Some Analytic Classes of Banach Function Spaces

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***Abstract*-We introduce a new class of functions, called the SK(p, q)−type spaces of analytic functions in the unit disk, then we give some characterizations of Bergman spaces by our SK(p, q) spaces.**

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I INTRODUCTION

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary and $H(\mathbb{D})$ be the class of all holomorphic functions in $\mathbb{D}$. For each $a \in \mathbb{D}$, the Green's function with logarithmic singularity at $a \in \mathbb{D}$ is denoted by $g(z,a) = \log\frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformations of $\mathbb{D}$. Assume that $K : [0,\infty) \to [0,\infty)$ is a right-continuous and nondecreasing function. The space $\mathcal{Q}_K$ of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{Q}_K} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))dA(z) < \infty, \text{ (see [15])}.$$

For an analytic function f on $\mathbb{D}$ and $0 < \alpha < \infty$. If

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty,$$

then f belongs to $\alpha-$Bloch space $\mathcal{B}^\alpha$. The little $\alpha-$Bloch space $\mathcal{B}^\alpha_0$ consists of analytic functions f on $\mathbb{D}$ such that

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |f'(z)| = 0.$$

For $0 < p < \infty$, the Bergman space $\mathcal{A}^p$ is the set of analytic functions f in the unit disk $\mathbb{D}$ with

$$\|f\|^p_{\mathcal{A}^p} = \frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

where $dA(z)$ denotes Lebesgue area measure. If $p > 1$, $\mathcal{A}^p$ is a Banach space with norm $\|.\|_{\mathcal{A}^p}$. If $0 < p < 1$, it is a complete metric space, where the metric is given by $d(f,g) = \|f-g\|^p_{\mathcal{A}^p}$. $\mathcal{A}^2$ is a Hilbert space with inner product

$$\langle f,g\rangle = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z)$$

and reproducing kernel $k_a(z) = \frac{1}{(1-\bar{a}z)^2}$ at $a \in \mathbb{D}$. It actually turns out that this holds for $f \in \mathcal{A}^p$, $1 \le p < \infty$. For more information about Bergman spaces we refer to [3, 4, 5, 6, 7, 14, 16].
A space closely related to $\mathcal{A}^p$ is $\mathcal{A}^{-n}$ $(n > 0)$, which consists of functions f analytic in $\mathbb{D}$ with

$$\|f\|_{\mathcal{A}^{-n}} = \sup_{z\in\mathbb{D}} |f(z)|(1-|z|^2)^n < \infty.$$

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$\mathcal{A}^{-n}$ is also a Banach space. It is easy to see that for any $\delta > 0$, $\mathcal{A}^{-\frac{1}{p-\delta}} \subseteq \mathcal{A}^p$. One can also check that $\mathcal{A}^p \subseteq \mathcal{A}^{-\frac{2}{p}}$. The space $\mathcal{A}^{-n}$ has been studied extensively (see [5, 10, 12, 13] and others). In this work, if $n = 1$ we call that $f \in \mathcal{A}^{-1}$. It should be remarked here that a function f is in the Bloch space if the derivative of f is in $\mathcal{A}^{-1}$ (see [5]). Palmberg in [10] introduced the $\mathcal{N}_p-$spaces, (with $p \in (0, \infty)$) consist of $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{N}_p} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2(1 - |\varphi_a(z)|^2)^p dA(z) < \infty,$$

when $p = 1$, $\mathcal{N}_p$ coincides $\mathcal{N}_1$. The $\mathcal{N}_1-$space was introduced in [8].
Very recently El-Sayed and Bahkit [1, 2] introduced the space $\mathcal{N}_K$ of holomorphic functions. This space consists of $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{\mathcal{N}_K} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 K(g(z, a))dA(z) < \infty.$$

If

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f(z)|^2 K(g(z, a))dA(z) = 0,$$

then f is said to belong to $\mathcal{N}_{0,K}$. Clearly, if $K(t) = t^p$, then $\mathcal{N}_K = \mathcal{N}_p$; since $g(z, a) \approx (1 - |\varphi_a(z)|^2)$. For $K(t) = 1$ it gives the Bergman space $\mathcal{A}^2$.

In this paper, we introduced the new space $\mathcal{S}_K(p, q)$ by the right continuous and nondecreasing function $K : [0,\infty) \to [0,\infty)$. For $0 < p < \infty, -2 < q < \infty$, we say that a function f analytic in $\mathbb{D}$ belongs to the space $\mathcal{S}_K(p, q)$ if

$$\|f\|^p_{\mathcal{S}_K(p,q)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p(1 - |z|^2)^q K(g(z, a))dA(z) < \infty.$$

Clearly, if $p = 2, q = 0$, then $\mathcal{S}_K(p, q) = \mathcal{N}_K$. For $K(t) = 1$ it gives the Bergman-type space $\mathcal{A}^p_q$. It is easy to check that $\|\cdot\|_{\mathcal{S}_K(p,q)}$ is a complete semi-norm on $\mathcal{S}_K(p, q)$ and it is Möbius invariant in the sense that $\|f \circ \phi\|_{\mathcal{S}_K(p,q)} = \|f\|_{\mathcal{S}_K(p,q)}$ whenever $f \in \mathcal{S}_K(p, q)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$, where $\mathrm{Aut}(\mathbb{D})$ is the group of all Möbius maps of the unit disk.

In this paper we show some relations between $\mathcal{S}_K(p, q)$ and $\mathcal{A}_\alpha$ for a nondecreasing K, and give a general way to construct different spaces $\mathcal{S}_{K_1}(p, q)$ and $\mathcal{S}_{K_2}(p, q)$ by using some functions K_1 and K_2. We assume from this paper that

(1)

It is also clear that $\mathcal{S}_K(p, q)$ contains all constant functions. If $\mathcal{S}_K(p, q)$ consists of just the constant functions, we say that it is trivial.

An important tool in the study of $\mathcal{S}_K(p, q)$ spaces is the auxiliary function ϕ_K defined by

$$\phi_K(s) = \sup_{0<t<1} \frac{K(st)}{K(t)}, \qquad 0 < s < \infty. \tag{2}$$

The following condition has played a crucial role in the study of $\mathcal{S}_K(p, q)$ spaces:

$$\int_1^\infty \phi_K(s)\frac{ds}{s^2} < \infty. \tag{3}$$

The function theory of $\mathcal{S}_K(p,q)$ obviously depends on the properties of K. Given two weight functions K_1 and K_2, we are going to write $K_1 \lesssim K_2$ if there exists a constant $C>0$, independent of t, such that $K_1(t) \le K_2(t)$ for all t. The notation $K_1 \gtrsim K_2$ is used in a similar fashion. When $K_1 \lesssim K_2 \lesssim K_1$, we write $K_1 \approx K_2$.

Two quantities A_f and B_f, both depending on an analytic function f on $\mathbb{D}$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on $\mathbb{D}$ we have:

$$\frac{1}{C}B_f \le A_f \le CB_f \quad \text{(see [11])}.$$

If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$. Given two weight functions K_1 and K_2, we are going to write $K_1 \lesssim K_2$ if there exists a constant $C>0$, independent of t, such that $K_1(t) \lesssim CK_2(t)$ for all t. The notation $K_1 \lesssim K_2$ is used in a similar fashion. When $K_1 \lesssim K_2 \lesssim K_1$, we write $K_1 \approx K_2$.

Recall that the function

$$f(r) = \sum_k^{\infty} a_k r^{n_k} \quad (\text{with } \ n_k \in \mathbb{N}\,;\ \text{for all } \ k \in \mathbb{N})$$

is said to belong to the Hadamard gap class (also known as lacunary series) if there exists a constant $c>1$ such that $\frac{n_{k+1}}{n_k} \ge c$ for all $k \in \mathbb{N}$ (see [9]).

Before proving theorems we recall a few facts about the Möbius function φ_a. First, the function φ_a is easily seen to be it own inverse under composition:

$$(\varphi_a \circ \varphi_a)(z) = z \quad \text{for all} \quad z \in \mathbb{D}$$

The following identity can be obtained by straight forward computation:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}\ , \quad (a, z \in \mathbb{D}).$$

A slightly different form in which we will apply the above identity is:

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = |\varphi_a'(z)|\ , \quad (a, z \in \mathbb{D}). \tag{4}$$

For $a \in \mathbb{D}$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by $dA(w) = |\varphi_a'(z)|^2 dA(z)$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on $\mathbb{D}$ we thus have the following change-of-variable formula:

$$\int_{D(0,r)} h(\varphi_a(w))dA(w) = \int_{D(a,r)} h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 dA(z)\ . \tag{5}$$

Proposition 1.1 *For f is non-constant function. If (1) does not hold, then $f \notin \mathcal{S}_K(p,q)$.*

Proof: By assumption there exists at least one point $a \in \mathbb{D}$ such that $f(a) \neq 0$. By

$$(1-|a|^2)^p |f(a)|^p \le \frac{1}{2\pi}\int_0^{2\pi} |(f \circ \varphi_a)(re^{i\theta})|^p d\theta, \qquad 0<r<1,$$

we conclude that

$$\begin{aligned}
&\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\
=\ & \int_{\mathbb{D}} |(f\circ\varphi_a)(z)|^p(1-|\varphi_a(z)|^2)^{q+2} \ \times\ (1-|z|^2)^{-2}K(\log\frac{1}{|z|})dA(z)\\
\geq\ & C\,(1-|a|^2)^{|q+2|}\int_0^1\left(\int_0^{2\pi}|(f\circ\varphi_a)(re^{i\theta})|^p d\theta\right)\ \times\ (1-|r|^2)^q K(\log\frac{1}{r})rdr\\
\geq\ & C\,(1-|a|^2)^{|q+2|+p}|f(a)|^p\int_0^1 (1-r^2)^q K(\log\frac{1}{r})rdr\ =\ \infty,
\end{aligned}$$

which means that $f\notin\mathcal{S}_K(p,q)$.

III HOLOMORPHIC SK(p, q) CLASSES

In this section we state some basic Banach space properties of the $\mathcal{S}_K(p,q)-$spaces.

Theorem 2.1 *Let* $0<p<\infty, -2<q<\infty$. *Then*
(i) $\mathcal{S}_K(p,q)\subset\mathcal{A}_{\frac{q+2}{p}}$.
(ii) $\mathcal{S}_K(p,q)=\mathcal{A}_{\frac{q+2}{p}}$ *if and only if*

$$\int_0^1 (1-r^2)^{-2}\ K(\log(1/r))rdr<\infty. \tag{6}$$

Proof: (i) For a fixed $r\in(0,1)$, let

$$E(a,r)=\{z\in\mathbb{D},\ |z-a|<r(1-|a|)\}.$$

We know that $E(a,r)\subset D(a,r)$ and for any $z\in E(a,r)$, we have

$$(1-r)(1-|a|)\leq 1-|z|\leq(1-r)(1-|a|),$$

which mans that $1-|z|^2\approx 1-|a|^2$ for any $z\in E(a,r)$. Since $|f(z)|^p$ is subharmonic, for $a\in\mathbb{D}$,

$$\begin{aligned}
\int_{\Delta}|f(z)|^p(1-|z|^2)^q\ K(g(z,a))dA(z)\ &\geq\ K(\log\frac{1}{r})\int_{D(a,r)}|f(z)|^p(1-|z|^2)^q dA(z)\\
&\geq\ \pi r^2\ K(\log\frac{1}{r})(1-|a|^2)^{q+2}|f(a)|^p.
\end{aligned}$$

If $f\in\mathcal{S}_K(p,q)$, then by estimate above we have

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{q+2}|f(a)|^p<\infty.$$

Hence $f\in\mathcal{A}_{\frac{q+2}{p}}$. The proof of (i) is complete
(ii) Let (4) hold. To prove $\mathcal{S}_K(p,q)=\mathcal{A}_{\frac{q+2}{p}}$. We only need to show that $\mathcal{A}_{\frac{q+2}{p}}\subset\mathcal{S}_K(p,q)$.
For $f\in\mathcal{A}_{\frac{q+2}{p}}$ we notice that

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^p (1-|a|^2)^q \, K(g(z,a)) dA(z) &\leq \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} \int_{\mathbb{D}} (1-|z|^2)^{-2} \, K(g(z,a)) dA(z) \\ &= \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} \int_{\mathbb{D}} (1-|w|^2)^{-2} \, K(\log(1/|w|)) dA(w) \\ &= 2\pi \, \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} \int_0^1 (1-r^2)^{-2} \, K(\log(1/r)) r dr \;<\; \infty.\end{aligned}$$

Hence $f \in \mathcal{A}_{\frac{q+2}{p}} \subset \mathcal{S}_K(p,q)$ and we have proved that (4) is a sufficient condition that $\mathcal{S}_K(p,q) = \mathcal{A}_{\frac{q+2}{p}}$. Conversely, we assume that $\mathcal{S}_K(p,q) = \mathcal{A}_{\frac{q+2}{p}}$. To prove that (4) is a necessary condition, we study the Hadamard gap function $f_a(z) = \sum_{j=1}^{\infty} a_j \, 2^{\frac{j(q+2-p)}{p}} z^{2^j}$, we assume that (4) dos not hold; that is,

$$\int_0^1 (1-r^2)^{-2} \, K(\log(1/r)) r dr = \infty.$$

Thus we find a continuous strictly decreasing function $h : [0,1) \to [0,0)$ tending to zero at 1 such that

$$\int_0^1 h(r)(1-r^2)^{-2} \, K(\log(1/r)) r dr = \infty. \tag{7}$$

It is easy to see that

$$r^{2^{k+2}-2} \geq \exp\{-2^{k+2}(1-r)\}, \quad r \in \left[\frac{1}{2}, 1\right). \tag{8}$$

We know that for $\alpha > 0, t^{2\alpha} \exp\{-4t\}$ is decreasing in $[\frac{\alpha}{2}, \frac{\alpha+2}{2}]$ and

$$\sup_{t>0} t^{2\alpha} \exp\{-4t\} = t^{2\alpha} \exp\{-4t\}|_{t=\frac{\alpha}{2}} = \left(\frac{\alpha}{2}\right)^{2\alpha} e^{-2\alpha}.$$

There exists an integer k for $\frac{3}{4} \leq r < 1$, such that $\frac{\alpha}{2} \leq 2^k(1-r) < \frac{\alpha+2}{2}$ and

$$2^{2\alpha k} \exp\{-2^{k+2}(1-r)\} \geq \left(\frac{\alpha+2}{2}\right)^{2\alpha} e^{-2(\alpha+1)}(1-r)^{-2\alpha}. \tag{9}$$

For $\frac{3}{4} \leq r < 1$ and k above we define

$$f_a(z) = \sum_{j=1}^{\infty} a_j \;\; 2^{j\frac{(q+2)}{p}} z^{2^j+1},$$

where $a_j = [h(1 - \frac{2p+q+2}{2^{(j+1)}p})]^{\frac{1}{p}}, \quad j = 0,1,2\cdots$. By (6) and (7) we conclude that

$$\begin{aligned} M_2^2(r, f_a) &= \int_0^{2\pi} |f_a(re^{i\theta})|^2 d\theta = 2\pi \sum_{k=1}^{\infty} a_k^2 2^{2k(\frac{q+2}{p})} z^{2^{k+1}-2} \\ &\geq 2\pi h^{\frac{2}{p}}(r) 2^{2k(\frac{q+2}{p})} \exp\{-2^{k+2}(1-r)\} \\ &\geq C h^{\frac{2}{p}}(r)(1-r)^{\frac{-2(q+2)}{p}}. \end{aligned}$$

Since f_a is defined by a Hadamard gap series, we have

$$M_2(r, f_a) \approx M_p(r, f_a),$$

where

$$M_p(r, f_a) = \left(\int_0^{2\pi} |f_a(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f_a(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
\geq\ &\int_0^1 M_p^p(r,f_a)(1-r^2)^qK(\log\frac{1}{r})rdr\\
\approx\ &\int_0^1 M_2^p(r,f_a)(1-r^2)^qK(\log\frac{1}{r})rdr\\
\geq\ & C\int_{\frac{3}{4}}^1 h(r)(1-r)^{-2}K(\log\frac{1}{r})rdr=\infty.
\end{aligned}$$

It means that $f_a\in\mathcal{A}_{\frac{q+2}{p}}\backslash\mathcal{S}_K(p,q)$, which is a contradiction. Hence (4) holds.

Theorem 2.2 *Let* $0<p<\infty, -2<q<\infty$. *Then*
(i) $\mathcal{S}_{0,K}(p,q)\subset\mathcal{A}_{0,\frac{q+2}{p}}$.
(ii) $\mathcal{S}_{0,K}(p,q)=\mathcal{A}_{0,\frac{q+2}{p}}$ *if and only if* (4) *holds.*

Proof: (i) Without loss of generality, we assume that $K(1)>0$, from the proof of Theorem 2.1 we have

$$\begin{aligned}
\pi(1/e)^2\,K(1)(1-|a|^2)^{q+2}|f(z)|^p &\leq K(1)\int_{E(a)}|f(z)|^p(1-|z|^2)^qdA(z)\\
&\geq K(1)\int_{D(a,1/e)}|f(z)|^p(1-|z|^2)^qdA(z)\\
&\geq \int_D|f(z)|^p(1-|z|^2)^q\,K(g(z,a))\,dA(z),
\end{aligned}$$

where $E(a)=\{z\in\mathbb{D}:|z-a|<\frac{1}{e}(1-|a|)\}$. If $f\in\mathcal{S}_{0,K}(p,q)$, we have

$$\lim_{|a|\to1}(1-|a|^2)^{\frac{q+2}{p}}|f(a)|=0.$$

Hence $f\in\mathcal{A}_{0,\frac{q+2}{p}}$. The proof of (i) is complete
(ii) We only need to show that $\mathcal{A}_{0,\frac{q+2}{p}}\subset\mathcal{S}_{0,K}(p,q)$. Assume that

$$A=\int_0^1(1-r^2)^{-2}\,K(\log(1/r))rdr<\infty.$$

For given $\varepsilon>0$ there exists an $r_1, 0<r_1<1$, such that

$$\int_{r_1}^1(1-r^2)^{-2}\,K(\log(1/r))rdr<\ \varepsilon\,. \tag{10}$$

Then we have

$$\begin{aligned}
&\int_{\mathbb{D}\backslash D(a,r_1)}|f(z)|^p(1-|a|^2)^q\,K(g(z,a))dA(z)\\
\leq\ &\|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}}\int_{\mathbb{D}\backslash D(a,r_1)}(1-|z|^2)^{-2}\,K(g(z,a))dA(z)\\
=\ &\|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}}\int_{r_1<|w|<1}(1-|w|^2)^{-2}\,K(\log(1/|w|))dA(w)
\end{aligned}$$

$$
\begin{aligned}
&= 2\pi \, \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} \int_{r_1}^1 (1-r^2)^{-2} \, K(\log(1/r)) r dr \\
&< 2\pi \, \varepsilon \, \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} .
\end{aligned}
$$

Similarly, if $f \in \mathcal{A}_{0,\frac{q+2}{p}}$, we have that

$$|f(\varphi_a(w))|(1-|\varphi_a(w)|^2)^{\frac{q+2}{p}} \to 0$$

uniformly for $|w| \le r$ if $|a| \to 1$, where r is fixed and $0 < r < 1$. Then

$$
\begin{aligned}
& \lim_{|a|\to 1} \int_{D(a,r_1)} |f(z)|^p \, (1-|z|^2)^q \, K(g(z,a)) dA(z) \\
= \; & \lim_{|a|\to 1} \int_{|w|<r_1} |f(\varphi_a(w))|^p \, (1-|\varphi_a(w)|^2)^{q+2} (1-|w|^2)^{-2} \, K(\log \frac{1}{|w|}) dA(z) \\
\le \; & C \lim_{|a|\to 1} \sup_{|w|<r_1} |f(\varphi_a(w))|^p \, (1-|\varphi_a(w)|^2)^{q+2} = 0.
\end{aligned}
$$

By the above it is easy to see that

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f(z)|^p \, (1-|z|^2)^q \, K(g(z,a)) dA(z) = 0.$$

Conversely, by the proof conversely of Theorem 2.1, we have

$$
\begin{aligned}
& \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_a(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
\ge \; & \int_0^1 M_p^p(r,f_a)(1-r^2)^q K(\log \frac{1}{r}) r dr \\
\approx \; & \int_0^1 M_2^p(r,f_a)(1-r^2)^q K(\log \frac{1}{r}) r dr \\
\ge \; & C \int_{\frac{3}{4}}^1 h(r)(1-r)^{-2} K(\log \frac{1}{r}) r dr = \infty.
\end{aligned}
$$

It means that $f_a \in \mathcal{A}_{0,\frac{q+2}{p}} \backslash \mathcal{S}_{0,K}(p,q)$, which is a contradiction. Hence (4) holds.

III WEIGHTED FUNCTIONS ON SK(p, q) CLASSES

The following result means that the weight functions K can be chosen as bounded.

Theorem 3.1 *We assume that $K_1(1) > 0$ and $K_2(r) = \inf\{K_1(r), K_1(1)\}$. Then $\mathcal{S}_{K_2}(p,q) = \mathcal{S}_{K_1}(p,q)$.*

If $f \in \mathcal{S}_{K_2}(p,q)$, f must be

$$
\begin{aligned}
& \int_{D(a,1/e)} |f(z)|^p \, (1-|z|^2)^q \, K_1(g(z,a)) dA(z) \\
\le \; & \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} \int_{D(a,1/e)} (1-|z|^2)^{-2} \, K_1(g(z,a)) dA(z) \\
= \; & \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} \int_{D(0,1/e)} (1-|w|^2)^{-2} \, K_1(\log(1/|w|)) dA(w) \le \; C \, \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} .
\end{aligned}
$$

Hence $f \in \mathcal{S}_{K_1}(p,q)$ and the theorem is proved.

Corollary 3.1 *Let* $0<p<\infty, -2<q<\infty$. *Then* $f \in \mathcal{S}_K(p,q)$ *if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z) < \infty. \tag{11}$$

Lemma 3.1 *Let* $0<p<\infty, -2<q<\infty$. *Then*

(i) $f \in \mathcal{A}_{\frac{q+2}{p}}$ *if and only if there exists* $\delta \in (0,1)$ *such that*

$$\sup_{a\in\mathbb{D}} \int_{D(a,\delta)} |f(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \infty; \tag{12}$$

(ii) $f \in \mathcal{A}_{0,\frac{q+2}{p}}$ *if and only if there exists* $\delta \in (0,1)$ *such that*

$$\lim_{|a|\to 1} \int_{D(a,\delta)} |f(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) = 0. \tag{13}$$

Proof: (i) Assume $f \in \mathcal{A}_{\frac{q+2}{p}}$. For any $\delta \in (0,1)$ and $a \in \mathbb{D}$, we have

$$\begin{aligned}
& \int_{D(a,\delta)} |f(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) \\
= & \int_{D(0,\delta)} |f(\varphi_a(z))|^p(1-|(\varphi_a(z))|^2)^{q+2}(1-|z|^2)^{-2} K\left(\log\frac{1}{|z|}\right)dA(z) \\
\le & \ \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}} \int_{D(0,r)} (1-r^2)^{-2}K(\log\frac{1}{r})rdr \\
\le & \ C\,\|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}}.
\end{aligned}$$

Conversely, suppose that (10) holds for some δ, $0<\delta<1$. By the proof of Theorem 2.1 we have

$$\begin{aligned}
& \int_{D(a,\delta)} |f(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) \\
\ge & \ K\left(\log\frac{1}{\delta}\right)\int_{D(a,\delta)} |f(z)|^p(1-|z|^2)^q dA(z) \\
\ge & \ K\left(\log\frac{1}{\delta}\right)\int_{E(a,\delta)} |f(z)|^p(1-|z|^2)^q dA(z) \\
\ge & \ \delta^2 K\left(\log\frac{1}{\delta}\right)|f(a)|^p(1-|a|^2)^{q+2},
\end{aligned}$$

which shows that $f \in \mathcal{A}_{\frac{q+2}{p}}$.

(ii) We omit the proof since its proof is similar to that of (i) above.

Theorem 3.2 *Let $0 < p < \infty, -2 < q < \infty$. Assume $K_1(r) \le K_2(r)$ for $r \in (0,1)$ and $\frac{K_1(r)}{K_2(r)} \to 0$ as $r \to 0$. If (4) is divergent for K_2, then*

$$\mathcal{N}_{K_2}(p,q) \subsetneqq \mathcal{N}_{K_1}(p,q).$$

Proof: It is clear that $\mathcal{S}_{K_2}(p,q) \subset \mathcal{S}_{K_1}(p,q)$. Suppose that $\mathcal{S}_{K_2}(p,q) = \mathcal{S}_{K_1}(p,q)$. From the open mapping theorem, we know that the identity mapping from one of these spaces into the other one is continuous. Thus there exists a constant C such that

$$\|\cdot\|_{\mathcal{S}_{K_2}(p,q)} \le C\|\cdot\|_{\mathcal{S}_{K_1}(p,q)}.$$

Since $\frac{K_1(r)}{K_2(r)} \to 0$ as $r \to 0$, then there exists $r_0 \in (0,1)$ such that $K_1(r) \le \frac{1}{2C}K_2(r)$ for $0 < r < r_0$. Choose $t_0 = e^{-r_0}$ and deduce that if $f \in \mathcal{S}_{K_2}(p,q)$, then

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^q K_2(g(z,a))dA(z) \\ \le\ & C \sup_{a\in\mathbb{D}} \int_{D(a,t_0)} |f(z)|^p(1-|z|^2)^q K_1(g(z,a))dA(z) \\ +\ & \frac{1}{2} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^q K_2(g(z,a))dA(z). \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^q K_2(g(z,a))dA(z) \\ \le\ & 2C \sup_{a\in\mathbb{D}} \int_{D(a,t_0)} |f(z)|^p(1-|z|^2)^q K_1(g(z,a))dA(z). \end{aligned}$$

By Lemma 3.1 there exists a constant C' such that for $f \in \mathcal{S}_{K_2}(p,q)$,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^q K_2(g(z,a))dA(z) \le C' \|f\|^p_{\mathcal{A}_{\frac{q+2}{p}}}. \tag{14}$$

If $h \in \mathcal{A}_{\frac{q+2}{p}}$ and $h_r(z) = h(rz), 0 < r < 1$, then $\|h_r\|_{\mathcal{A}_{\frac{q+2}{p}}} \le \|h\|_{\mathcal{A}_{\frac{q+2}{p}}}$. Since $h_r \in \mathcal{S}_{K_2}(p,q)$, we can choose $f = h_r$ in the inequality (12). Using Fatou's lemma, we deduce that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h(z)|^p(1-|z|^2)^q K_2(g(z,a))dA(z) \le C' \|h\|^p_{\mathcal{A}_{\frac{q+2}{p}}}.$$

We have proved that $h \in \mathcal{S}_{K_2}(p,q)$. It means that

$$\mathcal{S}_{K_2}(p,q) = \mathcal{A}_{\frac{q+2}{p}}.$$

It follows from Theorem 2.1 that the integral (4) with $K = K_2$ must be convergent, a contradiction. We obtain that

$$\mathcal{S}_{K_2}(p,q) \subsetneqq \mathcal{S}_{K_1}(p,q).$$

This completes the proof.

IV REFERENCES

[1] A. El-Sayed Ahmed and M. A. Bakhit, Holomorphic $\mathcal{N}_K$ and Bergman-type spaces, in JJ Grobler, LE Labuschagne and M. Möller (Eds), Proceedings of IWOTA 2007 conference, Birkhäuser Series on Operator Theory, Advances and Applications, (Birkhäuser Verlarg Publisher Basel/Switzerland), Vol 195 (2009), 121-138.

[2] A. El-Sayed Ahmed and M. A. Bahkit, Hadamard products and $\mathcal{N}_K$ space, Mathematical and computer Modelling, 51(1-2)(2010), 33-43.

[3] A. Borichev, H. Hedenmalm and K. Zhu, Bergman spaces and related topics in complex Analysis, American Mathematical Society, (2006).

[4] B. R. Choe, H. Koo and M. Stessin, Carleson measures for Bergman spaces and their dual Berezin transforms, Proc. Am. Math. Soc. 137(12)(2009), 4143-4155.

[5] P. Duren and A. Schuster, Bergman spaces, Mathematical surveys and Monographs 100, American Mathematical Society, Providence RI, (2004).

[6] E. G. Kwon, Quantities equivalent to the norm of a weighted Bergman space, J. Math. Anal. Appl. 338(2)(2008), 758-770.

[7] H. Hedenmalm, B. Korenblum and K. Zhu, T heory of Bergman spaces, Springer, New York, (2000).

[8] M. Lindström and N. Palmberg, Spectra of composition operators on BMOA, Integr. Equ. Oper. Theory, 53 (2005), 7586.

[9] J. Miao, A property of Analytic functions with Hadamard gaps, Bull. Austral. Math. Soc. 45(1992), 105-112.

[10] N. Palmberg, Composition operators acting on $\mathcal{N}_p$-spaces, Bull. Belg. Math. Soc.- Simon Stevin, 14 (2007) 545-554.

[11] K. Stroethoff, Besov-type characterisations for the Bloch space, Bull. Austral. Math. Soc. 39 (1989), 405420.

[12] A. P. Schuster, The homogeneous approximation property in the Bergman space, Houston J. Math. 24(4)(1998), 707-722.

[13] K. Seip, Beurling type density theorems in the unit disk, Invent. Math. 113(1)(1993), 21-39.

[14] N. L. Vasilevski, Commutative algebras of Toeplitz operators on the Bergman space. Operator Theory: Advances and Applications 185. Basel: Birkh¨auser. xxix (2008).

[15] H. Wulan and P.Wu, Characterizations of $\mathcal{Q}_T$ spaces, J.Math. Anal. Appl. 254 (2001), 484-497.

[16] K. Zhu, Operator theory in function spaces 2nd ed., Mathematical Surveys and Monographs 138. Providence, RI: American Mathematical Society (AMS). xvi (2007).