\kappa\frac{s^{\alpha+\delta\upsilon-\mu}}{(1-as^{\upsilon})^{\delta}}\right]^{-1}+\eta F(s)s^{\alpha}\left[1-\kappa\frac{s^{\alpha+\delta\upsilon-\mu}}{(1-as^{\upsilon})^{\delta}}\right]^{-1}$$

,…………………………………………………. (2.6)

where it is tacitly assumed that

$$\left|\left\{1-\kappa\frac{s^{\alpha+\delta\upsilon-\mu}}{(1-as^{\upsilon})^{\delta}}\right\}\right|<1\ .$$

By taking the inverse Sumudu transform, we get the required result

Setting $\mu=\upsilon-1$, $\delta=1$ and replace 'a' by (-a)

Corollary 2.1. Under the various relevant hypotheses of *Theorem 2.1* , a unique continuous solution of the Cauchy-type problem involving the Volterra-type integro-differential equation

$$D_{\tau}^{\alpha}h(\tau)=\kappa\int_0^{\tau}h(\tau-\xi)R_{\upsilon,\mu}(a,\xi)d\xi+\eta f(\tau),$$

……………………………………………………(2.7)

Where $0\le\tau\le1;\kappa,\upsilon,\eta,\mu\in C$ and $\Re\upsilon>0$, $\Re\mu>0$, $\Re(\upsilon-\mu)>0.$, together with the initial condition (2.2), is given by

$$h(\tau)=\sum_{k=1}^{N}a_k\zeta_k(\tau)+\eta\int_0^{\tau}\Psi(\tau-\xi)f(\xi)d\xi,$$

……………………………………………………(2.8)

Where

$$\zeta_k(\tau)=\sum_{r=0}^{\infty}R_{\upsilon,\rho_{r'}}(a,\tau),$$

and $\rho_{r'}=\mu_r+k-\alpha(1+r)-1$,

$$\Psi(\tau)=t^{\alpha-1}\sum_{r=0}^{\infty}R_{\upsilon,\rho_{r''}}(a,\tau)\text{ ,and }\rho_{r''}=\mu_r-\alpha(1+r)-1. \qquad (2.9)$$

Corollary 2.2. Under the various relevant hypotheses of *Theorem 2.1*, a unique continuous solution of the Cauchy-type problem involving the Volterra-type integro-differential equation

$$D_\tau^\alpha h(\tau)=\kappa\int_0^{\tau}h(\tau-\xi)E_\upsilon(-a,\xi^\upsilon)d\xi+\eta f(\tau),$$

……………………………………………………(2.10)

where $0\le\tau\le1;\kappa,\upsilon,\eta\in C$ $\Re\upsilon>0.$,together with the initial condition (2.2), is given by

$$h(\tau)=\sum_{k=1}^{N}a_k\Omega_k(\tau)+\eta\int_0^{\tau}\Phi(\tau-\xi)f(\xi)d\xi,$$

where

$$\Omega_k(\tau)=t^{\alpha+k-1}\sum_{r=0}^{\infty}t^{(\alpha+1)r}E_{\upsilon,\alpha(1+r)+r+k}(-at^\upsilon),$$

and

$$\Phi(\tau)=t^{\alpha-1}\sum_{r=0}^{\infty}t^{(\alpha+1)r}E_{\upsilon,\alpha(1+r)+r}(-at^\upsilon).$$

3. Solution of the generalized Volterra integral equation.

Theorem 3.1. The *Volterra type integral equation*

$$D_\tau^{-\lambda}h(\tau)=\kappa\int_0^{\tau}h(\xi)G_{\upsilon,\mu,\delta}(a,\xi)d\xi+\eta f(\tau),$$

……………………………………………………(3.1)

has its solution given explicitly by

$$h(\tau)=\eta \sum_{r=0}^{\infty} \kappa^{r} \int_{0}^{\tau} F(\tau-\xi) G_{\upsilon, \mu r, \delta r}(a, \xi) d \xi$$

………………………………………………….(3.2)

where $0 \le \tau \le 1; \kappa, \mu, \upsilon, \delta, \eta \in C$ and $\Re\upsilon > 0,\ \Re\mu>0,\ \Re(\upsilon-\mu)>0.$

Proof: Now taking the Sumudu transform on both the sides of Volterra integral equation (3.1), we get

$$H(s)=\eta F(s) s^{-\lambda}\left[1-\kappa \frac{s^{\delta \upsilon-\mu-\lambda}}{\left(1-a s^{\upsilon}\right)^{\delta}}\right]^{-1}$$

……………………………………………….(3.3)where it is tacitly assumed that

$$\left|\left\{1-\kappa \frac{s^{\delta \upsilon-\mu-\lambda}}{\left(1-a s^{\upsilon}\right)^{\delta}}\right\}\right|<1,$$

and F(s) and H(s) denote the Sumudu transform of $f(\tau)$ and $h(\tau)$ respectively.

$$H(s)=\eta s F(s) \sum_{r=0}^{\infty} \kappa^{r} \frac{s^{(\delta \upsilon-\mu-\lambda) r-\lambda-1}}{\left(1-a s^{\upsilon}\right)^{r \delta}}$$

…………………………………………..(3.4)

Now taking the inverse Sumudu transform

$$h(\tau)=\eta \sum_{r=0}^{\infty} \kappa^{r} \int_{0}^{\tau} F(\tau-\xi) G_{\upsilon, r \mu, r \delta}(a, \xi) d \xi,$$

………………………………………………….(3.5)

where $\mu(r)=(\mu+\lambda) r+\lambda$ and $\Re\upsilon > 0,\ \Re\mu>0,\ \Re(\upsilon-\mu)>0.$ by using Lemma 1.1.

If we set $\delta=1$ in Theorem 3.1 we get,

Corollary 5.1. The Volterra type integral equation

$$D_{\tau}^{-\lambda} h(\tau)=\kappa \int_{0}^{\tau} h(\xi) R_{\upsilon, \mu}(a, \xi) d \xi+\eta f(\tau),$$

has its solution given explicitly by

$$h(\tau)=\eta \sum_{r=0}^{\infty} \kappa^{r} \int_{0}^{\tau} F(\tau-\xi) G_{\upsilon, \mu(r), r}(a, \xi) d \xi,$$

provided $\Re\upsilon > 0,\ \Re\mu>0,\ \Re(\upsilon-\mu)>0.$

If we set $\mu=0$, $\delta=1$ and replace a by $-a$ in Theorem 3.1

Corollary 5.2. The Volterra type integral equation which is given by,

$$D_{\tau}^{-\lambda} h(\tau)=\kappa \int_{0}^{\tau} h(\xi) F_{\upsilon}(-a, \xi) d \xi+\eta f(\tau),$$

has its solution is given explicitly by

$$h(\tau)=\eta \sum_{r=0}^{\infty} \kappa^{r} \int_{0}^{\tau} F(\tau-\xi) G_{\upsilon, \lambda(r+1), r}(-a, \xi) d \xi.$$

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