On New Solutions To Lame's Differential Equation

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Abstract
We extends the works of [1] and [2] in obtaining new polynomial solutions Lame di
erential equation. We show that the derivative of solution of Lame's equation can be expressed in terms of solution di
erent from the former. However the new solutions are expressed in terms of polynomials as suggested in [1].
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1 Introduction

The Heun's differential equation is a natural extension of the Riemann hypergeometric differential equation which can be written as [4]

$$P_3(z)\,y''(z) + P_2(z)\,y'(z) + P_1(z)\,y(z) = 0, \tag{1.1}$$

where $P_i(z)$ are arbitrary polynomials of degree $i(i = 3, 2, 1)$ in the complex variable z. Replacing the variable z by the variable x, the above equation reads as

$$\mathcal{D}^2\, y \,+\, (\frac{\gamma}{x} + \frac{\delta}{x-1} \,+\, \frac{\epsilon}{x-a})\mathcal{D}\, y \,+\, \frac{(\alpha\beta x - q)}{x(x-1)(x-a)}\, y \,=\, 0, \tag{1.2}$$

where $D = \frac{d}{dx}$, $\{\alpha, \beta, \gamma, \delta, \epsilon, a, q\}$ $(a \neq 0, 1)$ are parameters, generally complex and arbitrary, linked by the Fuschian constraint $\alpha + \beta + 1 = \gamma + \delta + \epsilon$. This equation has four regular singular points at $\{0, 1, a, \infty\}$, with the exponents of these singularities being respectively, $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\epsilon\}$, and $\{\alpha, \beta\}$.

With the following assumptions

$$\gamma = \delta = \epsilon = \frac{1}{2},\ \ \alpha = -\frac{1}{2}v,\ \ \ \beta = \frac{1}{2}(v+1),\ \ q = -\frac{1}{4}dh, \tag{1.3}$$

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equation (1.2) above reduces to

$$\frac{d^2u}{dt^2} + \frac{1}{2}\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-d}\right)\frac{du}{dt} + \frac{dh - v(v+1)z}{4t(t-1)(t-d)}u = 0. \tag{1.4}$$

This equation is called the **Lame's** equation in one of its algebraic form. The parameter v is called the order of the equation and many special features arises when v is an integer. The Heun's equation arises from the separation of variables for Schrodinger equation in certain system. This equation arises from the separation of variables in Laplace's equation, ellipsoidal or elliptic-conal coordinates. Indeed, it is the only equation of non-confluent Heun's type.

In this paper, the works in [1] and [2] are combined together to obtain polynomial solutions to the **Lame's** equation.

In [1], a newly method to solve linear differential equations was developed. Their approach to Heun equation is very transparent in obtaining solution; in particular the polynomial solutions come out very naturally as it was shown below. A single variable differential equation, after appropriate manipulation, can be cast in the form

$$\left(F(D) + P(x, \frac{d}{dx})\right) y(x) = 0, \tag{1.5}$$

where $D \equiv x\frac{d}{dx}$, and $F(D) = \sum_n a_n D^n$ is a diagonal operator in the space of monomials with a_n's being some parameters. $P(x, \frac{d}{dx})$ is an arbitrary polynomial function of x and d/dx, excluding the diagonal operator. It was shown that the following ansatz,

$$y_\lambda(x) = \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)} P(x, \frac{d}{dx})\right]^m x^\lambda \tag{1.6}$$

is a solution of the above equation provided and the coefficient of x^λ in $y(x) - x^\lambda$ is zero, in order to ensure that the solution,$y(x)$ is non - singular. The fact that D is diagonal in the space of monomials x^n, makes $1/F(D)$ well defined in the above expression.

In [2], it was shown that any differential equation (1.4), say, can be rewritten in the general form

$$P\mathcal{D}^2 y + Q\mathcal{D} y + R y = 0, \tag{1.7}$$

where P, Q,R, are polynomial functions. Setting $\mathcal{D}y = z$, taking the derivative of
(1.7), subtracting the equation (1.7) multiplied by some function $\psi(x)$, we obtain the differential equation of the form

$$P\mathcal{D}^2 z + (P' + Q - \psi(x)P)\mathcal{D}z + (Q' + R - \psi(x)Q)z + (R' - \psi(x)R)y = 0, \quad (1.8)$$

where $V' := \mathcal{D}V$. The condition which render (1.7) invariant under derivative operation, i.e, allowing to write (1.8) in the form

$$\bar{P}\mathcal{D}^2 z + \bar{Q}\mathcal{D}z + \bar{R}z = 0, \quad (1.9)$$

where $\bar{P}, \bar{Q}, \bar{R}$ being appropriate polynomials, are of great interest in the investigation of solutions of the **Lame's** equation [4]. The suitable amsatz reads $\psi = \frac{R'}{R}$. In this case, (1.7) reads

$$P\mathcal{D}^2 z + (P' + Q - \frac{R'}{R}P)\mathcal{D}z + (Q' + R - \frac{R'}{R}Q)z = 0, \quad (1.10)$$

where $R(x) = xR' + R(0)$ is a polynomial of degree 1 which can be written as

$$R(x) = R'(x - c), \qquad \text{with} \qquad c = -\frac{R(0)}{R'}. \quad (1.11)$$

Therefore, P/R and Q/R have to be polynomials defining the singularities of the equation (1.10). In general, such a transformation leads to more singular points than in the initial equations. When the singular points coincide with already existing ones, the number of singularities to that of the initial **Lame's** equation increases by one . The derived equation can be transformed to another **Lame's** equation (1.4) and the derivative of the solution to the initial **Lame's** equation can be expressed in terms of a solution to another **Lame's** equation . This property may lead to interesting series solutions to **Lame's** equation.

After appopriate manipulation, the **Lame's** equation (1.4) can be cast in form (1.10) and the polynomial $\bar{P}, \bar{Q}, \bar{R}$ identified. For any such transformed equation, the pole can be eliminated setting either $R' = 0$ or $R(0) = 0$.

Let us denote $\mathcal{L}_n(\alpha, \beta, \gamma, \delta; x)$, the corresponding solutions to **Lame's**. In the next section we proceeds to obtaining the derived polynomial solutions of the **Lame's** at the various singularities.

2 Derived solutions to Lame's Equation

Considering the Lame's equation (1.4) and (1.10), we have

$$(x) = -\frac{v(v+1)}{ah - v(v+1)},$$

leading to an extra singularity

$$z = \frac{ah}{v(v+1)}.$$

This extra singularity coincides with the singularity $z = 0$ when $ah = 0$ and with $z = 1$ when $ah = v(v+1)$, and $z = \infty$ when $v(v+1) = 0$. These conditions leads to three cases to investigate.

2.1 Case I: Singularity at $t = 0$

In this case we have the following characteristics ;

$$ah = 0, \;\; \psi(x) = 1/z$$

leading to the equation

$$A(z)D^2 z + B(z)Dz + C(z)z = 0, \tag{2.1}$$

where

$$A(z) = z^3 - z^2(a+1) + az, \;\; B(z) = \frac{7}{2}z^3 - 2z^2(a+1) + \frac{1}{2}az$$

and

$$C(z) = z^2(\frac{3}{2} - \frac{v(v+1)}{4}) - z\frac{ah}{4} - \frac{a}{2}$$

Applying (1.6) to (2.1), we obtain

$$a_0 = -\frac{a}{2}, \;\; a_1 = 1 + \frac{3}{2}a, \;\; a_2 = -(1+a), \;\; \lambda = 1 \text{ or } \frac{a}{2(a+1)},$$

where the λ's are the exponents of the singularity at $z = 0$ of equation (2.1).

Since $z := \mathcal{D}y$, the polynomial solution is

$$\begin{aligned} \mathcal{D}y_\lambda &= \mathcal{D}\mathcal{L}_n(\alpha,\beta,\gamma,\delta;z) = \sum_{m=0}^{\infty}(-1)^m \left[\frac{1}{(D-1)(D+\frac{a}{2(a+1)})}\right]^m \times \\ &\left[(z^3+az)\frac{d^2}{dz^2} + (\frac{7}{2}z^3 - 2z^2(a+1))\frac{d}{dz} + z^2(\frac{3}{2} - \frac{v(v+1)}{4}) - z\frac{ah}{4})\right]^m z^\lambda dz : \\ &\lambda = 1 \text{ or } \frac{a}{2(a+1)} \end{aligned} \tag{2.2}$$

2.2 Case II: Singularity at $t = 1$

When $x = 1$, the corresponding characteristics are

$$\sigma = 4p\alpha, \quad \psi(x) = \frac{1}{x-1}.$$

The corresponding differential equation is

$$\begin{aligned} x(x-1)^2 D^2 z &+ \quad (1+\gamma+4px)(x-1)^2 + \delta x(x-1)\big) \, Dz \\ &+(1+\alpha)4p(x-1)^2 - \delta)z = 0. \end{aligned} \tag{2.3}$$

With the application of $t = x-1$, we obtain the equation

$$(t^2+t^3)D^2 z + \ (1+4p+\gamma+\delta)t^2 + 4pt^3 + \delta t\,\big)\, Dz + \ 4p(1+\alpha)t^2 - \delta\big)\, z = 0, \tag{2.4}$$

leading to $\lambda = -1$ or $\lambda = -\delta$. The corresponding polynomial solution is

$$\begin{aligned} \mathcal{D}y_\lambda &= \mathcal{D}\mathcal{C}_n(\alpha,\beta,\gamma,\delta;t) = \sum_{m=0}^{\infty}(-1)^m \left[\frac{1}{(D-1)(D+\delta)}\right]^m \times \\ &\left[t^3\frac{d^2}{dt^2} + \ (1+4p+\gamma+\delta)t^2 + 4pt^3)\,\frac{d}{dt} + 4p(1+\alpha)t^2\right]^m t^\lambda dt \ \lambda = 1 \ \text{ or } -\delta \end{aligned} \tag{2.5}$$

2.3 Case III: Singularity at $t = \infty$

When $x = \infty$, the corresponding characteristics are

$$4p\alpha = 0, \quad \psi(x) = 0.$$

The corresponding differential equation with the introduction of transformation $x = 1/t$ and assumption of $p = 0$ for possible solution is

$$t^2(1-t)D^2 z + \ (\gamma-1)t^2 + 2t(1-t) - t(1+\gamma+\delta) \ \ Dz + (\gamma+\delta-\sigma)\, z = 0, \tag{2.6}$$

leading to
$$\Big)$$
$$\lambda = \frac{1+\gamma+\delta \pm \sqrt{(1+\delta+\gamma)^2 - 4(\gamma+\delta-\sigma)}}{2}.$$

The corresponding polynomial solution is

$$\begin{aligned} \mathcal{D}y_\lambda &= \mathcal{D}\mathcal{C}_n(\alpha,\beta,\gamma,\delta;t) = \sum_{m=0}^{\infty}(-1)^m \left[\frac{1}{(D-1)(D+\delta)}\right]^m \times \\ &\left[t^3\frac{d^2}{dt^2} + \ (1+4p+\gamma+\delta)t^2 + 4pt^3)\,\frac{d}{dt} + 4p(1+\alpha)t^2\right]^m t^\lambda dt : \end{aligned} \tag{2.7}$$

If we assume that $\sigma = 0$, we obtain $\lambda = 1$ or $\lambda = \gamma + \delta$ with the polynomial solution

$$\mathcal{D}y_\lambda = \mathcal{D}\mathcal{C}_n(\alpha,\beta,\gamma,\delta;t) = \sum_{m=0}^{\infty}(-1)^m\left[\frac{1}{(D-1)(D-(\delta+\gamma)}\right]^m \times \left[t^3\frac{d^2}{dt^2} + \ (1+4p+\gamma+\delta)t^2+4pt^3)\frac{d}{dt}+4p(1+\alpha)t^2\right]^m t^\lambda dt : \tag{2.8}$$

These solutions are new solutions of **Lame** equation in relation to the derivatives of the old ones.

3 Concluding remarks

Derivatives of old solutions of the **Lame** equation has been expressed in terms of new ones. The polynomial solutions obtained are from the series solutions to the derived differential equation terminated at some positive integer say m. This method of obtaining polynomial solutions is being extended to some other confluent forms of diff erential equations, in particular other various confluent forms of the Heun's differential equation.

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