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Uniformly Starlike And Uniformly Convexity Properties For Certain Special Functions

V. B. L. Chaurasia[1], Yaghvendra Kumawat[2]

Abstract: The aim of the present paper is to establish the sufficient condition for the function $yz\left\{{}_p\overline{\psi}_q(z)\right\}$ to be in the classes of uniformly starlike and uniformly convex functions. Similar types of result using integral operators also obtained.+

Keywords and Phrases : Generalized Fox-Wright function, uniformly starlike function, uniformly convex function, univalent.

I. Introduction and definitions

Let C denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

where $a_n \geq 0$ and $n \in N$, that are analytic in the open unit disk $U = \{z : |z| < 1\}$.

A function $f \in C$ is said to be starlike univalent of order α, $0 \leq \alpha < 1$, if and only if $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$, $z \in U$. Also $f(z)$ of the form (1) is uniformly starlike, whenever $\operatorname{Re}\left(\frac{f(z)-f(\xi)}{(z-\xi)f'(z)}\right) \geq 0$, $(z,\xi) \in U \times U$. This class of all uniformly starlike functions is denoted by UST ([3]) (see also [11], [14] and [7]).

The function $f \in C$ of the form (1) is uniformly convex in U whenever $\operatorname{Re}\left(1 + (z-\xi)\frac{f''(z)}{f'(z)}\right) \geq 0$, where $(z,\xi) \in U \times U$. This class of all uniformly convex functions is denoted by UCV ([2]) (also refer [1], [12], [4] and [6]). Further it is said to be in the class $UCV(\alpha), \alpha \geq 0$, if $\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \left|\frac{zf''(z)}{f'(z)}\right| + \alpha.$

H. Silverman [8] introduced a subclass B of C consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \; a_n \geq 0. \tag{2}$$

A function f of the form (2) is said to be in the class $USTN(\alpha), 0 \leq \alpha < 1$, if $\operatorname{Re}\left(\frac{f(z)-f(\xi)}{(z-\xi)f'(z)}\right) \geq \alpha$, where $(z,\xi) \in U \times U$.

In the present paper, we shall use analogues of the lemmas in [8] and [9]. Respectively in the following form:

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Lemma 1. A function f of the form (1) is in the class $UST(\alpha)$, if $\sum_{n=2}^{\infty}[(3-\alpha)n-2]|a_n| \le (1-\alpha)N$, where $N>0$ is a suitable constant. In particular, $f \in UST$ whenever $\sum_{n=2}^{\infty}(3n-2)|a_n| \le N$.

Lemma 2. A sufficient condition for a function f of the form (1) to be in the class $UCV(\alpha)$ is that $\sum_{n=2}^{\infty} n[(\alpha+1)n-\alpha]a_n \le N$, where $N>0$ is a suitable constant. In particular, $f \in UCV$ whenever $\sum_{n=2}^{\infty} n^2 a_n \le N$.

Lemma 3. A necessary and sufficient condition for a function f of the form (2) to be in the class $UCV(\alpha)$ is that $\sum_{n=2}^{\infty} n[(\alpha+1)n-\alpha]a_n \le N$.

The generalized Fox-Wright function is defined by ([5], p.271, eqn.(7))

$$ {}_p\overline{\Psi}_q(z) = {}_p\overline{\Psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;\mathbf{B}_j)_{1,q};\end{matrix}\ z\right] == \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j n)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j n)\}^{B_j}}\frac{z^n}{n!} \tag{3}$$

Where $1+\sum_{j=1}^{q}\beta_j > \sum_{j=1}^{p}\alpha_j$, α_j (j = 1,…,p) and β_j (j = 1,…,q) are real and positive and A_j (j = 1,…,p) and B_j (j = 1,…,q) can take non-integer values.

It is interesting to note that ${}_p\overline{F}_q$ ([5], p.271, eqn.(9)) is obtained by taking $\alpha_i = \beta_j = 1$ (i = 1,…,p; j = 1,…,q) in eqn. (3) and ${}_pF_q$ can also be obtained by taking $A_i = B_j = \alpha_j = \beta_j = 1$ (i = 1,…,p; j = 1,…,q)s in eqn (1.3).

For the sake of brevity, we use here the following notation:

$$ y = \frac{\prod_{j=1}^{q}\{\Gamma(b_j)\}^{B_j}}{\prod_{j=1}^{p}\{\Gamma(a_j)\}^{A_j}}. $$

II. Main Result

Theorem 2.1 If $a_i > 0 (i=1,\dots,p)$, $b_j > 0 (j=1,\dots,q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j + 1$ and $1+\sum_{j=1}^{q} B_j\beta_j > \sum_{j=1}^{p} A_j\alpha_j$, then a sufficient condition for the function $yz\{{}_p\overline{\psi}_q(z)\}$ to be in the class $UST(\alpha)$, $0 \le \alpha < 1$, is that

$$ \left(\frac{3-\alpha}{1-\alpha}\right) {}_p\overline{\psi}_q\left[\begin{matrix}(a_j+\alpha_j,\alpha_j;A_j)_{1,p};\\(b_j+\beta_j,\beta_j;B_j)_{1,q};\end{matrix}\ 1\right] + {}_p\overline{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}\ 1\right] \le y^{-1}(1+N). \tag{4}$$

Proof. Since

$$yz\left\{{}_p\overline{\psi}_q(z)\right\} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p}\left\{\Gamma\left(a_j + \alpha_j(n-1)\right)\right\}^{A_j} yz^n}{\prod_{j=1}^{q}\left\{\Gamma\left(b_j + \beta_j(n-1)\right)\right\}^{B_j}(n-1)!},$$

according to Lemma 1, we need only to show that,

$$\sum_{n=2}^{\infty}[(3-\alpha)n-2]\left|\frac{\prod_{j=1}^{p}\left\{\Gamma\left(a_j + \alpha_j(n-1)\right)\right\}^{A_j} y}{\prod_{j=1}^{q}\left\{\Gamma\left(b_j + \beta_j(n-1)\right)\right\}^{B_j}(n-1)!}\right| \le (1-\alpha)N.$$

Now $\sum_{n=2}^{\infty}[(3-\alpha)n-2]\left|\frac{\prod_{j=1}^{p}\left\{\Gamma\left(a_j + \alpha_j(n-1)\right)\right\}^{A_j} y}{\prod_{j=1}^{q}\left\{\Gamma\left(b_j + \beta_j(n-1)\right)\right\}^{B_j}(n-1)!}\right|$

$$= y(3-\alpha)\sum_{n=0}^{\infty}\left|\frac{\prod_{j=1}^{p}\left\{\Gamma\left(a_j + \alpha_j(n+1)\right)\right\}^{A_j}}{\prod_{j=1}^{q}\left\{\Gamma\left(b_j + \beta_j(n+1)\right)\right\}^{B_j} n!}\right| + y(1-\alpha)\sum_{n=1}^{\infty}\left|\frac{\prod_{j=1}^{p}\left\{\Gamma\left(a_j + \alpha_j n\right)\right\}^{A_j}}{\prod_{j=1}^{q}\left\{\Gamma\left(b_j + \beta_j n\right)\right\}^{B_j} n!}\right|$$

$$= y(3-\alpha)\,{}_p\overline{\psi}_q\left[\begin{matrix}\left(a_j+\alpha_j,\alpha_j;A_j\right)_{1,p};\\ \left(b_j+\beta_j,\beta_j;B_j\right)_{1,q};\end{matrix}1\right] + y(1-\alpha)\,{}_p\overline{\psi}_q\left[\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,p};\\ \left(b_j,\beta_j;B_j\right)_{1,q};\end{matrix}1\right] - (1-\alpha)$$

$$\le (1-\alpha)N,$$

this gives the desired result from the Lemma 1.

Theorem 2.2 If $a_i > 0(i=1,\ldots,p)$, $b_j > 0(j=1,\ldots,q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j + 1$ and $1 + \sum_{j=1}^{q} B_j\beta_j > \sum_{j=1}^{p} A_j\alpha_j$,

then a sufficient condition for the function $yz\left\{{}_p\overline{\psi}_q(z)\right\}$ to be in the class $USTN(\alpha)$, $0 \le \alpha < 1$, is that

$$\left(\frac{3-\alpha}{1-\alpha}\right){}_p\overline{\psi}_q\left[\begin{matrix}\left(a_j+\alpha_j,\alpha_j;A_j\right)_{1,p};\\ \left(b_j+\beta_j,\beta_j;B_j\right)_{1,q};\end{matrix}1\right] + {}_p\overline{\psi}_q\left[\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,p};\\ \left(b_j,\beta_j;B_j\right)_{1,q};\end{matrix}1\right] \le y^{-1}(1+N). \tag{5}$$

Proof. The proof directly follows from the Theorem 1.

Theorem 2.3 If $a_i > 0(i=1,\ldots,p)$, $b_j > 0(j=1,\ldots,q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j + 2$ and $1 + \sum_{j=1}^{q} B_j\beta_j > \sum_{j=1}^{p} A_j\alpha_j$,

then a sufficient condition for the function $yz\left\{{}_p\overline{\psi}_q(z)\right\}$ to be in the class $UCV(\alpha)$, $0 \le \alpha < 1$, is that

$$(1+\alpha)\,{}_p\overline{\psi}_q\left[\begin{matrix}\left(a_j+2\alpha_j,\alpha_j;A_j\right)_{1,p};\\ \left(b_j+2\beta_j,\beta_j;B_j\right)_{1,q};\end{matrix}1\right] + (3+2\alpha)\,{}_p\overline{\psi}_q\left[\begin{matrix}\left(a_j+\alpha_j,\alpha_j;A_j\right)_{1,p};\\ \left(b_j+\beta_j,\beta_j;B_j\right)_{1,q};\end{matrix}1\right]$$

$$+ {}_p\overline{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}1\right] \le y^{-1}(1+N). \qquad (6)$$

Proof. By Lemma 2, it suffices to show that

$$\sum_{n=2}^{\infty} n\left[(\alpha+1)n-\alpha\right]\frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j}\, y}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j(n-1))\}^{B_j}(n-1)!} \le N .$$

Now, $\displaystyle\sum_{n=2}^{\infty} n\left[(\alpha+1)n-\alpha\right]\frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j}\, y}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j(n-1))\}^{B_j}(n-1)!}$

$$= y(1+\alpha)\sum_{n=2}^{\infty}(n-1)^2\frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j(n-1))\}^{B_j}(n-1)!}$$

$$+ y(2+\alpha)\sum_{n=2}^{\infty}(n-1)\frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j(n-1))\}^{B_j}(n-1)!} + y\sum_{n=2}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j(n-1))\}^{B_j}(n-1)!}$$

$$= y(1+\alpha)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma(a_j+2\alpha_j+\alpha_j n)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+2\beta_j+\beta_j n)\}^{B_j}n!} + y(3+2\alpha)\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j+\alpha_j n)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j+\beta_j n)\}^{B_j}n!}$$

$$+ y\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j n)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j n)\}^{B_j}n!} - 1$$

$$= y(1+\alpha)\,{}_p\overline{\psi}_q\left[\begin{matrix}(a_j+2\alpha_j,\alpha_j;A_j)_{1,p};\\(b_j+2\beta_j,\beta_j;B_j)_{1,q};\end{matrix}1\right] + y(3+2\alpha)\,{}_p\overline{\psi}_q\left[\begin{matrix}(a_j+\alpha_j,\alpha_j;A_j)_{1,p};\\(b_j+\beta_j,\beta_j;B_j)_{1,q};\end{matrix}1\right]$$

$$+ y\,{}_p\overline{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}1\right] - 1,$$

this last expression is bounded above by N if and only if (6) holds. Hence the Theorem 3 is proved.

III. An integral operators

In this section we obtain sufficient conditions for the function $y\,{}_p\overline{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}z\right]=y\int_0^z {}_p\overline{\psi}_q(x)dx$ to be in the classes UST and UCV.

Theorem 3.1 If $a_i>0(i=1,\dots,p)$, $b_j>0(j=1,\dots,q)$, $\sum_{j=1}^q b_j>\sum_{j=1}^p a_j$ and $1+\sum_{j=1}^q B_j\beta_j>\sum_{j=1}^p A_j\alpha_j$, then a sufficient condition for the function $y\{{}_p\overline{\phi}_q(z)\}=y\int_0^z {}_p\overline{\psi}_q(x)dx$ to be in the class UST, is that

$$3\,{}_p\overline{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}1\right]-2\,{}_p\overline{\psi}_q\left[\begin{matrix}(a_j-\alpha_j,\alpha_j;A_j)_{1,p};\\(b_j-\beta_j,\beta_j;B_j)_{1,q};\end{matrix}1\right]$$
$$+2\frac{\prod_{j=1}^p\{\Gamma(a_j-\alpha_j)\}^{A_j}}{\prod_{j=1}^q\{\Gamma(b_j-\beta_j)\}^{B_j}}\le y^{-1}(1+N). \tag{7}$$

Proof. we have,

$$y\{{}_p\overline{\phi}_q(z)\}=y\int_0^z {}_p\overline{\psi}_q(x)dx=z+\sum_{n=2}^{\infty}\frac{\prod_{j=1}^p\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j}y\,z^n}{\prod_{j=1}^q\{\Gamma(b_j+\beta_j(n-1))\}^{B_j}n!}. \tag{8}$$

Now,

$$\sum_{n=2}^{\infty}(3n-2)\frac{\prod_{j=1}^p\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j}y}{\prod_{j=1}^q\{\Gamma(b_j+\beta_j(n-1))\}^{B_j}n!}$$
$$=3y\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p\{\Gamma(a_j+\alpha_j n)\}^{A_j}}{\prod_{j=1}^q\{\Gamma(b_j+\beta_j n)\}^{B_j}n!}-2y\sum_{n=0}^{\infty}\frac{\prod_{j=1}^p\{\Gamma(a_j-\alpha_j+\alpha_j n)\}^{A_j}}{\prod_{j=1}^q\{\Gamma(b_j-\beta_j+\beta_j n)\}^{B_j}n!}+2y\frac{\prod_{j=1}^p\{\Gamma(a_j-\alpha_j)\}^{A_j}}{\prod_{j=1}^q\{\Gamma(b_j-\beta_j)\}^{B_j}}-1$$
$$=3y\,{}_p\overline{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}1\right]-2y\,{}_p\overline{\psi}_q\left[\begin{matrix}(a_j-\alpha_j,\alpha_j;A_j)_{1,p};\\(b_j-\beta_j,\beta_j;B_j)_{1,q};\end{matrix}1\right]$$
$$+2y\frac{\prod_{j=1}^p\{\Gamma(a_j-\alpha_j)\}^{A_j}}{\prod_{j=1}^q\{\Gamma(b_j-\beta_j)\}^{B_j}}-1, \tag{9}$$

which in view of Lemma 1, (9) gives the result (7).

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Theorem 3.2 If $a_i > 0(i=1,...,p)$, $b_j > 0(j=1,...,q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j$ and $1+\sum_{j=1}^{q} B_j\beta_j > \sum_{j=1}^{p} A_j\alpha_j$, then a sufficient condition for the function $y\{{}_p\bar{\phi}_q(z)\} = y\int_0^z {}_p\bar{\psi}_q(x)dx$ to be in the class UCV, is that

$$ {}_p\bar{\psi}_q\left[\begin{matrix}(a_j+\alpha_j,\alpha_j;A_j)_{1,p};\\(b_j+\beta_j,\beta_j;B_j)_{1,q};\end{matrix}1\right] + {}_p\bar{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}1\right] \le y^{-1}(1+N). \qquad (10)$$

Proof. we have,

$$\sum_{n=2}^{\infty} n^2 \frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j} y}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j(n-1))\}^{B_j} n!}$$

$$= y\sum_{n=2}^{\infty} \frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j(n-1))\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j(n-1))\}^{B_j}(n-2)!} + y\sum_{n=1}^{\infty} \frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j n)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j n)\}^{B_j} n!}$$

$$= y\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j(n+1))\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j(n+1))\}^{B_j} n!} + y\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j n)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j n)\}^{B_j} n!} - 1$$

$$= y\,{}_p\bar{\psi}_q\left[\begin{matrix}(a_j+\alpha_j,\alpha_j;A_j)_{1,p};\\(b_j+\beta_j,\beta_j;B_j)_{1,q};\end{matrix}1\right] + y\,{}_p\bar{\psi}_q\left[\begin{matrix}(a_j,\alpha_j;A_j)_{1,p};\\(b_j,\beta_j;B_j)_{1,q};\end{matrix}1\right] - 1, \qquad (11)$$

Theorem 3.2 follows from (10), (11) and Lemma 2.

IV. Special Cases

If we take $\alpha_j = \beta_k = 1\,(j=1,...,p;\ k=1,...,q)$ in Theorems (2.1), (2.2), (2.3), (3.1) and (3.2) respectively, the ${}_p\bar{\psi}_q$ function will reduce to the ${}_p\bar{F}_q$ function; we get the following Theorems:

Theorem 4.1 If $a_i > 0(i=1,...,p)$, $b_j > 0(j=1,...,q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j + 1$, then a sufficient condition for the function $yz\{{}_p\bar{F}_q(z)\}$ to be in the class $UST(\alpha)$, $0 \le \alpha < 1$, is that

$$\left(\frac{3-\alpha}{1-\alpha}\right){}_p\bar{F}_q\left[\begin{matrix}(a_j+1,1;A_j)_{1,p};\\(b_j+1,1;B_j)_{1,q};\end{matrix}1\right] + {}_p\bar{F}_q\left[\begin{matrix}(a_j,1;A_j)_{1,p};\\(b_j,1;B_j)_{1,q};\end{matrix}1\right] \le y^{-1}(1+N). \qquad (12)$$

Theorem 4.2 If $a_i > 0(i=1,...,p)$, $b_j > 0(j=1,...,q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j + 1$, then a sufficient condition for the function $yz\{{}_p\bar{F}_q(z)\}$ to be in the class $USTN(\alpha)$, $0 \le \alpha < 1$, is that

$$\left(\frac{3-\alpha}{1-\alpha}\right){}_p\bar{F}_q\left[\begin{matrix}(a_j+1,1;A_j)_{1,p};\\(b_j+1,1;B_j)_{1,q};\end{matrix}1\right] + {}_p\bar{F}_q\left[\begin{matrix}(a_j,1;A_j)_{1,p};\\(b_j,1;B_j)_{1,q};\end{matrix}1\right] \le y^{-1}(1+N). \qquad (13)$$

Theorem 4.3 If $a_i > 0(i = 1,..., p)$, $b_j > 0(j = 1,..., q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j + 2$, then a sufficient condition for the function $yz\{{}_p\overline{F}_q(z)\}$ to be in the class $UCV(\alpha)$, $0 \le \alpha < 1$, is that

$$(1+\alpha)_p\overline{F}_q\left[\begin{matrix}(a_j+2,1;A_j)_{1,p};\\(b_j+2,1;B_j)_{1,q};\end{matrix}1\right]+(3+2\alpha)_p\overline{F}_q\left[\begin{matrix}(a_j+1,1;A_j)_{1,p};\\(b_j+1,1;B_j)_{1,q};\end{matrix}1\right]$$

$$+{}_p\overline{F}_q\left[\begin{matrix}(a_j,1;A_j)_{1,p};\\(b_j,1;B_j)_{1,q};\end{matrix}1\right]\le y^{-1}(1+N). \qquad (14)$$

Theorem 4.4 If $a_i > 0(i = 1,..., p)$, $b_j > 0(j = 1,..., q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j$, then a sufficient condition for the function $y\{{}_p\overline{G}_q(z)\} = y\int_0^z {}_p\overline{F}_q(x)dx$ to be in the class UST, is that

$$3_p\overline{F}_q\left[\begin{matrix}(a_j,1;A_j)_{1,p};\\(b_j,1;B_j)_{1,q};\end{matrix}1\right]-2_p\overline{F}_q\left[\begin{matrix}(a_j-1,1;A_j)_{1,p};\\(b_j-1,1;B_j)_{1,q};\end{matrix}1\right]$$

$$+2\frac{\prod_{j=1}^{p}\{\Gamma(a_j-1)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j-1)\}^{B_j}}\le y^{-1}(1+N). \qquad (15)$$

Theorem 4.4 If $a_i > 0(i = 1,..., p)$, $b_j > 0(j = 1,..., q)$, $\sum_{j=1}^{q} b_j > \sum_{j=1}^{p} a_j$, then a sufficient condition for the function $y\{{}_p\overline{G}_q(z)\} = y\int_0^z {}_p\overline{F}_q(x)dx$ to be in the class UCV, is that

$${}_p\overline{F}_q\left[\begin{matrix}(a_j+1,1;A_j)_{1,p};\\(b_j+1,1;B_j)_{1,q};\end{matrix}1\right]+{}_p\overline{F}_q\left[\begin{matrix}(a_j,1;A_j)_{1,p};\\(b_j,1;B_j)_{1,q};\end{matrix}1\right]\le y^{-1}(1+N).$$

If we set $A_i = B_j = 1(i = 1,..., p;\ j = 1,..., q)$, $N \to My$, Theorems (2.1), (2.2), (2.3), (3.1) and (3.2) reduce to the results recently obtained by Chaurasia and Srivastava ([16]).

Further on taking $\alpha_k = \beta_l = 1(k = 1,..., p;\ l = 1,..., q)$ and $N = \frac{\prod_{j=1}^{p}\Gamma(a_j)}{\prod_{j=1}^{q}\Gamma(b_j)}$, we arrive at the results of Shanmugam, Ramachandran, Sivasubramanian and Gangadharan ([12]).

By specifying the parameters suitably, the results of this paper readily yield the results due to Dixit and Verma ([1]).

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