On Uniformly Spriallike Functions And A Corresponding Subclass Of Spirallike Functions

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***Abstract*-In this paper we introduce and study the class of uniformly-spirallike functions of order and a subclass of it. Also some convolution results are obtained.**

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I. Introduction

Let A denote the class of all analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

defined on the unit disk $\Delta = \{z : |z| < 1\}$ normalized by $f(0) = 0,\ f'(0) = 1.$ The function f ∈ A is spirallike if $Re\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\} > 0$ for all $z \in \Delta$ and for some a with $|\alpha| < \pi/2.$ Also f(z) is convex spirallike if if zf′ (z) is spirallike.

Definition 1.1-

(Uniformly α-spirallike functions of order β). The function f(z) of the form (1.1) is uniformly α-spirallike of order β) if the image of every circular arc Γ_z with center at ζ lying in Δ is α-spirallike of order β with respect to $f(\zeta).$ This condition is equivalent to arg $\left(\frac{z'(t)f'(z)}{f(z)-f(\zeta)} - \beta ie^{i\alpha}\right)$ lies between α and $\alpha+\pi.$

The class of all uniformly α-spirallike functions of order β is denoted by $USP(\alpha,\beta).$

Theorem 1.1.- Let $f \in \mathcal{A}.$ Then $f \in UCV(\beta)$ *iff*

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} > \left|\frac{zf''(z)}{f'(z)}\right| + \beta, z \in \Delta.$$

This class extends the class of UCV functions introduced and studied by various authors in [3, 4, 7, 12]. We now give the following analytic description for $USP(\alpha,\beta).$

Theorem 1.2- Let $|\alpha| < \pi/2,\ \beta > 0.$ A function $f \in \mathcal{A}.$ belongs to $USP(\alpha,\beta).$ iff.

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} \geq \beta, z \neq \zeta, z, \zeta \in \Delta.$$

Proof.-Describe Γ_z by $z(t) = \zeta + re^{it}, t \in [0, 2\pi].$ Then $z'(t) = i(z-\zeta).$ Now $USP(\alpha,\beta)$ *iff*

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} \geq \beta, z \neq \zeta, z, \zeta \in \Delta.$$

Since

$$\begin{aligned} arg\left(\frac{z'(t)f'(z)}{f(z)-f(\zeta)} - \beta ie^{i\alpha}\right) &= arg\left(\frac{i(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta ie^{i\alpha}\right) \\ &= \pi/2 + arg\left(\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta e^{i\alpha}\right) \\ &= \pi/2 + \alpha + arg\left(e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta\right), \end{aligned}$$

we have

$$-\pi/2 \leq arg\left(e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta\right) \leq \pi/2$$

Or

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} - \beta\right\} \geq 0$$

Or

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} \geq \beta. \tag{a}$$

This extends obviously the class of functions studied in [4]. An equivalent form of Theorem 1.2 in terms of Hadamard product is

Theorem 1.3-

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic in Δ then the Hadamard product of f and g is $f * g = \sum_{n=0}^{\infty} a_n b_n z^n.$

In particular, if f is a normalized analytic function in Δ then for all $\beta, \gamma,$ $0 \leq |\beta| \leq 1, 0 \leq |\gamma| \leq 1,$ (see [10]) we have

$$\frac{1}{\beta}f(\beta z) = f * \frac{z}{1-\beta z}$$

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and

$$\frac{f(\beta z)-f(\gamma z)}{\beta-\gamma}=f*\frac{z}{(1-\beta z)(1-\gamma z)}$$
$$zf'(\beta z)=f*\frac{z}{(1-\beta z)^2}.$$

Theorem 1.3-Let $f\in\mathcal{A}.$ Then $USP(\alpha,\beta)$ *iff*

$$Re\left\{e^{i\alpha}\frac{f*\dfrac{z}{(1-bz)(1-cz)}}{f*\dfrac{z}{(1-bz)^2}}\right\}\le\frac{1}{\beta}.$$

Proof-Let $w=bz,\zeta=cz.$
Now

$$f*\frac{z}{(1-bz)(1-cz)}=\frac{\dfrac{f(bz)-f(cz)}{b-c}}{zf'(bz)}=\frac{\dfrac{f(\omega)-f(\zeta)}{(\omega-\zeta)/z}}{zf'(\omega)}=\frac{f(\omega)-f(\zeta)}{(\omega-\zeta)f'(\omega)}.$$

$$Re\left\{e^{i\alpha}\frac{f*\dfrac{z}{(1-bz)(1-cz)}}{f*\dfrac{z}{(1-bz)^2}}\right\}=Re\left\{e^{i\alpha}\frac{f(\omega)-f(\zeta)}{(\omega-\zeta)f'(\omega)}\right\}\le\frac{1}{\beta}\qquad\text{(b)}$$

and the result follows from Theorem 1.2.

Theorem 1.4-If $f\in\mathcal{A}$ satisfies
$Re\left\{e^{i\alpha}\frac{f'(\omega)}{f'(z)}\right\}\le\frac{1}{\beta},\ z,\omega\in\Delta$ then $f\in USP(\alpha,\beta).$

Proof-Note that

$$\frac{f(z)-f(\zeta)}{e^{-i\alpha}(z-\zeta)f'(z)}=\frac{1}{e^{-i\alpha}f'(z)}\int_0^1 f'(tz+(1-t)\zeta)dt$$
$$=\int_\zeta^z e^{i\alpha}\frac{f'(\omega)}{f'(z)(z-\zeta)}d\omega,\quad\text{where }\omega=tz+(1-t)\zeta.$$

By hypothesis,

$$Re\left\{e^{i\alpha}\frac{f'(\omega)}{f'(z)}\right\}\le\frac{1}{\beta}$$

or

$$Re\left\{\frac{f(z)-f(\zeta)}{e^{-i\alpha}(z-\zeta)f'(z)}\right\}\le\frac{1}{\beta},$$

which is equivalent to

$$Re\left\{e^{-i\alpha}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\}\ge\beta.$$
$$\Rightarrow f\in USP(\alpha,\beta).\qquad\text{(a)}$$

II. UNIFORMLY CONVEX α-spiral FUNCTIONS OF ORDER β

Let Γ_ω be the image of an arc $\Gamma_z: z=z(t),\ (a\le t\le b)$ under the function f(z). The arc Γ_ω is convex α-spirallike of order β if

$$arg\left(\frac{z''(t)}{z'(t)}+z'(t)\frac{f''(z)}{f'(z)}-\beta ie^{i\alpha}\right)$$

lies between α and $\alpha+\pi.$

Definition 2.1-The function f(z) represented by (1.1) is uniformly convex α-spiral of order β if the image of every circular arc Γ_z with center at ζ lying in Δ is convex α-spirallike of order β.

The class of all uniformly convex α-spiral functions of order β is denoted by $UCSP(\alpha,\beta).$

We now give an analytic description of $UCSP(\alpha,\beta).$

Theorem 2.1. A function $f(z)\in\mathcal{A}$ is in $UCSP(\alpha,\beta)$ *iff*

$$Re\left\{e^{-i\alpha}\left(1+\frac{(z-\zeta)f''(z)}{f'(z)}\right)\right\}\ge\beta,z\ne\zeta,z,\zeta\in\Delta.$$

Proof-Let $f(z)\in UCSP(\alpha,\beta).$ Then we have

$$\alpha\le arg\left(\frac{z''(t)}{z'(t)}+z'(t)\frac{f''(z)}{f'(z)}-\beta ie^{i\alpha}\right)\le\alpha+\pi$$

where the curve Γ_z is given by $z(t)=\zeta+re^{it},\ 0\le t\le 2\pi.$
Then $f(z)\in UCSP(\alpha,\beta)$ iff

$$\alpha\le arg\left(i+i(z-\zeta)\frac{f''(z)}{f'(z)}-\beta ie^{i\alpha}\right)\le\alpha+\pi.$$

Since

$$arg\left(i+i(z-\zeta)\frac{f''(z)}{f'(z)}-\beta ie^{i\alpha}\right)=\frac{\pi}{2}+arg\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}-\beta e^{i\alpha}\right),$$
$$\alpha-\frac{\pi}{2}\le arg\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}-\beta e^{i\alpha}\right)\le\alpha+\frac{\pi}{2}.$$

(or)

$$-\frac{\pi}{2}\le-\alpha+arg\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}-\beta e^{i\alpha}\right)\le\frac{\pi}{2}.$$

Hence $Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}-\beta e^{i\alpha}\right)\right\}\ge 0.$

or $Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\}\ge\beta,\ z\ne\zeta,\ z,\zeta\in\Delta.$ (a)

We now prove a single variable characterization of the class $UCSP(\alpha,\beta).$

Theorem 2.2-A function f(z) ∈ A is in $UCSP(\alpha,\beta)$ *iff*

$$Re\left\{e^{-i\alpha}\left(1+z\frac{f''(z)}{f'(z)}\right)\right\}\ge\left|z\frac{f''(z)}{f'(z)}\right|+\beta,z\in\Delta.$$

Proof-If $f \in UCSP(\alpha,\beta)$. then we have

$$Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\} \geq \beta.$$

Equivalently,

$$Re\left\{e^{-i\alpha}\left(1+z\frac{f''(z)}{f'(z)}\right)\right\} \geq Re\left\{e^{-i\alpha}\zeta\frac{f''(z)}{f'(z)}\right\}+\beta$$

for every $z \neq \zeta \in \Delta$.

Choose ζ = $e^{ir}z$ such that

$$Re\left|\left\{e^{-i\alpha}\zeta\frac{f''(z)}{f'(z)}\right\}+\beta = \left|z\frac{f''(z)}{f'(z)}\right|+\beta.\right.$$

Then we have

$$Re\left\{e^{-i\alpha}\left(1+z\frac{f''(z)}{f'(z)}\right)\right\} \geq \left|z\frac{f''(z)}{f'(z)}\right|+\beta.$$

Conversely assume the inequality is satisfied. Let $\zeta \in \Delta$ be arbitrary. Since $Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\}$ is harmonic in Δ it is enough to prove that $Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\} \geq \beta$ for all $z \in \Delta$ for which $|z| > |\zeta|$. Now

$$\begin{aligned} Re\left\{e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} &\geq \left|\frac{zf''(z)}{f'(z)}\right|+\beta \\ &\geq \left|e^{-i\alpha}\zeta\frac{f''(z)}{f'(z)}\right|+\beta \\ &\geq Re\left\{e^{-i\alpha}\zeta\frac{f''(z)}{f'(z)}\right\}+\beta \end{aligned}$$

$$\Rightarrow Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right)\right\} \geq \beta.$$

$$\Rightarrow f(z) \in \mathcal{A} \text{ is in } UCSP(\alpha,\beta).$$

(b)

Definition 2.2-A function f(z) ∈ *A* is in $SP_p(\alpha,\beta)$ *iff*

$$Re\left\{e^{-i\alpha}\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)}-1\right|+\beta,\ z\in\Delta.$$

Geometrically it means that $\frac{zf'(z)}{f(z)}$ lies in the parabolic region

$$\Omega_\alpha = \{\omega : Re\{e^{-i\alpha}\omega\} > |\omega-1|+\beta\}.$$

If $\omega \in \Omega_\alpha$, then $Re\{e^{-i\alpha}\omega\} \geq \frac{\cos\alpha+\beta}{2}$ and therefore the functions in the class $SP_p(\alpha,\beta)$ are α-spirallike of order $\frac{\cos\alpha+\beta}{2}$. When $\beta = 0,\ SP_p(\alpha,\beta)$ reduces to $SP_p(\alpha)$ *[11].*

The class $UCSP(0,\beta)$ is the class $UCV(\beta)$. In fact, every function in the class $UCSP(\alpha,\beta)$ is related to the class U $UCV(\beta)$. as in the following theorem.

Theorem 2.3-Let f(z) ∈ A and s(z) be defined by $f'(z) = (s'(z))e^{ia\cos a}$. Then f(z) ∈ $UCSP(\alpha,\beta)$ iff s(z) $\in UCV(\beta)$.

Proof-The result follows since

$$1+\frac{zs''(z)}{s'(z)} = \frac{e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)+i\sin\alpha}{\cos\alpha}.$$

We now prove a two variable characterization of the function in the class $SP_p(\alpha,\beta)$.

Theorem 2.4-The function f is in $SP_p(\alpha,\beta)$ *iff*

$$Re\left\{e^{-i\alpha}\left((z-\zeta)\frac{f'(z)}{f(z)}+\frac{\zeta}{z}\right)\right\} \geq \beta.$$

Proof-We have f(z) is in $SP_p(\alpha,\beta)$ iff $\int_0^z \frac{f(z)}{z}dz \in UCSP(\alpha,\beta)$.

By the two variable characterization for the class $UCSP(\alpha,\beta)$ for

$$F(z) = \int_0^z \frac{f(z)}{z}dz \in UCSP(\alpha,\beta),$$

$$Re\left\{e^{-i\alpha}\left(1+(z-\zeta)\frac{F''(z)}{F'(z)}\right)\right\} \geq \beta.$$

$$\Rightarrow Re\left\{e^{-i\alpha}\left((z-\zeta)\frac{f'(z)}{f(z)}+\frac{\zeta}{z}\right)\right\} \geq \beta.$$

(a)

Theorem 2.5- The function

$$f(z) = z + a_n z^n \text{ is in } SP_p(\alpha,\beta) \text{ iff}$$

$$|a_n| \leq \frac{\cos\alpha-\beta}{n(1+\cos\alpha)-1-\beta}.$$

Proof-The function $f(z) = z + a_n z^n \in SP_p(\alpha,\beta)$ iff

$$\left|\frac{zf''(z)}{f'(z)}-1\right|+\beta \leq Re\left\{e^{-i\alpha}\frac{zf''(z)}{f'(z)}\right\}, |z|<1. \tag{1}$$

It suffices to prove (2.1) for |z| = 1. Let |an| = r $a_n z^{n-1} = re^{i\theta}$. Then equation (2.1) becomes

$$\left|\frac{(n-1)re^{i\theta}}{1+re^{i\theta}}\right|+\beta \leq Re\left\{e^{-i\alpha}\frac{1+nre^{i\theta}}{1+re^{i\theta}}\right\}. \tag{2}$$

Simplifying and separating the real part of the expression on the right hand side of (2.2) we get

$$Re\left\{e^{-i\alpha}\frac{1+nre^{i\theta}}{1+re^{i\theta}}\right\}=\frac{[1+(n+1)r\cos\theta+nr^2]\cos\alpha+\sin\alpha(n-1)r\sin\theta}{1+r^2+2r\cos\theta}.$$

Therefore inequality (2.2) gives

$$(n-1)r+\beta(1+r^2+2r\cos\theta)^{1/2}\le\frac{(1+(n+1)r\cos\theta+nr^2)\cos\alpha+\sin\alpha(n-1)r\sin\theta}{(1+r^2+2r\cos\theta)^{1/2}}.$$

The minimum for the expression in the right hand side of the above equation occurs at $\theta=\pi$ and this minimum value is $\cos\alpha(1-nr)$. Therefore the necessary and sufficient condition for $f(z)=z+a_nz^n$ to be in $SP_p(\alpha,\beta)$ is that $(n-1)r+\beta(1-r)\le\cos\alpha(1-nr)$.

Solving this equation for r = $|a_n|$ we have

$$|a_n|\le\frac{\cos\alpha-\beta}{n(1+\cos\alpha)-1-\beta}. \qquad (a)$$

Since $f\in UCSP(\alpha,\beta)$ iff $zf'(z)\in SP_p(\alpha,\beta)$ we have the following.

Corollary 2.1-The function

$$f(z)=z+a_nz^n \text{ is in } UCSP(\alpha,\beta) \text{ iff}$$

$$|a_n|\le\frac{\cos\alpha-\beta}{n[n(1+\cos\alpha)-1-\beta]}.$$

Theorem 2.6-Let $f_i(z)\in SP_p(\alpha,\beta),\ i=1,2,\ldots,n$ and F(z) be given by

$F(z)=\int_0^z\prod_{i=1}^n\left(\frac{f_i(z)}{z}\right)^{\alpha_i}dz$ *where* $\alpha_i\ge0$ *and* $\sum_{i=1}^n\alpha_i=1$. *Then* $F(z)\in UCSP(\alpha,\beta)$.

Proof.-

Since $1+\frac{zF''(z)}{F'(z)}=\sum_{i=1}^n\alpha_i\frac{zf_i'(z)}{f_i(z)}$, we have

$$\begin{aligned}Re\left\{e^{-i\alpha}\left(1+\frac{zF''(z)}{F'(z)}\right)\right\}&=\sum_{i=1}^n\alpha_iRe\left(e^{-i\alpha}\frac{f_i'(z)}{f_i(z)}\right)\\&\ge\sum_{i=1}^n\alpha_i\left[\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\beta\right]\\&\ge\left|\sum_{i=1}^n\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)\right|+\beta\\&\ge\left|\frac{zF''(z)}{F'(z)}\right|+\beta.\end{aligned}$$

III. Convolution Theorems

Theorem 3.1-If $f(z)=z+\sum_{n=2}^\infty a_nz^n$ and $g(z)=z+\sum_{n=2}^\infty b_nz^n$ *are* elements of $UCSP(\alpha,\beta)$ then $(f*g)(z)=h(z)=z+\sum_{n=2}^\infty a_nb_nz^n$ *is in* $UCSP(\alpha,r)$ *where*

$$r\le\frac{\cos\alpha(32+3\cos^2\alpha-20\cos\alpha-8\beta+3\beta^2+2\beta\cos\alpha)-4\beta^2}{32+\cos^2\alpha+\beta^2-16\cos\alpha-16\beta+6\beta\cos\alpha},$$

$0\le\alpha<1,\ \beta\ge0.$

Proof. Since $f(z)$ and $g(z)$ are in $UCSP(\alpha,\beta)$ we have $\sum_{n=2}^\infty n(2n-\cos\alpha-\beta)|a_n|\le\cos\alpha-\beta$ and $\sum_{n=2}^\infty n(2n-\cos\alpha-\beta)|b_n|\le\cos\alpha-\beta$. We wish to find the largest r such that $\sum_{n=2}^\infty n(2n-\cos\alpha-r)|a_nb_n|\le\cos\alpha-r$. Equivalently we want to show that the conditions

$$\sum_{n=2}^\infty\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}n|a_n|\le1 \qquad (1)$$

and

$$\sum_{n=2}^\infty\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}n|b_n|\le1 \qquad (2)$$

imply that

$$\sum_{n=2}^\infty\frac{2n-\cos\alpha-r}{\cos\alpha-r}n|a_nb_n|\le1 \qquad (3)$$

for all

$$r\le\frac{\cos\alpha(32+3\cos^2\alpha-20\cos\alpha-8\beta+3\beta^2+2\beta\cos\alpha)-4\beta^2}{32+\cos^2\alpha+\beta^2-16\cos\alpha-16\beta+6\beta\cos\alpha}.$$

From (3.1) and (3.2) and by means of Cauchy-Schwarz inequality, we get that

$$\sum_{n=2}^\infty\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}n\sqrt{|a_nb_n|}\le1. \qquad (4)$$

Hence it is enough if we prove

$$\frac{2n-\cos\alpha-r}{\cos\alpha-r}n|a_nb_n|\le\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}n\sqrt{|a_nb_n|}\ \text{ for } n=2,3,\ldots$$

(or)

$$\sqrt{|a_nb_n|}\le\left(\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}\right)\left(\frac{\cos\alpha-r}{2n-\cos\alpha-r}\right).$$

From (3.4) it follows that

$$n\sqrt{|a_nb_n|}\le\frac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\ \text{for all } n. \qquad (5)$$

The above inequality is equivalent to

$$\frac{r+\cos\alpha}{2} \le \frac{\cos\alpha - \left[\frac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\right]^2}{1-\frac{1}{n}\left[\frac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\right]^2}. \qquad (6)$$

The right hand side of (3.6) is an increasing function of n, (n = 2, 3, . . .). By taking n = 2 in (3.6) we get

$$r \le \frac{\cos\alpha(32+3\cos^2\alpha-20\cos\alpha-8\beta+3\beta^2+2\beta\cos\alpha)-4\beta^2}{32+\cos^2\alpha+\beta^2-16\cos\alpha-16\beta+6\beta\cos\alpha}. \qquad \text{(a)}$$

Theorem 3.2-For $For\ f(z) \in UCSP(\alpha,\beta_1)$ and $g(z) \in UCSP(\alpha,\beta_2)$ $f(z)*g(z) \in UCSP(\alpha,r)$ where we have

$$r \le \frac{\cos\alpha\left(32-20\cos\alpha+3\cos^2\alpha-4(\beta_1+\beta_2)+3\beta_1\beta_2+(\beta_1+\beta_2)\cos\alpha\right)-4\beta_1\beta_2}{32+\cos^2\alpha+\beta_1\beta_2-16\cos\alpha-8(\beta_1+\beta_2)+3\cos\alpha(\beta_1+\beta_2)}.$$

Proof-Proceeding as in the proof of Theorem 3.1 we get

$$\frac{\cos\alpha+r}{2} \le \frac{\cos\alpha-\left[\frac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\frac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}{1-\frac{1}{n}\left[\frac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\frac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}. \qquad (7)$$

The right hand side of (3.7) is an increasing function of n = 2, 3, ... Setting n = 2 we get

$$r \le \frac{\cos\alpha\left(32-20\cos\alpha+3\cos^2\alpha-4(\beta_1+\beta_2)+3\beta_1\beta_2+(\beta_1+\beta_2)\cos\alpha\right)-4\beta_1\beta_2}{32+\cos^2\alpha+\beta_1\beta_2-16\cos\alpha-8(\beta_1+\beta_2)+3\cos\alpha(\beta_1+\beta_2)}. \qquad (8)$$

Theorem 3.3-Let $f(z), g(z), h(z) \in UCSP(\alpha,\beta).$ Then $Then\ f(z)*g(z)*h(z) \in UCSP(\alpha,r_1)\ where$

$$r_1 \le \frac{\begin{array}{l}6\cos^5\alpha+\cos^4\alpha(28\beta-100)+\cos^3\alpha(240-276\beta+18\beta^2)\\ +\cos^2\alpha(-672+514\beta-108\beta^2+10\beta^3)\\ +\cos\alpha(1024-768\beta+144\beta^2-28\beta^3)+16\beta^2\end{array}}{\begin{array}{l}1024-768\beta+184\beta^2-4\beta^3+10\cos^4\alpha+\cos^3\alpha(-116+18\beta)\\ +\cos^2\alpha(512-228\beta+32\beta^2)+\cos\alpha(-1280+256\beta-156\beta^2+6\beta^3)\end{array}}$$

Proof. From Theorem 3.1 we get $f(z)*g(z) \in UCSP(\alpha,r)$ where

$$r \le \frac{\cos\alpha(32+3\cos^2\alpha-20\cos\alpha-8\beta+3\beta^2+2\beta\cos\alpha)-4\beta^2}{32+\cos^2\alpha+\beta^2-16\cos\alpha-16\beta+6\beta\cos\alpha}.$$

$f(z)*g(z)*h(z) \in UCSP(\alpha,r_1)$ where

$$r_1 \le \frac{32\cos\alpha-20\cos^2\alpha-4\cos\alpha(r+\beta)+3\cos^3\alpha+(r+\beta)\cos^2\alpha+3r\beta\cos\alpha-4r\beta}{32+\cos^2\alpha+r\beta-16\cos\alpha-8(r+\beta)+3(r+\beta)\cos\alpha}. \qquad \text{(a)}$$

Substituting for r and simplifying we get the required result.

Theorem 3.4-Let

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = z + \sum_{n=2}^{\infty} b_n z^n$
$f, g \in SP_p(\alpha,\beta).\ Then\ f*g \in SP_p(\alpha,r)\ for$

$$r \le \frac{\cos\alpha[8+\cos^2\alpha+\beta^2-6\cos\alpha+\beta\cos\alpha-\beta]-2\beta^2}{2(4-2\cos\alpha-2\beta+\beta\cos\alpha)}.$$

Proof. Since f(z) and g(z) are in $SP_p(\alpha,\beta)$ we have

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)|a_n| \le \cos\alpha-\beta \text{ and}$$
$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)|b_n| \le \cos\alpha-\beta.$$

We wish to find the largest r such that

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-r)|a_n b_n| \le \cos\alpha-r.$$

Equivalently we want to show that the conditions

$$\sum_{n=2}^{\infty}\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}|a_n| \le 1 \qquad (9)$$

imply that

$$\sum_{n=2}^{\infty}\frac{2n-\cos\alpha-r}{\cos\alpha-r}|a_n b_n| \le 1$$

for all

$$r \le \frac{\cos\alpha[8+\cos^2\alpha+\beta^2-6\cos\alpha+\beta\cos\alpha-\beta]-2\beta^2}{2(4-2\cos\alpha-2\beta+\beta\cos\alpha)}.$$

From (3.8) and (3.9) and by Cauchy-Schwarz inequality, we get that

$$\sum_{n=2}^{\infty}\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}\sqrt{|a_n b_n|} \le 1. \qquad (10)$$

Hence it is enough if we prove

$$\frac{2n-\cos\alpha-r}{\cos\alpha-r}|a_n b_n| \le \frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}\sqrt{|a_n b_n|}.$$

(or)

$$\sqrt{|a_n b_n|} \le \left(\frac{2n-\cos\alpha-\beta}{\cos\alpha-\beta}\right)\left(\frac{\cos\alpha-r}{2n-\cos\alpha-r}\right).$$

From (3.10) it follows that

$$\sqrt{|a_n b_n|} \le \frac{\cos\alpha-\beta}{2n-\cos\alpha-\beta} \text{ for all } n.$$

The above inequality is equivalent to

$$\frac{r+\cos\alpha}{2} \leq \frac{\cos\alpha - n\left[\dfrac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\right]^2}{1-\left[\dfrac{\cos\alpha-\beta}{2n-\cos\alpha-\beta}\right]^2}. \quad (11)$$

The right hand side of (3.11) is an increasing function of n (n = 2, 3, . . .).By taking n = 2 in (3.11) we get

$$r \leq \frac{\cos\alpha[8+\cos^2\alpha+\beta^2-6\cos\alpha+\beta\cos\alpha-\beta]-2\beta^2}{2(4-2\cos\alpha-2\beta+\beta\cos\alpha)}. \quad \text{(a)}$$

Theorem 3.5-For $f(z) \in SP_p(\alpha,\beta_1)$ *and* $g(z) \in SP_p(\alpha,\beta_2)$ and $g(z) \in SP_p(\alpha,\beta_2)$ we have

$f(z) * g(z) \in SP_p(\alpha, r)$ *where*

$$r \leq \frac{\cos\alpha(8+\cos^2\alpha+\beta_1\beta_2-6\cos\alpha)-2\beta_1\beta_2}{8-4\cos\alpha-2(\beta_1+\beta_2)+(\beta_1+\beta_2)\cos\alpha}.$$

Proceeding as in the proof of Theorem 3.4 we get

$$\frac{r+\cos\alpha}{2} \leq \frac{\cos\alpha - n\left[\dfrac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\dfrac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}{1-\left[\dfrac{\cos\alpha-\beta_1}{2n-\cos\alpha-\beta_1}\right]\left[\dfrac{\cos\alpha-\beta_2}{2n-\cos\alpha-\beta_2}\right]}.$$

The right hand side is an increasing function of n = 2, 3, ... Setting n = 2 we get

$$r \leq \frac{\cos\alpha(8+\cos^2\alpha+\beta_1\beta_2-6\cos\alpha)-2\beta_1\beta_2}{8-4\cos\alpha-2(\beta_1+\beta_2)+(\beta_1+\beta_2)\cos\alpha}.$$

Corollary 3.1-Let $f(z), g(z), h(z) \in SP_p(\alpha,\beta)$. Then $f(z) * g(z) * h(z) \in SP_p(\alpha, r_1)$ *where*

$$r_1 \leq \frac{\begin{array}{c}4\beta^3+\cos^4\alpha(3\beta-4)+\cos^3\alpha(32+\beta^2)+\cos^2\alpha(\beta^3-3\beta^2+60\beta-80)\\ +\cos\alpha(-4\beta^3+2\beta^2-48\beta+64)\end{array}}{\begin{array}{c}64+8\beta^2-48\beta+\cos^4\alpha+\cos^3\alpha(\beta-8)+\cos^2\alpha(36-15\beta+3\beta^2)\\ +\cos\alpha(80+50\beta-12\beta^2)\end{array}}.$$

From Theorem 3.4 we get $f(z) * g(z) \in SP_p(\alpha, r)$ *where*

$$r \leq \frac{\cos\alpha[8+\cos^2\alpha+\beta^2-6\cos\alpha+\beta\cos\alpha-\beta]-2\beta^2}{2(4-2\cos\alpha-2\beta+\beta\cos\alpha)}.$$

$f(z) * g(z) * h(z) \in SP_p(\alpha, r_1)$ *where*

$$r_1 \leq \frac{\cos\alpha[8+\cos^2\alpha+\beta r-6\cos\alpha)-2\beta r}{8-4\cos\alpha-2\beta-2r+(\beta+r)\cos\alpha}.$$

Substituting for r and simplifying we get the required result.

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