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A Blow up Result In The Cauchy Problem For A Semi-Linear Accretive Wave Equation

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A BLOW UP RESULT IN THE CAUCHY PROBLEM FOR A SEMI-LINEAR ACCRETIVE WAVE EQUATION

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I. Introduction

A very rich literature has been done on the semi-linear wave equation

$$u_{tt} - \Delta u = a\,|u_t|^{p-1}\,u_t + b\,|u|^{q-1}\,u, \tag{1}$$

where a and b are real numbers. Some special cases for the coefficients a and b have being considered by many authors:

1) When $a \le 0$ and $b = 0$, the damping term $|u_t|^{p-1}u_t$ ensures global existence for arbitrary data (See, for instance, Haraux and Zuazua [5]).
2) When $a = 0$ and $b \ge 0$, the source term $|u|^{q-1}u$ is responsible for finite blow up of the global nonexistence of solutions with negative initial energy (See Ball [2]; Kalantarov and Ladyzhenskaya [7]; and, Yordanov and Zhang [12]).
3) When $a \le 0, b > 0$ and $p > q$ or when $a \le 0, b > 0$ and $p = 1$, the global solutions (in time) under negative energy condition exist (Georgiev and Todorova [3] and Messaoudi [9]).
4) The case $a > 0$ is more complicated. For instance, a local existenceuniqueness solutions are guaranteed only for small values of p and regular initial data. This is due to the fact that the non linear term $|u_t|^{p-1}u_t$ has bad sign and is not locally Lipschitz continuous on $L^2(\Omega)$, where Ω is a bounded open domain of $\mathbb{R}^n$. This problem was studied by Haraux [4]. He showed that (with $b = 0$ on bounded domain) there is no nontrivial global and bounded solution. He also constructed blow up solutions with arbitrary small initial data. The same problem was considered by Jazar and Kiwan (See [6] and the references therein for the same equation on bounded domain).
5) For the case when $a = a(x,t)$ is a positive function, the author (see Ref. [10]) proved that any strong solution, with $\int u_t dx \ge C$, where C is a positive constant depends only of p, n, and R, blows up in finite time, when $supp(u_0) \cup supp\,(u_1) \subset B_R(0)$ (the ball of radius R).

In this paper, we consider the semi-linear wave equation with $a = 1$ and $b = 0$:

$$\begin{cases} u_{tt} - \Delta u = |u_t|^{p-1}\,u_t \quad (x,t) \in \mathbb{R}^N \times [0,T)\,, \\ u\,(x,0) = u_0\,(x) \in H^1_{loc,\,u}\left(\mathbb{R}^N\right), \\ u_t\,(x,0) = u_1\,(x) \in L^2_{loc,\,u}\left(\mathbb{R}^N\right). \end{cases} \tag{2}$$

and show that given any time $T > 0$, there exist initial data with sufficiently negative energy for which the solution blows up in a time $t^* \le T$. To achieve this goal, we will follow the same approach of Zaag and Merle [MZ1] by comparing, for our case, the growth u_t and k, where k is a solution of the explosively EDO $k_{tt} = |k_t|^{p-1}k_t$ associated with the equation (2). Unfortunately, the presence of the viscous term $|u_t|^{p-1}u_t$ makes our task more difficult. To overcome this difficulty, we draw attention to the work of Rivera and Fator [11] and rewrite (2) as follows:

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$$\begin{cases} u_{tt} - \int_0^t \Delta u_t(\tau)\, d\tau - \Delta u_0 = |u_t|^{p-1} u_t \quad (x,t) \in \mathbb{R}^N \times [0,T), \\ u(x,0) = u_0(x) \in H^1_{loc,\,u}(\mathbb{R}^N), \\ u_t(x,0) = u_1(x) \in L^2_{loc,\,u}(\mathbb{R}^N). \end{cases} \tag{3}$$

Then, we substitute the following change of variable:

$$v(x,t) = u_t(x,t), \tag{4}$$

in (3) to obtain the integro-differential equation

$$\begin{cases} v_t - \int_0^t \Delta v(\tau)\, d\tau - \Delta u_0(x) = |v|^{p-1} v, \; (x,t) \in \mathbb{R}^N \times [0,T) \\ v(x,0) = u_t(x,0) = u_1(x) =: v_0 \in L^2_{loc,\,u}(\mathbb{R}^N). \end{cases} \tag{5}$$

Now, we introduce $w := u_t/k$, where $k := \kappa (T-t)^{-\beta}$ with $\beta := \frac{1}{p-1}$ and $\kappa := \beta^\beta$. Using the following transformation defined by:

For $a \in \mathbb{R}^N$ and $T > 0$

$$z = x - a, \; s = -\log(T-t), \; v(t,x) = \frac{1}{(T'-t)^\beta} \theta_{T',\,a}(s,z) \tag{6}$$

and

$$u(x,0) =: \frac{1}{(T')^{\beta+!}} \theta_{a,00} \quad , \quad v(0,z) =: \frac{1}{(T')^\beta} \theta(s_0,y) =: \frac{1}{(T')^\beta} \theta_{a,0},$$

where $s_0 = -log(T)$. We then see that the function $\theta_a = \theta_{T,a}$ (we write θ for simplicity) satisfies for all $s \geq -Log(T)$ and all $z \in \mathbb{R}^N$

$$g(s)\theta_s + \beta g(s)\theta - \int_{s_0}^s g_2(\tau) \Delta\theta d\tau - g(s_0) \Delta\theta_{00} = g(s) |\theta|^{p-1} \theta \tag{7}$$

Where $g(s) = e^{(\beta+1)s}$ and $g(s) = e^{(\beta-1)s}$.

In the new set of variables (s,z), the behavior of u_t as $t \uparrow T$ is equivalent to the behavior of θ as $s \to \infty$. As far as we know, no local existence of solution was given for our problem (2). For this reason, we assume that there exists a set $A \subset R$ for which our problem (2) admits solutions for some $p \in A \subset \mathbb{R}$. In this work, We do not consider the same condition as in [10]. First let us provide the following assumption. H_1 we assume that $\alpha > \max\left(2, \frac{\beta}{2}(\beta+1)\right)$.

Our main result in this paper is:

Theorem 1 : Let be $p \in A \cap \left(1, \frac{N+3}{N-1}\right)$, and assume that the hypothesis H_1 is satisfied and θ a solution for (7) on B such that $E(\theta)(s_0) < 0$ for some $s_0 \in \mathbb{R}$, then θ blows up in $H^1(B) \times L^2(B)$ in time $s^* \leq s$, where B is the unit ball and E is the functional of energy associated to the equation (7). The above theorem implies directly the following blowing - up result for (5).

Proposition 2 : Let $p \in A \cap \left(1, \frac{N+3}{N-1}\right)$, and suppose that the hypothesis H_1 holds and v is a solution of (5) on B as $\Xi_{T,a}(v)(t) = E(\theta_{T,a})(-\log(T-t)) < 0$ for some $0 \leq t \leq T$ and $a \in \mathbb{R}^N$, then v blows up in finite time $T' < T$. The paper is organized as follows. In section 2 we define an associated decreasing energy to equation (7)(see Lemma 3) and in the section 3 we provide proofs for Theorem 1 and Proposition 2.

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II. The Associated Energy

In this section we start first by defining a weighted energy associated to the equation (7) and then, prove the lemma 3 . The wighted energy is given by

$$\begin{aligned} E(s) &= -\frac{\beta}{2}\int_B g(s)\,\rho^\alpha\theta^2 dz + \frac{1}{p+1}\int_B g(s)\,\rho^\alpha\,|\theta|^{p+1}\,dz \\ &+\frac{1}{8}\int_{s_0}^{s}\int_B \rho^\alpha g_2(\tau)\left\{|4\nabla\theta(\tau)-\nabla\theta(s)|^2-|\nabla\theta(s)|^2\right\}dzd\tau \\ &+\alpha\int_{s_0}^{s}\int_B g_2(\tau)\left[(N\rho-2(\alpha-1))\,|z|^2\right]\rho^{\alpha-2}\left\{|\theta(s)-\theta(\tau)|^2-|\theta(s)|^2\right\}dzd\tau \\ &+\alpha\int_{s_0}^{s}\int_B g(\tau)\,\rho^{\alpha-1}\left\{\left[e^{-2\tau}z\nabla\theta(s)-\theta(\tau)\right]^2-\left[e^{-2\tau}z\nabla\theta(s)\right]^2\right\}dzd\tau \\ &-\frac{g(s_0)}{2}\left\{\int_B\rho^\alpha\,|\nabla\theta(s)+\nabla\theta_{00}|^2\,dz-\int_B\rho^\alpha\,|\nabla\theta(s)|^2\,dz\right\} \\ &-\alpha g(s_0)\left\{\int_B\rho^{\alpha-1}\left[\theta(s)-z\nabla\theta_{00}\right]^2dz-\int_B\rho^{\alpha-1}\left[\theta(s)\right]^2dz\right\}. \end{aligned} \tag{8}$$

where B denotes the unit ball, α is any number satisfying $\alpha > \max\left(2,\frac{\beta}{2}(\beta+1)\right)$, and $\rho(z) := 1-|z|^2$.

Lemma 3 : The energy $s \to E(s)$ is a decreasing function of $s \geq s_0$. Moreover, we have

$$\begin{aligned} &E(s+1)-E(s) \\ &= -\tfrac{(\beta+1)}{p+1}\int_s^{s+1}\int_B g(s)\,\rho^\alpha\,|\theta(s')|^{p+1}\,dzds' \\ &-\int_s^{s+1}\int_B g(s)\,\rho^\alpha\theta_s^2(s')\,dzds' \\ &-\left[\alpha-\tfrac{\beta}{2}(\beta+1)\right]\int_s^{s+1} g(s')\int_B\rho^\alpha\theta^2(s')\,dzds' \\ &-\alpha\int_s^{s+1}\int_B g(s')\,\rho^{\alpha-1}\,|z|^2\,|\theta(s')|^2\,dzds' \\ &-\int_s^{s+1}\int_B g_2(s')\,\rho^\alpha\,|\nabla\theta(s')|^2\,dzds', \end{aligned} \tag{9}$$

where $\alpha > \max\left(2,\frac{\beta}{2}(\beta+1)\right)$.

Proof. To calculate the derivative of E, we multiply equation (7) by $\rho^\alpha\theta_s$ and integrate the equation over B

$$\begin{aligned} &\tfrac{1}{p+1}\tfrac{d}{ds}\int_B g(s)\,\rho^\alpha\,|\theta|^{p+1}\,dz-\tfrac{(\beta+1)}{p+1}\int_B g(s)\,\rho^\alpha\,|\theta|^{p+1}\,dz \\ &=\int_B g(s)\,\rho^\alpha\theta_s^2dz+\tfrac{\beta}{2}\tfrac{d}{ds}\int_B g(s)\,\rho^\alpha\theta^2dz-\tfrac{\beta}{2}(\beta+1)\int_B g(s)\,\rho^\alpha\theta^2dz \\ &-\int_B\int_{s_0}^s g(\tau)\,\Delta\theta(\tau)\,\rho^\alpha\theta_s(s)\,d\tau dz-\ g(s_0)\int_B\rho^\alpha\theta_s\Delta\theta_{00}dz \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\tfrac{\beta}{2}\tfrac{d}{ds}\int_B g(s)\,\rho^\alpha\theta^2dz-\tfrac{1}{p+1}\tfrac{d}{ds}\int_B g(s)\,\rho^\alpha\,|\theta|^{p+1}\,dz \\ &+\int_B\int_{s_0}^s g(\tau)\,\rho^\alpha\nabla\theta(\tau)\,\nabla\theta_s(s)\,d\tau dz-2\alpha\int_B\int_{s_0}^s g(\tau)\,\rho^{\alpha-1}z\nabla\theta(\tau)\,\theta_s(s)\,d\tau dz \\ &+g(s_0)\int_B\rho^\alpha\nabla\theta_{00}\nabla\theta_sdz-2\alpha g(s_0)\int_B\rho^{\alpha-1}z\nabla\theta_{00}\theta_sdz \\ &=-\tfrac{(\beta+1)}{p+1}\int_B g(s)\,\rho^\alpha\,|\theta|^{p+1}\,dz-\int_B g(s)\,\rho^\alpha\,[\theta_s]^2\,dz+\tfrac{\beta}{2}(\beta+1)\int_B g(s)\,\rho^\alpha\theta^2dz. \end{aligned}$$

The last equation can be written as

$$\begin{aligned} &\tfrac{\beta}{2}\tfrac{d}{ds}\int_B g(s)\,\rho^\alpha\theta^2dz-\tfrac{1}{p+1}\tfrac{d}{ds}\int_B g(s)\,\rho^\alpha\,|\theta|^{p+1}\,dz+I_1+I_2+I_3+I_4 \\ &=-\tfrac{(\beta+1)}{p+1}\int_B g(s)\,\rho^\alpha\,|\theta|^{p+1}\,dz-\int_B g(s)\,\rho^\alpha\,[\theta_s]^2\,dz+\tfrac{\beta}{2}(\beta+1)\int_B g(s)\,\rho^\alpha\theta^2dz, \end{aligned} \tag{10}$$

where

$$\begin{aligned}
I_1 &= \int_B \int_{s_0}^{s} g_2(\tau)\,\rho^{\alpha}\nabla\theta(\tau)\,\nabla\theta_s(s)\,d\tau dz\\
&= -\tfrac{1}{2}\tfrac{d}{ds}\left\{\int_B \int_{s_0}^{s} \rho^{\alpha} g_2(\tau)\left|2\nabla\theta(\tau)-\tfrac{1}{2}\nabla\theta(s)\right|^2 d\tau dz\right\}\\
&+\tfrac{1}{2}\left\{\int_B \rho^{\alpha} g_2(s)\left|2\nabla\theta(s)-\tfrac{1}{2}\nabla\theta(s)\right|^2 d\tau dz\right\}\\
&+\tfrac{1}{8}\tfrac{d}{ds}\int_{s_0}^{s} g_2(\tau)d\tau \int_B \rho^{\alpha}\left|\nabla\theta(s)\right|^2 dz\\
&-\tfrac{1}{8}\, g_2(s)\int_B \rho^{\alpha}\left|\nabla\theta\right|^2 dz,\\
&= -\tfrac{1}{2}\tfrac{d}{ds}\left\{\int_B \int_{s_0}^{s} \rho^{\alpha} g_2(\tau)\left|2\nabla\theta(\tau)-\tfrac{1}{2}\nabla\theta(s)\right|^2 d\tau dz\right\}\\
&+\tfrac{1}{8}\tfrac{d}{ds}\int_{s_0}^{s} g_2(\tau)d\tau \int_B \rho^{\alpha}\left|\nabla\theta(s)\right|^2 dz\\
&+ g_2(s)\int_B \rho^{\alpha}\left|\nabla\theta\right|^2 dz,
\end{aligned}$$

$$\begin{aligned}
I_3 &= g(s_0)\int_B \rho^{\alpha}\nabla\theta_{00}\nabla\theta_s dz\\
&= \tfrac{g(s_0)}{2}\tfrac{d}{ds}\left\{\int_B \rho^{\alpha}\left|\nabla\theta_{00}+\nabla\theta\right|^2 dz - \int_B \rho^{\alpha}\left|\nabla\theta\right|^2 dz\right\},
\end{aligned}$$

$$\begin{aligned}
I_4 &= -2\alpha g(s_0)\int_B \rho^{\alpha-1} y\nabla\theta_{00}\theta_s dz\\
&= \alpha g(s_0)\tfrac{d}{ds}\left\{\int_B \rho^{\alpha-1}\left[z\nabla\theta_{00}-\theta\right]^2 dz - \int_B \rho^{\alpha-1}\left[\theta\right]^2 dz\right\}.
\end{aligned}$$

$$\begin{aligned}
I_2 &= -2\alpha\int_B \int_{s_0}^{s} g(\tau)\,\rho^{\alpha-1}\left[\nabla\theta(\tau)\, zg(\tau)\,\theta_s(s)\right] d\tau dz\\
&= 2\alpha\int_B \int_{s_0}^{s} g_2(\tau)\left[\theta(\tau)\,\nabla\left(z\rho^{\alpha-1}\theta_s(s)\right)\right] d\tau dz\\
&= 2\alpha\int_B \int_{s_0}^{s} g_2(\tau)\,\theta(\tau)\, N\rho^{\alpha-1}\theta_s(s)\,d\tau dz\\
&-4\alpha(\alpha-1)\int_B \int_{s_0}^{s} g_2(\tau)\,\theta(\tau)\,|z|^2\rho^{\alpha-2}\theta_s(s)\,d\tau dz\\
&+2\alpha\int_B \int_{s_0}^{s} g_2(\tau)\,\theta(\tau)\, z\rho^{\alpha-1}\nabla\theta_s(s)\,d\tau dz\\
&= A_1 + A_2 + A_3
\end{aligned}$$

And

$$\begin{aligned}
A_1 &= 2\alpha N\int_B \int_{s_0}^{s} g_2(\tau)\left[\theta(\tau)\,\rho^{\alpha-1}\theta_s(s)\right] d\tau dz\\
&= -\alpha N\tfrac{d}{ds}\int_B \int_{s_0}^{s} g_2(\tau)\,\rho^{\alpha-1}\left[\theta(\tau)-\theta(s)\right]^2 d\tau dz\\
&+\alpha N\tfrac{d}{ds}\left\{\int_{s_0}^{s} g_2(\tau)\,d\tau\int_B \rho^{\alpha-1}\theta^2 dz\right\}\\
&-\alpha N\int_B g_2(s)\,\rho^{\alpha-1}\theta^2 dz,
\end{aligned}$$

$$\begin{aligned}
A_2 &= -4\alpha(\alpha-1)\int_B \int_{s_0}^{s} g_2(\tau)\,\theta(\tau)\left(|z|^2\rho_s^{\alpha-2}\theta(s)\right) d\tau dz\\
&= 2\alpha(\alpha-1)\tfrac{d}{ds}\int_B \int_{s_0}^{s} g_2(\tau)\,|z|^2\rho^{\alpha-2}\left[\theta(\tau)-\theta(s)\right]^2 d\tau dz\\
&-2\alpha(\alpha-1)\tfrac{d}{ds}\left[\int_{s_0}^{s} g_2(\tau)\,d\tau\int_B |z|^2\rho^{\alpha-2}\theta^2 dz\right]\\
&+2\alpha(\alpha-1)\int_B g_2(s)\,|z|^2\rho^{\alpha-2}\theta^2 dz,
\end{aligned}$$

$$\begin{aligned}
A_3 &= 2\alpha\int_B \int_{s_0}^{s} g(\tau)\,\theta(\tau)\, z\rho^{\alpha-1}\nabla\left(e^{-2\tau}\theta\right)_s(s)\,d\tau dz\\
&= -\alpha\tfrac{d}{ds}\int_B \int_{s_0}^{s} g(\tau)\,\rho^{\alpha-1}\left[\theta(\tau)-e^{-2\tau}z\nabla\theta(s)\right]^2 d\tau dz\\
&+\alpha\tfrac{d}{ds}\left\{\int_{s_0}^{s} e^{-4\tau} g(\tau)\,d\tau\int_B \rho^{\alpha-1}(z\nabla\theta)^2 dz\right\}\\
&-\alpha\int_B e^{-4s} g(s)\,\rho^{\alpha-1}(z\nabla\theta)^2 dz\\
&+\alpha\int_B g(s)\,\rho^{\alpha-1}\left[\theta(s)-e^{-2s}z\nabla\theta(s)\right]^2 dz\\
&= -\alpha\tfrac{d}{ds}\int_B \int_{s_0}^{s} g(\tau)\,\rho^{\alpha-1}\left[\theta(\tau)-e^{-2\tau}z\nabla\theta(s)\right]^2 d\tau dz\\
&+\alpha\tfrac{d}{ds}\left\{\int_{s_0}^{s}\int_B e^{-4s} g(\tau)\,d\tau\rho^{\alpha-1}(z\nabla\theta)^2 dz\right\}\\
&+\alpha\int_B g(s)\,\rho^{\alpha-1}\left[\theta(s)\right]^2 dz - \alpha\int_B g_2(s)\,\rho^{\alpha-1}z\nabla\left(\theta(s)^2\right) dz
\end{aligned}$$

$$
\begin{aligned}
&= -\alpha\frac{d}{ds}\int_B\int_{s_0}^{s} g(\tau)\,\rho^{\alpha-1}\left[\theta(\tau) - e^{-2\tau}z\nabla\theta(s)\right]^2 d\tau dz\\
&+\alpha\frac{d}{ds}\left\{\int_{s_0}^{s}\int_B e^{-4\tau}\,g(\tau)\,d\tau\rho^{\alpha-1}(z\nabla\theta)^2\,dz\right\}\\
&+\alpha\int_B g(s)\,\rho^{\alpha-1}[\theta(s)]^2\,dz + \alpha\int_B g_2(s)\,\nabla.\left(\rho^{\alpha-1}z\right)\theta(s)^2\,dz\\
&= -\alpha\frac{d}{ds}\int_B\int_{s_0}^{s} g(\tau)\,\rho^{\alpha-1}\left[\theta(\tau) - e^{-2\tau}z\nabla\theta(s)\right]^2 d\tau dz\\
&+\alpha\frac{d}{ds}\left\{\int_{s_0}^{s}\int_B e^{-4s}\,g(\tau)\,d\tau\rho^{\alpha-1}(z\nabla\theta(\tau))^2\,dz\right\}\\
&+\alpha\int_B g(s)\,\rho^{\alpha-1}[\theta(s)]^2\,dz + \alpha N\int_B g_2(s)\,\rho^{\alpha-1}[\theta(s)]^2\,dz\\
&-2\alpha(\alpha-1)\int_B g_2(s)\,\rho^{\alpha-2}|z|^2\,\theta(s)^2\,dz.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_2 &= -\alpha N\frac{d}{ds}\int_B\int_{s_0}^{s} g_2(\tau)\,\rho^{\alpha-1}[\theta(\tau)-\theta(s)]^2\,d\tau dz\\
&+\alpha N\frac{d}{ds}\left\{\int_{s_0}^{s}\int_B g_2(\tau)\,d\tau\rho^{\alpha-1}\theta^2 dz\right\}\\
&+2\alpha(\alpha-1)\frac{d}{ds}\int_B\int_{s_0}^{s} g_2(\tau)\,|z|^2\rho^{\alpha-2}[\theta(\tau)-\theta(s)]^2\,d\tau dz\\
&-2\alpha(\alpha-1)\frac{d}{ds}\int_{s_0}^{s} g_2(\tau)\,d\tau\int_B |z|^2\rho^{\alpha-2}[\theta(s)]^2\,dz\\
&-\alpha\frac{d}{ds}\int_B\int_{s_0}^{s} g(\tau)\,\rho^{\alpha-1}\left[\theta(\tau)-e^{-2\tau}z\nabla\theta(s)\right]^2 d\tau dz\\
&+\alpha\frac{d}{ds}\left\{\int_{s_0}^{s}\int_B e^{-4\tau}\,g(\tau)\,d\tau\rho^{\alpha-1}(z\nabla\theta)^2\,dz\right\}\\
&+\alpha\int_B g(s)\,\rho^{\alpha-1}[\theta(s)]^2\,dz.
\end{aligned}
$$

Substitute $I_0, ..., I_4$ in equation (10) we finally obtain

$$
\begin{aligned}
\frac{d}{ds}E(s) &= -\frac{(\beta+1)}{p+1}\int_B g(s)\,\rho^{\alpha}|w|^{p+1}\,dz\\
&-\left(\alpha-\frac{\beta}{2}(\beta+1)\right)\int_B g(s)\,\rho^{\alpha}[\theta(s)]^2\,dz\\
&-\alpha\int_B g(s)\,\rho^{\alpha-1}|z|^2\theta^2 dz - \int_B g(s)\,\rho^{\alpha}\theta_s^2 dz\\
&-\ g_2(s)\int_B \rho^{\alpha}|\nabla\theta|^2\,dz.
\end{aligned}
$$

We choose $\alpha > \max\left(2, \frac{\beta}{2}(\beta+1)\right)$. So we deduce (8) . This completes the proof of the lemma.

III. Proof The Main Result

In this section, we prove results of explosion for equation (7) and (5), using the method Georgiev and Todorova.

Proof of Proposition 2 : Suppose that there exist $T>0,\ 0<t_0<T$, and $a\in\mathbb{R}^n$ such that $\Xi_{T,a}(v)(t_0)<0$. Let $s_0=-\log(T-t_0)$, then $E(w_{T,a})(s_0)<0$. By applying Theorem 3 (see bellow), we find that the solution θ of (7) blows up in finite time $s^*<\infty$. Since $v(t,x)=\frac{1}{(T-t)^{\beta}}\theta(s,y)$, we deduce that v blows-up in finite time T' such that $s^* = -\log(T-t^*) \geq -\log(T-T')$, so we have $T'\leq T-e^{-s^*}<T$.

Proof of Theorem 1 : Since $E(s_0)<0$ and $E(s)$ is decreasing and then $E(s)<0$ for all $s\geq s_0$.

By setting $h(s)=-E(s)$, it follows that $h(s)\geq h(s_0)$ for all $s\geq s_0$.

Consider two different cases:

1. Assume that $h(s)$ is bounded. Then, we deduce that all the right terms in the following equation

$$
\begin{aligned}
&\frac{(\beta+1)}{p+1}\int_{s_0}^{s}\int_B g(\tau)\,\rho^{\alpha}|\theta|^{p+1}\,dzd\tau + \int_{s_0}^{s}\int_B g(\tau)\,\rho^{\alpha}\theta_\tau^2 dzd\tau\\
&+\left(\alpha-\frac{\beta}{2}(\beta+1)\right)\int_{s_0}^{s}\int_B g(\tau)\,\rho^{\alpha}\theta^2 dzd\tau\\
&+\alpha\int_{s_0}^{s}\int_B \rho^{\alpha-1}g(\tau)\,(z\theta)^2\,dzd\tau + \frac{1}{2}\int_{s_0}^{s}\int_B g_2(\tau)\,\rho^{\alpha}|\nabla\theta|^2\,dzd\tau\\
&= h(s)-h(s_0).
\end{aligned}
$$

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are bounded. It means that

$\int_{s_0}^{s}\int_B g(\tau)\,\rho^\alpha\theta^2 dzd\tau$, $\int_{s_0}^{s}\int_B g(\tau)\,\rho^\alpha\,|\theta|^{p+1}\,dzd\tau$ and $\int_{s_0}^{s}\int_B g_2(\tau)\,\rho^\alpha\,|\nabla\theta|^2\,dzd\tau$ are bounded for $p<\frac{N+3}{N-1}$.

Now, we introduce the following functional defined by

$$\varphi(s)=(h(s))^{1-\delta}+\varepsilon\int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\,|\theta(s)|^2\,dz.$$

where $0<\delta<1$ and ε are positive constants to be determined later.
We note that

$$\begin{aligned}
[\varphi(s)]^{\frac{1}{1-\delta}} &= C\left(h(s)+\varepsilon\int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\,|\theta(s)|^2\,dz\right)^{\frac{1}{1-\delta}}\\
&\le C\left(h(s)^{\frac{1}{1-\delta}}+\int_{s_0}^{s}g(\tau)d\tau\left(\int_B\rho^{\alpha}\,|\theta(s)|^2\,dz\right)^{\frac{1}{1-\delta}}\right)\\
&\le C\left(1+g(s)\left(\int_B\rho^{\alpha}\,|\theta(s)|^{\frac{2}{1-\delta}}\,dz\right)\right),
\end{aligned}$$

we choose δ such that $\frac{2}{1-\delta}\le p+1$ so we have $\delta\le\frac{p-1}{p+1}\ \in(0,1)$.

The derivative of this functional is given by

$$\begin{aligned}
\varphi'(s)&=(1-\delta)\,(h(s))^{-\delta}\,h(s)'+2\varepsilon\textstyle\int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\theta(s)\,\theta_s(s)\,dz\\
&\quad+2\varepsilon g(s)\textstyle\int_B\rho^{\alpha+1}\,|\theta(s)|^2\,dz\\
&\ge(1-\delta)\,M_0^{-1}h(s)'+I_0+2\varepsilon g(s)\textstyle\int_B\rho^{\alpha+1}\,|\theta(s)|^2\,dz.
\end{aligned}\tag{11}$$

because h is bounded.
From (7), it follows that

$$\begin{aligned}
I_0 &= \int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\theta(s)\,\theta_s(s)\,dz\\
&= -\beta\int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\theta^2dz\\
&\quad+\left(\int_{s_0}^{s}g(\tau)d\tau\right)g^{-1}(s)\int_B\rho^{\alpha+1}\theta(s)\left(\int_{s_0}^{s}g_2(\tau)\,\Delta\theta(\tau)\,d\tau\right)dz\\
&\quad+\int_{s_0}^{s}g(\tau)d\tau\left[g(s_0-s)\int_B\rho^{\alpha+1}\theta(s)\,\Delta\theta_{00}dz+\int_B\rho^{\alpha+1}\,|\theta(s)|^{p+1}\,dz\right],
\end{aligned}\tag{12}$$

then from the Green's formula we can write

$$I_0=-\beta\int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\theta^2dz+I_1+I_2+I_3+I_4+\int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\,|\theta(s)|^{p+1}\,dz,$$

Where

$$\begin{aligned}
I_1 &= -\left(\int_{s_0}^{s}g(\tau)d\tau\right)g^{-1}(s)\int_B\rho^{\alpha+1}\nabla\theta(s)\int_{s_0}^{s}g_2(\tau)\,\nabla\theta(\tau)\,d\tau dz\\
&\ge -\left|\int_B\rho^{\alpha+1}\nabla\theta(s)\int_{s_0}^{s}g_2(\tau)\,\nabla\theta(\tau)\,d\tau dz\right|\\
&\ge -\left[\sigma_1\ g_2(s)\int_B\rho^{\alpha}\,|\nabla\theta(s)|^2\,dz+\sigma_1^{-1}\int_{s_0}^{s}\int_B g_2(\tau)\,\rho^{\alpha}\,|\nabla\theta(\tau)|^2\,dzd\tau\right].
\end{aligned}\tag{13}$$

Using Young's inequality, we obtain

$$\begin{aligned} I_2 &= +2(\alpha+1)\left(\int_{s_0}^{s} g(\tau)d\tau\right) g^{-1}(s)\int_B z\rho^{\alpha}\theta(s)\left(\int_{s_0}^{s} g_2(\tau)\nabla\theta(\tau)d\tau\right)dz \qquad (14)\\ &\geq -2(\alpha+1)\left[\sigma_2\, g(s)\int_B \rho^{\alpha}|z\theta(s)|^2 dz+\sigma_2^{-1}\int_{s_0}^{s}\int_B g_2(\tau)\rho^{\alpha}|\nabla\theta(\tau)|^2 dzd\tau\right]. \end{aligned}$$

Similarly, we find

$$\begin{aligned} I_3 &= -g(s_0-s)\int_{s_0}^{s} g(\tau)d\tau\int_B \rho^{\alpha+1}\nabla\theta(s)\nabla\theta_{00}dz \qquad (15)\\ &\geq -\int_B \rho^{\alpha}\left[\sigma_3|\nabla\theta_{00}|^2+\sigma_3^{-1}g_2(s)|\nabla\theta(s)|^2\right]dz \end{aligned}$$

and

$$\begin{aligned} I_4 &= 2(\alpha+1)\, g(s_0-s)\int_{s_0}^{s} g(\tau)d\tau\int_B z\rho^{\alpha}\theta(s)\nabla\theta_{00}dz \qquad (16)\\ &\geq -2(\alpha+1)\left[\sigma_4^{-1}\int_B \rho^{\alpha}|\nabla\theta_{00}|^2+\sigma_4\ g(s)\int_B \rho^{\alpha}(z\theta(s))^2 dz\right]. \end{aligned}$$

Substituting (13)-(16) into (11) we obtain

$$\begin{aligned} \varphi'(s) \geq\ & 2\varepsilon g(s)\int_B \rho^{\alpha+1}|\theta(s)|^2 dz\\ &+\left[M_1\left(\alpha-\frac{\beta}{2}(\beta+1)\right)-2\beta\varepsilon\right]\int_B g(s)\rho^{\alpha}[\theta(s)]^2 dz\\ &+\left[M_1\alpha-4(\alpha+1)\varepsilon(\sigma_4+\sigma_2)\right]\int_B g(s)\rho^{\alpha-1}[z\theta(s)]^2 dz\\ &+\left[\frac{M_1}{2}-2\varepsilon\left(\sigma_3^{-1}+\sigma_1\right)\right]g_2(s)\int_B \rho^{\alpha}|\nabla\theta|^2 dz\\ &-2\varepsilon\left[2(\alpha+1)\sigma_4^{-1}+\sigma_3\right]\int_B \rho^{\alpha}\nabla\theta_{00}^2 dz\\ &-2\varepsilon\left(\sigma_1^{-1}+\sigma_2^{-1}\right)\int_{s_0}^{s}\int_B \rho^{\alpha+1}g_2(\tau)|\nabla\theta(\tau)|^2 dzd\tau\\ &+\frac{(\beta+1)}{p+1}M_1\int_{s_0}^{s}\int_B g(s)\rho^{\alpha}|\theta|^{p+1}dzd\tau \end{aligned}$$

where $M_1=\frac{(1-\delta)}{M_0}$.

Now, the first we choose δ such that

$$\delta\leq\min\left(\frac{\left(\alpha-\frac{\beta}{2}(\beta+1)\right)-2\beta\varepsilon M_0}{\left(\alpha-\frac{\beta}{2}(\beta+1)\right)},\frac{p-1}{p+1}\right).$$

So

$$M_1\left(\alpha-\frac{\beta}{2}(\beta+1)\right)-2\beta\varepsilon\geq 0.$$

After we choose $\sigma_1,\sigma_2,\sigma_3$ and σ_4 such that the following coefficients are Positive

$$\begin{aligned} \left[M_1\alpha-4(\alpha+1)\varepsilon(\sigma_4+\sigma_2)\right] &\geq 0,\\ \left[\frac{M_1}{2}-2\varepsilon\ \left(\sigma_3^{-1}+\sigma_1\right)\int_B \rho^{\alpha}|\nabla\theta(\tau)|^2 d\tau\right] &\geq 0. \end{aligned}$$

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Then

$$\begin{aligned}\varphi'(s) &\geq \frac{(\beta+1)}{p+1}M_1\int_{s_0}^{s}\int_B g(s)\rho^{\alpha}|\theta|^{p+1}dz\\ &-2\varepsilon\left(2(\alpha+1)\sigma_4^{-1}+\sigma_3\right)g(s_0)\int_B\rho^{\alpha+1}\nabla\theta_{00}^2\\ &-\left(\sigma_1^{-1}+\sigma_2^{-1}\right)2\varepsilon\int_B\int_{s_0}^{s}\rho^{\alpha+1}g_2(\tau)|\nabla\theta(\tau)|^2d\tau dz\\ &\geq C\varphi^{\frac{1}{1-\alpha}}(s)-2\varepsilon\left(2(\alpha+1)\sigma_4^{-1}+\sigma_3\right)\int_B\rho^{\alpha+1}\nabla\theta_{00}^2dz\\ &-\left(\sigma_1^{-1}+\sigma_2^{-1}\right)2\varepsilon\int_B\int_{s_0}^{s}\rho^{\alpha+1}g_2(\tau)|\nabla\theta(\tau)|^2d\tau dz\end{aligned}$$

and as $\int_B\int_{s_0}^{s}\rho^{\alpha+1}g_2(\tau)|\nabla\theta(\tau)|^2d\tau dz$ is bounded, then we can choose ε small enough such that

$$\begin{gathered}C\varphi^{\frac{1}{1-\delta}}(s)-2\varepsilon\left[2(\alpha+1)\sigma_4^{-1}+\sigma_3\right]\int_B\rho^{\alpha+1}\nabla\theta_{00}^2dz\\ -\left(\sigma_1^{-1}+\sigma_2^{-1}\right)2\varepsilon\int_B\int_{s_0}^{s}\rho^{\alpha+1}g_2(\tau)|\nabla\theta(\tau)|^2d\tau dz\geq 0,\end{gathered}$$

This implies that there exists ε' such that

$$\varphi'(s)\geq\varepsilon'\varphi^{\frac{1}{1-\alpha}}(s).$$

So we deduce that

$$\varphi(s)=(h(s))^{1-\alpha}+\varepsilon\int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\theta^2(s)dz$$

blows-up in finite time s^*, It follows that $\int_{s_0}^{s}g(\tau)d\tau\int_B\rho^{\alpha+1}\theta^2(s)dz$ blows up also in finite time because $h(s)$ is bounded. Thus $\|\theta\|_{L^2(B)}$ blows-up also in finite time.

2. We assume that $h(s)$ blows-up in finite time s^* and since

$$h(s)\leq\sup_{s_0\leq s\leq s^*}\left[\|\theta\|_{H^1(B)}+\|\theta_t\|_{L^2(B)}\right],$$

then the solution θ blows - up in finite time

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