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Metric Boolean Algebras and an Application To Propositional Logic

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METRIC BOOLEAN ALGEBRAS AND AN APPLICATION TO PROPOSITIONAL LOGIC

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Metric Boolean Algebras and an Application To Propositional Logic

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Abstract - Let $\boldsymbol{B}$ be a Boolean algebra and $\boldsymbol{\Omega}$ be the set of all homomorphisms from $\boldsymbol{B}$ into $\boldsymbol{D}$, and μ be a probability measure on $\boldsymbol{\Omega}$. We introduce the concepts of sizes of elements of $\boldsymbol{B}$ and similarity degrees of pairs of elements of $\boldsymbol{B}$ by means of μ, and then define a metric on $\boldsymbol{B}$. As an application, we propose a kind of approximate reasoning theory for propositional logic.

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I. Preliminaries

Let L be a distributive lattice and D be the lattice $\{0,1\}$. Assume Ω is the set of all homomorphism from L to D, then $\forall f\in\Omega, f$ is order-preserving, $i.e.,\ a,b\in L, a\leq b$ if and only if $f(a)\leq f(b)$.
In fact, if $a,b\in L, a\nleq b$, then there exists an $f\in\Omega$, such that $f(a)=1$ and $f(b)=0$ (see [4]). Therefore, if we think of elements of L as functions from Ω to D, $i.e.,\ \forall a\in L$ defining $a:\Omega\to\{0,1\}, a(f)=f(a)(f\in\Omega)$, then the partial order $\leq$ or L will have a representation by means of the partial order among function of 2^{Ω}.This fact is also true for Boolean algebras. We use the same symbol Ω to denote the set of all (Boolean) homomorphisms from a Boolean algebra B into D.

Proposition 1 : Suppose that B is a Boolean algebra, $D=\{0,1\}$, and Ω is the set of all homomorphism from L to D, then $a,b\in B, a\leq b$ if and only if $\forall f\in\Omega, f(a)\leq f(b)$.Throughout this paper, assume that B is a Boolean algebra and a probability measure μ is given on Ω, we will introduce the concepts of sizes of elements of B and similarity degrees between elements pairs of elements of B by means of μ and proposition 1, and finally define a metric ρ on B therefrom. Especially, this can be done the Lindenbaum algebra $B_L=F(S)/\approx$, where $F(S)$ is the set consisting of all logic propositions and $\approx$ is the congruence relation of logically equivalence[6] , and Ω is the set consisting of all valuations of $F(S)$ of which a probability measure can naturally be introduced. Finally, a kind of approximate reasoning theory can be developed on $F(S)$ because there have been distances among propositions.

II. Metric Boolean Algebras

Suppose that μ is a probability measure on Ω

Definition 1 : Define $\tau: B\to[0,1]$ as follows:

$$\tau(a)=\mu(\{f\in\Omega\mid a(f)=1\}),\quad a\in B$$

then $\tau(a)$ is called the *size* of *a* with respect to μ, or briefly , *size* of a if no confusion arises.
The following proposition is obvious.

Proposition 2 : (i) $0\leq\tau(a)\leq 1, a\in B$;

(ii) $\tau(1_B)=1,\tau(0_B)=0,$ where 1_B and 0_B are the greatest element and the least element of B respectively;
(iii) $\tau(a')=1-\tau(a), a\in B$
(iv) If $a\leq b$, then $\tau(a)\leq\tau(b), a,b\in B$.

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Example 1 : Consider the Boolean algebra $\mathcal{P}(X)$, where X is non-empty finite set , and $\mathcal{P}(X)$ is the power set of X. It is easy to verify that a mapping $f:\mathcal{P}(X)\to D$ is a homomorphism iff there exists an (unique) element x of X such that $f^{-1}(1)=\{A\subset X \mid x\in A\}$. Hence, $|\Omega|=|X|$. Assume that μ is the evenly distributed probability measure on Ω, then the size of an element A of $\mathcal{P}(X)$ is the cardinal of A over $|X|$, $i.e., \tau(A)=|A|/|X|$ $(A\in\mathcal{P}(X))$.

Proposition 3 : $\tau(a\vee b)=\tau(a)+\tau(b)-\tau(a\wedge b), a,b\in B$.

Assume that $a\to b=a'\vee b (a,b\in B)$ (see 5), then we have:

Proposition 4 : $\tau(a)\ge\alpha, \tau(a\to b)\ge\beta$, then $\tau(b)\ge\alpha+\beta-1$, where $\alpha,\beta\in[0,1]$.

Proposition 5 : If $\tau(a\to b)\ge\alpha, \tau(b\to c)\ge\beta$, then $\tau(a\to c)\ge\alpha+\beta-1$.

Definition 2 : Define $\eta: B\times B\to[0,1]$ as follows:

$$\eta(a,b)=\tau(a\to b)\wedge(b\to a), a,b\in B.$$

then $\eta(a,b)$ is called the **similarity degree** between a and b, and η the similar relation on B with respect to μ.

Example 2 : Consider the Boolean algebra $\mathcal{P}(X)$ where X is any non-empty set, and μ is any probability measure on Ω. Then

(1) $\eta(A, X-A)=0, (A\in\mathcal{P}(X))$. In fact, $A\to(X-A)=(X-A)\cup(X-A)=X-A, (X-A)\to A=A\cup A=A$

Hence, $\eta(A,X-A)=\tau((X-A)\cap A)=\tau(\emptyset)=0$.

(2) $\eta(A,B)+\eta(A,X-B)=1$. In fact, let $G=((X-A)\cup B)\cap((X-B)\cup A), H=((X-A)\cup(X-B))\cap(B\cup A)$. then it is routine to verify that $G\cup H=X, G\cap H=\emptyset$. Hence, $\eta(A,B)+\eta(A,X-B)=\mu(G)+\mu(H)=1$.

Proposition 6 : $\eta(a,b)=\mu(\{f\in\Omega \mid f(a)=f(b)\}), a,b\in B$.

Proposition 7 : (i) $\eta(a,b)=1$ if and only if $a=b$.
(ii) $\eta(a,a')=0$
(iii) $\eta(a,b)+\eta(a,b')=1$.
(iv) $\eta(a,c)\ge\eta(a,b)+\eta(b,c)-1$.

Proposition 8 : Suppose that $\eta(x,a)\ge\alpha, \eta(y,b)\ge\beta$ then $\eta(x\to y, a\to b)\ge\alpha+\beta-1$.

It follow from proposition 7 that the function ρ defined below is a metric on B.

Definition 3 : Define $\rho: B\times B$ as follows:

$$\rho(a,b)=1-\eta(a,b), a,b\in B.$$

Then ρ is a metric on B and (B,ρ) is called the **metric Boolean algebra** with respect to μ.

Theorem 1 : Let (B,ρ) be the metric Boolean algebra, then

(i) All the operations $', \vee, \wedge, \to$ are uniformly continuous.

(ii) If (B,ρ) is a complete metric space, then B is a $\omega-$complete lattice.

Proof : (i) By proposition 7, it follows that $\rho(a',b')=1-\eta(a',b')=\eta(a',b)=1-\eta(a,b)=\rho(a,b)$. If $\rho(a,b)\le\varepsilon$, then $\rho(a',b')\le\varepsilon$, where ε is any given positive number. This proves the continuity of the operation $': B\to B$.

Suppose that $\rho(x,a)\le\varepsilon, \rho(y,b)\le\varepsilon$, then $\eta(x,a)>1-\varepsilon, \eta(y,b)>1-\varepsilon$, and it follows proposition 8, having $\eta(x\to y, a\to b)\ge(1-\varepsilon)+(1-\varepsilon)-1=1-2\varepsilon$, hence $\rho(x\to y, a\to b)\le 2\varepsilon$. This proves that the operation $\to: B\times B\to B$ is uniformly continuous.

Since $a\vee b=a'\to b, a\wedge b=(a'\vee b')'$, hence the operations $\vee$ and $\wedge$ also uniformly continuous.

(ii) Suppose that (B,ρ) is a complete metric space, and $\Delta=\{a_1,a_2,\cdots\}\subset B$. Let $b_n=\vee_{i=1}^n a_i$, then $b_1,b_2,\cdots$ is an increasing sequence and hence $\tau(b_1),\tau(b_2),\cdots$ is an increase sequence in [0,1] and is therefore Cauchy. Note that $b_n\to b_m=b_n'\vee b_m=1_B$, and $b_m'\wedge b_n=0_B$ whenever $b_n\le b_m$, then $\rho(b_m,b_n)=1-\eta(b_m,b_n)=1-\tau((b_m\to b_n)\wedge(b_n\to b_m))=1-\tau(b_m\to b_n)=1-\tau(b_m'\vee b_n)=1-[\tau(b_m')+\tau(b_n)-\tau(b_m'\wedge b_n)]=1-\tau(b_m')-\tau(b_n)=\tau(b_m)-\tau(b_n)$. hence $b_1,b_2,\cdots$ is a Cauchy sequence in (B,ρ) and converges to an element c of B. Fix an n we have from $b_n=b_n\wedge b_{n+k}$ and the continuity of $\wedge$ that

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$$b_n = lim_{k\to\infty}(b_n \wedge b_{n+k}) = b_n \wedge lim_{k\to\infty} b_{n+k} b_n \wedge c,$$

hence $b_n \leq c$ and c is an upper bound of $\Sigma = \{b_1, b_2, \cdots\}$. On the other hand, let e be any upper bound of Σ, then $b_n = b_n \wedge e$ and hence

$$c = lim_{n\to\infty} b_n = lim_{n\to\infty}(b_n \wedge e) = (lim_{n\to\infty} b_n) \wedge e = c \wedge e.$$

This means that $c \leq e$, hence $c = sup\,\Sigma$. It is clear that $sup\Delta = sup\Sigma = c$. We can prove that $inf\Delta$ exists in a similar way, hence B is $\omega-$ complete lattice.

III. An Application

Consider the set of all abstract formulas in propositional logic. Let $S = \{p_1, p_2, \cdots\}$ be a countable set and $F(S)$ be free algebra of type $(\neg, \rightarrow)$ generated by S where $\neg$ and $\rightarrow$ are unary and binary operations respectively. Homomorphisms from $F(S)$ to $D = \{0,1\}$ called valuations of $F(S)$, the set is consisting of valuations will be denoted Ω. Assume that A, B are formulas (propositions) of $F(S)$, A and B are logically equivalent if $\forall \nu \in \Omega, \nu(A) = \nu(B)$. It is well known that the logical equivalence relation $\approx$ is a congruence relation on $F(S)$ with respect to $(\neg, \rightarrow)$, and the quotient algebra $F(S)/=\approx$ is a Boolean algebra and called the Lindenbaum algebra[6], and denoted B_L. Let $X = \prod_{n=1}^{\infty} X_n$, where $X_n = \{0,1\}$, and μ_n be the evenly distributed probability measure on X_n, *i.e.*, $\mu(\emptyset) = 0$, $\mu(X_n) = 1$, and $\mu_n(\{0\}) = \mu_n(\{1\}) = \frac{1}{2}, (n = 1, 2, \cdots)$, and let μ be the infinite product of $\mu_1, \mu_2, \cdots$ on X (see[3]). Since $F(S)$ is the free algebra generated by S, a valuation $\nu : F(S) \rightarrow \{0,1\}$ is completely decided by its restriction $\nu|S$. Since $\nu|S = \{\nu(p_1), \nu(p_2), \cdots\}$ can be thought of as a point of X, there is a bijection between Ω and X and hence the measure μ on X can be transplanted on Ω. We use the same symbol μ to denote the measure on Ω, *i.e.*,

$$\mu(\Sigma) = \mu(\{(x_1, x_2, \cdots) \in X \mid \exists \nu \in \Sigma, x_n = \nu(p_n), n = 1, 2, \cdots\}) \Sigma \subset \Omega.$$

Assume that $A, B \in F(S)$ and $A \approx B$ and $\nu \in \Omega$, then $\nu(A) = \nu(B)$ hence ν induces a unique Boolean homomorphism $\nu^* : B_L = F(S)/\approx \rightarrow \{0,1\}$ defined by $\nu^*(a) = \nu(A)$, where a is the congruence class containing $A\,(a \in B_L)$. In the following ν^* will be simplified as ν and Ω can be thought of as the set consisting of all Boolean homomorphisms B_L to $\{0,1\}$. Therefore there is a metric ρ on the Lindenbaum algebra B_L and (B_L, ρ) will be called the **metric Lindenbaum algebra** . Denote the congruence class B_L containing A by $[A]$, where $A \in F(S)$ and define $d : F(S) \times F(S) \rightarrow [0,1]$ as follows:

$$d(A,B) = \rho([A],[B]), \quad A, B \in F(S),$$

then d is a pesudo-metric on $F(S)$. Now that there exists the concept of distances among formulas, an approximate reasoning theory can naturally be developed on $F(S)$. We give in the following only a short sketch.

Definition 4 : Let T be the set consisting of all theorems in $F(S)$ (see[2]) and $A \in F(S)$. If $d(A, \mathrm{T}) < \varepsilon$, then A is called a **theorem with error** ε, and denoted $(\varepsilon) \vdash A$. Moreover, let $D(\Gamma)$ be the set of all $\Gamma-$ conclusions, $i.e., D(\Gamma) = \{A \in F(S) \mid \Gamma \vdash A\}$.

If $inf\{H(D(\Gamma), \mathcal{T}) \mid \Gamma \vdash A\} < \varepsilon$, where $H(\Sigma, \Delta)$ is Hausdorff distance between Σ and Δ (Σ, Δ) are non-empty subsets of $F(S)$, for a general definition of the Hausdorff distance between two non-empty, bounded subsets of a metric space without the requirement of closedness see[1], and that of subsets of a pseudo-metric space, see[6]), then A is called $\varepsilon-$ **quasi theorem** and denoted $\vdash (\varepsilon)A$.

Theorem 2 : Suppose that A is any formula of $F(S)$, then

$$(\varepsilon) \vdash A \text{ if and only if } \vdash (\varepsilon)A$$

Definition 5 : Suppose that $\Gamma \subseteq F(S)$, let $Div(\Gamma) = sup\{\rho(A,B) \mid A, B \in D(\Gamma)\}$, where $sup\,\emptyset = 0$ is assumed, then $Div(\Gamma)$ is called **the divergence degree of** *Gamma*. Moreover , let $Dev(\Gamma) = H(D(\Gamma), \mathcal{T})$, then $Dev(\Gamma)$ is called the **deviation** of Γ.

Theorem 3 : Suppose that $\Gamma \subseteq F(S)$, then $Div(\Gamma) = Dev(\Gamma)$.

Proof : Suppose that $Dev(\Gamma) = \alpha$ and $\varepsilon > 0$, then it follows from

$$H(D(\Gamma),\mathcal{T}) = max(sup\{d(A,\mathcal{T})|A \in D(\Gamma)\}, sup\{d(T,D(\Gamma))|T \in \mathcal{T}\})$$
$$= sup\{d(A,\mathcal{T})|A \in D(\Gamma)\} = \alpha.$$

that there is an $A \in D(\Gamma)$ such that $d(A,\mathcal{T}) \geq \alpha - \varepsilon$. Choose any T from $\mathcal{T}$, then $A, T \in D(\Gamma)$ and we have $Div(\Gamma) \geq d(A,\mathcal{T}) \geq \alpha - \varepsilon$ Since ε is arbitrary, we have $Div(\Gamma) \geq \alpha = Dev(\Gamma)$.

Conversely, we have

$$Dev(\Gamma) = H(D(\Gamma),\mathcal{T}) = max(sup\{d(A,\mathcal{T})|A \in D(\Gamma)\}, sup\{d(T,\mathcal{T})|T \in D(\Gamma)\})$$
$$= sup\{d(A,\mathcal{T})|A \in D(\Gamma)\} = sup\{1-\tau([A])|A \in D(\Gamma)\}.$$

Moreover, assume that $A, B \in D(\Gamma)$. Note that $[A] \leq [B] \to [A]$, and $[B] \leq [A] \to [B]$, we have $d(A,B) = \rho([A],[B]) = 1-\eta([A],[B]) = 1-\tau(([A]\to[B]) \wedge [B]\to[A])) \leq 1-\tau([A]) \wedge \tau([B]) = (1-\tau([A])) \vee (1-\tau([A]))$.

Hence , $Div(\Gamma) = sup\{d(A,B)|A,B \in D(\Gamma)\} \leq D(\Gamma)$.

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