Some New Characterizations Of Strongly α-Preinvex Functions*

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***Abstract*-In this paper, we study some properties of strongly α-preinvex functions by introducing generalized strongly α-invex functions and generalized strongly $\alpha\eta$monotonicity. Our results generalize the results obtained by Muhammad Aslam Noor, Khalida Inayat Noor [J.Math.Anal.Appl. 316(2006) 697–706].**

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I. INTRODUCTION

It is well known that convexity and generalized convexity play a key role in many aspects of optimization, such as optimality conditions, saddle-point theorems, duality theorems, theorems of alternatives, and convergence of optimization algorithms, so the research on convexity and generalized convexity is one of the important aspects in mathematical programming. During the past several decades, to relax convexity assumptions imposed on the functions in theorems on optimality and duality, various generalized convexity concepts have been proposed. Among them, an important and significant generalization of convex function is introduction of invexity, which was introduced by Hanson [1, 3]. This concept is particularly interesting, since it is provides a broader setting to study the optimization problems. Later, Ben-Isreal and Mond [2] introduced a class of generalized convex functions, which is called the preinvex function. Mohan and Neogy [4]have shown that the preinvex functions and invex functions are equivalent under some conditions. In [5, 6], Weir, Mond and Noor have shown that the preinvex functions preserve some nice properties that convex functions
have. Jeyakumar, Mond [7] and Jeyakumar [8] introduced another class of generalized convex functions, which was called strongly α- -invex function, and some of its properties were studied. However, the concepts of the strongly α- -invex function in [7, 8] is misleading. To overcome this drawback, recently, M.A.Noor and K.I.Noor [9] also introduced a new class of generalized convex condition for strongly pseudo α- -invex function. However, the functions involved in [9] are all required to be differentiable.

To relax the differentiable Assumption, in this paper, by introducing the concepts of generalized strongly α--invex functions and generalized strongly $\alpha\eta$-monotonicity, we also study some properties of strongly α- -preinvex functions. Our results generalize the results obtained in [9].

II. PRELIMINARIES

Let K be a nonempty closed set in a real Hilbert space H. we denote by $\langle\cdot\rangle$ and $\|\cdot\|$ the inner product and norm, respectively. Let F : K −→ R and $\eta(\cdot,\cdot)$: K × K −→ H be continuous functions. Let $\alpha(\cdot,\cdot)$ K ×K −→ R\ {0} be a bifunction. First of all, we review some concepts as follows which have some relationships with this paper.

Definition 2.1-(See [9])Let u ∈ K Then the set K is said to be α-invex at u with respect to $\eta(\cdot,\cdot)$: and $\alpha(\cdot,\cdot)$ if, for all μ, ν ∈ K, t ∈ [0, 1], μ + t α (ν, μ) η (ν, μ) ∈ K. K is said to be an α- invex set with respect to η and α , if K is α -invex at each u ∈ K

Definition 2.2-(See [9])A function F on the set K is said to strongly α- preinvex , if there exists a constant μ > 0 such that

$$F(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu))\leq(1-t)F(\mu)+tF(\nu)-t(1-t)\mu\|\eta(\nu,\mu)\|^2,\quad\forall\mu,\nu\in K.$$

Definition 2.3-(See [9])A differentiable function F on the set K is said to strongly α- invex if there exists a constant μ > 0 such that

$$F(\nu)-F(\mu)\geq\langle\alpha(\nu,\mu)F'(\mu),\eta(\nu,\mu)\rangle+\mu\|\eta(\nu,\mu)\|^2,\quad\forall\mu,\nu\in K,$$

where F′(μ) is the differential of a function F at μ ∈ K.

Condition A. $F(\mu+\alpha(\nu,\mu)\eta(\nu,\mu))\leq F(\nu),\quad\forall\mu,\nu\in K.$

Condition C.

Let $\eta(\nu,\mu):K\times K\to H$ and $\alpha(\nu,\mu):K\times K\to R\setminus\{0\}$ satisfy the assumptions

$$\eta(\mu,\mu+t\alpha(v,\mu)\eta(v,\mu))=-t\eta(\nu,\mu),$$

$$\eta(\nu,\mu+t\alpha(v,\mu)\eta(v,\mu))=(1-t)\eta(\nu,\mu),\quad\forall\mu,\nu\in K,T\in[0,1].$$

In the sequel, we assume that F is directional differentiable, and following [10], we also assume that

$$\alpha(x,\mu)F'(x,\eta(x,\mu))=\lim_{\lambda\to0^+}\frac{F(x+\lambda\alpha(x,\mu)\eta(x,\mu))-F(x)}{\lambda},$$

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where $F'(\cdot,\cdot)$ is the directional derivative of F.

Definition 2.4-A directional differentiable function F on the set K is said to generalized strongly α- invex if there exists a constant μ > 0 such that

$$F(\nu)-F(\mu)\geq\alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu))+\mu\parallel\eta(\nu,\mu)\parallel^2,\quad\forall\mu,\nu\in K,$$

where F′(μ, η (v, μ)) is the directional derivative of a function F at μ ∈ K along the direction η (v, μ).

Remark 1-The definition generalized Definition 2.3.

Definition 2.5- (i) A directional differentiable operator T : K → H is said to generalized strongly $\alpha\eta$ -monotone, if there exists a constant α_1 > 0 such that

$$\alpha(\nu,\mu)T'(\mu,\eta(\nu,\mu))+\alpha(\mu,\nu)T'(\nu,\eta(\mu,\nu))\leq-\alpha_1\{\parallel\eta(\nu,\mu)\parallel^2+\parallel\eta(\mu,\nu)\parallel^2\},\quad\forall\mu,\nu\in K.$$

ii) A directional differentiable operator T : K → H is said to generalized strongly $\alpha\eta$-monotone, generalized $\alpha\eta$-pseudomonotone, if there exists a constant α_1 > 0 such that

$$\alpha(\mu,\nu)T'(\nu,\eta(\mu,\nu))+\alpha_1\parallel\eta(\nu,\mu)\parallel^2\geq0\Longrightarrow-\alpha(\mu,\nu)T'(\nu,\eta(\mu,\nu))\geq0\quad\forall\mu,\nu\in K.$$

Remark 2-We may also define generalized $\alpha\eta$- monotone, generalized strictly $\alpha\eta$- monotone, generalized $\alpha\eta$ -pseudomonotone, generalized strictly $\alpha\eta$ -pseudomonotone and generalized quasi $\alpha\eta$ -monotone in the similar way.

Definition 2.6-A directional differentiable function F on the set K is said to generalized strongly pseudo $\alpha\eta$-invex, if there exists a constant α_1 > 0 such that

$$\alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu))+\alpha_1\parallel\eta(\mu,\nu)\parallel^2\geq0\Longrightarrow F(\nu)-F(\mu)\geq0,\quad\forall\mu,\nu\in K.$$

Remark 3- We may also define generalized strongly quasi $\alpha\eta$- invex and generalized pseudo $\alpha\eta$- invex in the similar way.

III. SOME PROPERTIES OF STRONGLY α-PREINVEX FUNCTIONS

In this section, we discuss the relationship among strongly α - Preinvex , generalized strongly α - invex and generalized strongly $\alpha\eta$ --monotone under some conditions, Our results generalize the results obtained by Muhammad Aslam Noor, Khalida Inayat Noor in [9].

Theorem 3.1-Let F be a directional differentiable function on the α- invex set K. Let Condition C holds and α (μ, v) = α (v, μ) for all μ, _ ∈ K. Then the function F is strongly α- Preinvex if and only if F is generalized strongly α-invex.

Proof- Let F be a strongly α-preinvex function on the α-invex set K. Then, for all μ, v ∈ K, t ∈ [0, 1], one gets

$$\nu_t=\mu+t\alpha(\nu,\mu)\eta(\nu,\mu)\in K,$$

$$F(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu))\leq(1-t)F(\mu)+tF(\nu)-t(1-t)\mu\parallel\eta(\nu,\mu)\parallel^2,$$

which can be written as

$$F(\nu)-F(\mu)\geq\frac{F(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu))-F(\mu)}{t}+(1-t)\mu\parallel\eta(\nu,\mu)\parallel^2.$$

Taking into account F be a directional differentiable function on the α- invex set K, letting t → 0+ in above inequality, we have

$$F(\nu)-F(\mu)\geq\alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu))+\mu\parallel\eta(\nu,\mu)\parallel^2.$$

So F is generalized strongly α- invex

Conversely, let F is generalized strongly α- invex on the α- invex K. Then, using Condition C, we obtain

$$\begin{aligned}F(\nu)-F(\nu_t)&\geq\alpha(\nu,\nu_t)F'(\nu_t,\eta(\nu,\nu_t))+\mu\parallel\eta(\nu,\nu_t)\parallel^2\\&=(1-t)\alpha(\nu,\nu_t)F'(\nu_t,\eta(\nu,\mu))+\mu(1-t)^2\parallel\eta(\nu,\mu)\parallel^2.\end{aligned}\tag{1}$$

Similarly, we have

$$\begin{aligned}F(\mu)-F(\nu_t)&\geq\alpha(\mu,\nu_t)F'(\nu_t,\eta(\mu,\nu_t))+\mu\parallel\eta(\mu,\nu_t)\parallel^2\\&=-t\alpha(\mu,\nu_t)F'(\nu_t,\eta(\nu,\mu))+\mu t^2\parallel\eta(\nu,\mu)\parallel^2.\end{aligned}\tag{2}$$

Multiplying (3.1) by t and (3.2) by (1 − t) and adding the resultant, we get

$$F(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu))\leq(1-t)F(\mu)+tF(\nu)-t(1-t)\mu\parallel\eta(\nu,\mu)\parallel^2.$$

This shows that F be a strongly α- preinvex function on the α-invex set K.

Lemma 3.1-Let F be a directional differentiable function on the α- invex set K, g(t) = F(μ+t α (v, μ) η (v, μ)) and $F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu))+F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),-\eta(\nu,\mu))=0,\ \forall t_0\in[0,1]$. *Then $g(t)$ is differentiable for all $\mu,\nu\in K,\ t\in[0,1]$.*

Proof. For all μ, v ∈ K, t0 ∈ [0, 1], in view of F be a directional differentiable function on the α-invex set K, we have

$$\begin{aligned} g'_+(t_0) &= \lim_{t\to t_0} \frac{F(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu)) - F(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu))}{t-t_0} \\ &= \lim_{t-t_0\to 0^+} \frac{F(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu)+(t-t_0)\alpha(\nu,\mu)\eta(\nu,\mu)) - F(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu))}{t-t_0} \\ &= \alpha(\nu,\mu)F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu)), \end{aligned}$$

$$\begin{aligned} g'_-(t_0) &= \lim_{t\to t_0} \frac{F(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu)) - F(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu))}{t-t_0} \\ &= -\lim_{t_0-t\to 0^+} \frac{F(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu)+(t_0-t)\alpha(\nu,\mu)(-\eta(\nu,\mu))) - F(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu))}{t-t_0} \\ &= -\alpha(\nu,\mu)F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),-\eta(\nu,\mu)). \end{aligned}$$

In view of

$$F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu))+F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),-\eta(\nu,\mu))=0,\ \forall t_0\in[0,1].$$

This implies that g(t) is differentiable for all μ, ν ∈ K, t ∈ [0, 1].

Theorem 3.2- Let F be a directional differentiable function on the α-invex set K. If F is generalized strongly α- invex set K. If F is generalized strongly α - invex and $F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu))+F'(\mu+t_0\alpha(\nu,\mu)\eta(\nu,\mu),-\eta(\nu,\mu)) = 0,\ \forall t_0\in[0,1]$, then F is generalized strongly $\alpha\eta$-monotone. Conversely, if the function α (ν, μ) is a symmetric, that is, α (ν, μ) = α (μ, ν) for all μ, ν ∈ K, then F is generalized strongly α-invex under Condition A and C.

Proof-Let F is generalized strongly α- invex , then

$$F(\nu)-F(\mu)\geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu))+\mu\parallel \eta(\nu,\mu)\parallel^2,\quad \forall\mu,\nu\in K. \tag{3}$$

Exchange μ and ν in (3.3), we have

$$F(\mu)-F(\nu)\geq \alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu))+\nu\parallel \eta(\mu,\nu)\parallel^2,\quad \forall\mu,\nu\in K. \tag{4}$$

Adding (3.3) and (3.4), one gets

$$\alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu))+\alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu))\leq -\mu\{\parallel \eta(\nu,\mu)\parallel^2+\parallel \eta(\mu,\nu)\parallel^2\}, \tag{5}$$

which implies that F is generalized strongly $\alpha\eta$-monotone

Conversely, let F be generalized strongly $\alpha\eta$-monotone. Then there exists a constant $\alpha_1 > 0$ such that

$$\alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu))+\alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu))\leq -\alpha_1\{\parallel \eta(\nu,\mu)\parallel^2+\parallel \eta(\mu,\nu)\parallel^2\},$$

which can be written as

$$F'(\nu,\eta(\mu,\nu))\leq -F'(\mu,\eta(\nu,\mu))-\overline{\alpha}_1\{\parallel \eta(\nu,\mu)\parallel^2+\parallel \eta(\mu,\nu)\parallel^2\}, \tag{6}$$

Since the function α (ν, μ) is a symmetric and $\overline{\alpha}_1=\frac{\alpha_1}{\alpha(\mu,\nu)}$.

In view of K is a α-invex set, so, for all μ, ν∈ K, t ∈ [0, 1], one gets

$\nu_t=\mu+t\alpha(\nu,\mu)\eta(\nu,\mu)\in K.$

Taking $\nu = \nu_t$ in (3.6) and using Condition C, we obtain

$$\begin{aligned} F'(\nu_t,\eta(\mu,\nu_t)) &\leq -F'(\mu,\eta(\nu_t,\mu))-\overline{\alpha}_1\{\parallel \eta(\nu_t,\mu)\parallel^2+\parallel \eta(\mu,\nu_t)\parallel^2\} \\ &= -tF'(\mu,\eta(\nu,\mu))-2t^2\overline{\alpha}_1\parallel \eta(\nu,\mu)\parallel^2, \end{aligned}$$

which implies that

$$F'(\nu_t,\eta(\nu,\mu))\geq F'(\mu,\eta(\nu,\mu))+2t\overline{\alpha}_1\parallel \eta(\nu,\mu)\parallel^2 . \tag{7}$$

Let $g(t)=F(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu))$ for all $\mu,\nu\in K,\ t\in[0,1]$. Then from Lemma 3.1 and (3.7), one gets

$$\begin{aligned} g'(t) &= \alpha(\nu,\mu)F'(\mu+t\alpha(\nu,\mu)\eta(\nu,\mu),\eta(\nu,\mu)) \\ &\geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu))+2\alpha(\nu,\mu)t\overline{\alpha}_1\parallel \eta(\nu,\mu)\parallel^2 \qquad (3.8) \\ &= \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu))+2\alpha_1 t\parallel \eta(\nu,\mu)\parallel^2 . \end{aligned}$$

Integrating (3.8) between 0 and 1, we have

$$g(1) - g(0) \geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \alpha_1 \parallel \eta(\nu,\mu) \parallel^2,$$

that is,

$$F(\mu + \alpha(\nu,\mu)\eta(\nu,\mu)) - F(\mu) \geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \alpha_1 \parallel \eta(\nu,\mu) \parallel^2 .$$

Taking into account Condition A, we have

$$F(\nu) - F(\mu) \geq \alpha(\nu,\mu)F'(\mu,\eta(\nu,\mu)) + \alpha_1 \parallel \eta(\nu,\mu) \parallel^2 .$$

This shows that the function F is generalized strongly α-invex on the α-invex set K.

Remark 4-We can also obtain corresponding results such as Theorem 3.3, Theorem 3.4 in [9] in the same way as [9].

Theorem 3.3-Let the function F be generalized strongly $\alpha\eta$-pseudomonotone and F′(μ +t0 α (ν, μ) η (ν, μ), η (ν, μ))+F′(μ+t0 α (ν, μ) η (ν, μ), η (ν, μ)) = 0, ∀t0 ∈ [0, 1]. If Condition A and C hold, then the function F is generalized strongly pseudo $\alpha\eta$invex.

Proof- Let the function F be generalized strongly $\alpha\eta$-pseudomonotone. Then, for all μ, ν ∈ K,

$$\alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu)) + \alpha_1 \parallel \eta(\nu,\mu) \parallel^2 \geq 0 \Longrightarrow -\alpha(\mu,\nu)F'(\nu,\eta(\mu,\nu)) \geq 0 \quad \forall \mu,\nu \in K. \qquad (9)$$

Since K is an α-invex set, for all μ, ν∈ K, t ∈ [0, 1], $\nu_t = \mu + t\,\alpha(\nu,\mu)\,\eta(\nu,\mu) \in K$. Taking $\nu = \nu_t$ in (3.9) and using Condition C, we have

$$F'(\nu_t,\eta(\nu,\mu)) \geq 0. \qquad (10)$$

Let $g(t) = F(\nu_t) = F(\mu + t\,\alpha(\nu,\mu)\,\eta(\nu,\mu))$ for all μ, ν ∈ K, t ∈ [0, 1]. Then using (3.10) and Lemma 3.1, one gets

$$g'(t) = \alpha(\nu,\mu)F'(\nu_t,\eta(\nu,\mu)) \geq 0.$$

Integrating the above relation between 0 and 1, we have g(1) − g(0) ≥ 0, that is, $F(\nu_t) - F(\mu) \geq 0$. Using Condition A, we have F(ν)−F(μ) ≥ 0. This shows that the function F is generalized strongly pseudo $\alpha\eta$-invex.

Remark 5-We can also obtain corresponding results such as Corollary 3.1–Corollary 3.4 in [9] in the same way as [9].

IV. CONCLUDING REMARKS

In this paper, to relax the differentiable Assumption, by introducing the concepts of generalized strongly α-invex functions and generalized strongly $\alpha\eta$-monotonicity, we study some properties of strongly α-preinvex functions. Our results generalize the results obtained by Muhammad Aslam Noor, Khalida Inayat Noor in [9].

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