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On Hypergeometric Series Identities

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ON HYPERGEOMETRIC SERIES IDENTITIES

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On Hypergeometric Series Identities

M.P. Chaudhary[α], Salahuddin[Ω], Vinesh Kumar[β]

Abstract - H.Exton [J.Comput.Appl.Math.88(1997)269-274] obtained a quite general transformation involving hypergeometric functions by elementary manipulation of series, some of these results are erroneous. Four erroneous results have been corrected by Medhat A. Rakha et al, and made a remark on other three results in [4]. Here, we respond the remark and confirm that other three results are also erroneous.

Keywords : Gauss's second theorem, Vandermonde's theorem, Dixon's theorem.

I. Introduction

Exton in [3], discovered a number of hypergeometric identities, which were previously not recorded in the literature. He established them by applications of Gauss's second summation theorem and other known hypergeometric theorems. Medhat A. Rakha et al, observed that there are errors in four results of Exton's [3; p.272 {(2.5) and (3.1)}, and p.273 {(3.2) and(3.3)}], and they obtained correct forms for the same. They further observed that the result present by equation (3.7) is not new as the right hand sides of the results presented by equations (3.6) and (3.7) are same and yields a simple identity between two 2F1(12) functions. They also observed that Exton's results given in [3; p.272(2.9), p.273 {(3.4) and (3.6)}] are correct. They cannot derive the Exton's results given in [3; p.273(3.5), p.274 {(3.8) and (3.9)}] nor could verify them numerically, but remarked that these three results therefore should be taken as incorrect [4]. The purpose of this note is, as- (i). To present four erroneous results [3; p.272 {(2.5) and (3.1)}, and p.273 {(3.2) and(3.3)}], along with their correct form given in [4]. (ii). To present two correct results [p.273 {(3.6) and (3.7)}] in a single equation. (iii). To examine the three erroneous results [3; p.273(3.5), p.274 {(3.8) and (3.9)}], and confirm the same.

II. Preliminaries

The generalized hypergeometric function is defined in [1, p.41], as

$$ {}_pF_q\left[\begin{array}{ll} a_1, a_2, \cdots, a_p\ ; & \\ & x \\ b_1, b_2, \cdots, b_q\ ; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!} \tag{1}$$

where the Pochhammer symbol is defined as $(a)_n = (a, n) = \frac{\Gamma(a+n)}{\Gamma(a)}$. If $q = p,$ the series given by equation (1) is converges for $|x| < \infty$, but when $q = p - 1$, then the series is convergence for $|x| < 1$. But, when only one of the parameters a_j is a negative integer or zero, then the series given by equation (1.1) terminates and always converges since it becomes a polynomial in x of degree $-a_j$. Exton's investigation is based on following general transformation, which he obtained by techniques of elementary manipulation of of series [3, p. 270 (1.8)].

$$ \sum_{n=0}^{\infty} \frac{(c_1)_n (c_2)_n \cdots (c_p)_n}{(d_1)_n (d_2)_n \cdots (d_q)_n} \frac{(\frac{1}{2}a)_n (-2x)^n}{n!} {}_{p+1}F_q\left[\begin{array}{ll} a+2n, c_1+n, \cdots, c_p+n\ ; & \\ & x \\ d_1+n, \cdots, c_q+n\ ; & \end{array}\right]$$

$$ = {}_{2p+1}F_{2q}\left[\begin{array}{ll} \frac{1}{2}a, \frac{1}{2}c_1, \frac{1}{2}c_2, \cdots, \frac{1}{2}c_p, \frac{1}{2}(1+c_1), \frac{1}{2}(1+c_2), \cdots, \frac{1}{2}(1+c_p)\ ; & \\ & -4^{p-q}x^2 \\ \frac{1}{2}d_1, \frac{1}{2}d_2, \cdots, \frac{1}{2}d_q, \frac{1}{2}(1+d_1), \frac{1}{2}(1+d_2), \cdots, \frac{1}{2}(1+d_q)\ ; & \end{array}\right] \tag{2}$$

If one of the numerator parameters cj equals a negative integer, the resulting equation (2) involves finite sums and convergence at $x = \pm 1$ is assured.

Gauss's second summation theorem:[3, p.270(1.6)]

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$$ {}_2F_1\left[\begin{matrix} a, \quad b & ; \\ \frac{1}{2}(a+b+1) & ; \end{matrix} \quad \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})} \tag{3} $$

Vandermonde's theorem:[3, p.270(1.9)]

$$ {}_2F_1\left[\begin{matrix} a, -n & ; \\ c & ; \end{matrix} \quad 1\right] = \frac{(c-a,n)}{(c,n)} = \frac{\Gamma(c)\Gamma(c-a+n)}{\Gamma(c+n)\Gamma(c-a)} \tag{4} $$

Kummer's theorem:[3, p.271(1.10)]

$$ {}_2F_1\left[\begin{matrix} a, \ b & ; \\ 1+a-b & ; \end{matrix} \quad -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+a)\Gamma(1-b+\frac{a}{2})} \tag{5} $$

Dixon's theorem:[3, p.271(1.11)]

$$ {}_3F_2\left[\begin{matrix} a, \quad b, \quad -n & ; \\ 1+a-b, 1+a+n & ; \end{matrix} \quad 1\right] = \frac{[(1+a,n)(1+\frac{a}{2}-b,n)]}{[(1+\frac{a}{2},n)(1+a-b,n)]} $$

$$ = \frac{\Gamma(1+a-b)\Gamma(1+a-n)\Gamma(\frac{a}{2}+1)\Gamma(\frac{a}{2}+n-b+1)}{\Gamma(1+a)\Gamma(1+a-b+n)\Gamma(\frac{a}{2}+1-b)\Gamma(\frac{a}{2}+1+n)} \tag{6} $$

Erroneous results in [3] and corresponding corrected results in [4]

In [3, p.272 (2.5)], a result is recorded, as

$$ \frac{(d-a,N)}{(d,N)} \, {}_3F_2\left[\begin{matrix} -N \quad , \quad \frac{a}{2} \quad , \quad 1+a-d & ; \\ \frac{1}{2}+\frac{a}{2}-\frac{d}{2}-\frac{N}{2}, 1+\frac{a}{2}-\frac{d}{2}-\frac{N}{2} & ; \end{matrix} \quad \frac{1}{2}\right] $$

$$ = {}_3F_2\left[\begin{matrix} -\frac{N}{2}, \frac{1}{2}-\frac{N}{2} & ; \\ \frac{1}{2}+\frac{a}{2}\frac{d}{2}, \frac{1}{2}+\frac{d}{2} & ; \end{matrix} \quad -1\right] \tag{7} $$

this result has been corrected and recorded in [4], as

$$ \frac{(b-a)_N}{(b)_N} \, {}_3F_2\left[\begin{matrix} -N \quad , \quad \frac{1}{2}a \quad , \quad 1+a-b & ; \\ \frac{1}{2}(1+a-b-N), \frac{1}{2}(2+a-b-N) & ; \end{matrix} \quad \frac{1}{2}\right] $$

$$ = {}_3F_2\left[\begin{matrix} \frac{1}{2}N, \frac{1}{2}-\frac{1}{2}N, \frac{1}{2}a & ; \\ \frac{1}{2}b, \frac{1}{2}+\frac{1}{2}b & ; \end{matrix} \quad -1\right] \tag{8} $$

In [3, p.272 (3.1)], a result is recorded, as

$$ \frac{(1+a,N)}{(1+\frac{a}{2})} \, {}_2F_1\left[\begin{matrix} \frac{a}{2}, -N & ; \\ \frac{1}{2}+\frac{a}{2} & ; \end{matrix} \quad -\frac{1}{2}\right] = {}_3F_2\left[\begin{matrix} \frac{a}{2}, -\frac{N}{2}, \frac{1}{2}-\frac{N}{2} & ; \\ \frac{1}{2}+\frac{a}{2}+\frac{N}{2}, 1+\frac{a}{2}+\frac{N}{2} & ; \end{matrix} \quad -1\right] \tag{9} $$

this result has been corrected and recorded in [4], as

$$\frac{(1+a)_N}{(1+\frac{1}{2}a)_N}\ {}_2F_1\left[\begin{matrix} -N, \frac{1}{2}a; \\ \frac{1}{2}+\frac{1}{2}a; \end{matrix} \ \frac{1}{2}\right] = {}_3F_2\left[\begin{matrix} \frac{1}{2}N, \frac{1}{2}-\frac{1}{2}N, \frac{1}{2}a \qquad ; \\ \frac{1}{2}(1+a+N), \frac{1}{2}(2+a+N); \end{matrix} \ -1\right] \tag{10}$$

In [3, p.273 (3.2)], a result is recorded, as

$$\frac{(1+a,N)(1+\frac{a}{2}-b,N)}{(1+\frac{a}{2},N)(1+a-b,N)}\ {}_3F_2\left[\begin{matrix} b\ , \frac{a}{2}, -N \qquad ; \\ \frac{1}{2}+\frac{a}{2}, b-\frac{a}{2}-N; \end{matrix} \ \frac{1}{2}\right]$$

$$= {}_5F_4\left[\begin{matrix} \frac{b}{2}, \frac{b}{2}+\frac{1}{2}, \frac{a}{2}, -\frac{N}{2}, \frac{1}{2}-\frac{N}{2} \qquad ; \\ \frac{1}{2}+\frac{a}{2}-\frac{b}{2}, \frac{1}{2}+\frac{a}{2}-\frac{b}{2}, \frac{1}{2}+\frac{a}{2}+\frac{N}{2}, 1+\frac{a}{2}+\frac{N}{2}; \end{matrix} \ -1\right] \tag{11}$$

this result has been corrected and recorded in [4], as

$$\frac{(1+a)_N(1+\frac{1}{2}a-b)_N}{(1+\frac{1}{2}a)_N(1+a-b)_N}\ {}_3F_2\left[\begin{matrix} -N\ , \frac{1}{2}a, b \qquad ; \\ \frac{1}{2}+\frac{1}{2}a, b-\frac{1}{2}a-N; \end{matrix} \ \frac{1}{2}\right]$$

$$= {}_5F_4\left[\begin{matrix} -\frac{1}{2}N,\ \frac{1}{2}-\frac{1}{2}N,\ \frac{1}{2}a,\ \frac{1}{2}b,\ \frac{1}{2}+\frac{1}{2}b \qquad ; \\ \frac{1}{2}(1+a-b), \frac{1}{2}(2+a-b), \frac{1}{2}(1+a+N), \frac{1}{2}(2+a+N); \end{matrix} \ -1\right] \tag{12}$$

In [3, p.273 (3.3)], a result is recorded, as

$$(1+a,N)/\left(\frac{1}{2}+\frac{a}{2},N\right) \times {}_1F_0\left[-N;-;\left(\frac{1}{2}\right)^N\right] = (1+a,N)/\left[2^N\left(\frac{1}{2}+\frac{a}{2},N\right)\right]$$

$$= {}_4F_3\left[\begin{matrix} 1+\frac{a}{4}, \frac{a}{2}, -\frac{N}{2}, \frac{1}{2}-\frac{N}{2} \qquad ; \\ \frac{a}{4}, \frac{1}{2}+\frac{a}{2}+\frac{N}{2}, 1+\frac{a}{2}+\frac{N}{2}; \end{matrix} \ -1\right] \tag{13}$$

this result has been corrected and recorded in [4], as

$$\frac{(1+a)_N}{(\frac{1}{2}+\frac{1}{2}a)_N}\ {}_1F_0\left[\begin{matrix} -N\ ; \\ -\quad ; \end{matrix} \ \frac{1}{2}\right] = \frac{2^{-N}(1+a)_N}{(\frac{1}{2}+\frac{1}{2}a)_N}$$

$$= {}_4F_3\left[\begin{matrix} -\frac{1}{2}N,\ \frac{1}{2}-\frac{1}{2}N,\ \frac{1}{2}a, 1+\frac{1}{4}a \qquad ; \\ \frac{1}{4}a, \frac{1}{2}(1+a+N), \frac{1}{2}(2+a+N); \end{matrix} \ -1\right] \tag{14}$$

In [4], it is also observed that right hand sides of both equations in [3, p.273 {(3.6) and (3.7)}] are same, here we are writing both equations in joint form, as

$$\frac{(1+a,N)}{(1+a-b,N)} \times {}_2F_1\left[\begin{matrix} b, -N; \\ \frac{1}{2}+\frac{a}{2}; \end{matrix} \ \frac{1}{2}\right]$$

$$= \frac{[(\frac{1}{2}+\frac{a}{2}-b,N)(1+a,N)]}{[(\frac{a}{2}+\frac{1}{2},N)(1+a-b,N)]}{}_2F_1\left[\begin{matrix} b, -N \qquad ; \\ \frac{1}{2}+b-\frac{a}{2}-N; \end{matrix} \ \frac{1}{2}\right]$$

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$$= {}_6F_5\left[\begin{matrix} 1+\frac{a}{4}, & \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}, & \frac{a}{2}, & -\frac{N}{2}, & \frac{1}{2}-\frac{N}{2} & ; \\ \frac{a}{4}, \frac{1}{2}+\frac{a}{2}-\frac{b}{2}, 1+\frac{a}{2}-\frac{b}{2}, \frac{1}{2}+\frac{a}{2}+\frac{N}{2}, 1+\frac{a}{2}+\frac{N}{2}; \end{matrix} -1\right] \tag{15}$$

Examination of three erroneous results in [3]

In this section, we response the remark of Medhat A. Rakha et al on Exton's three results given in [3; p.273(3.5), p.274 {(3.8) and (3.9)}], and confirmed the same.

In [3, p.273 (3.5)], a result is recorded, as

$$\frac{[(a-2b,N)(1+\frac{a}{2}-b,N)(-b,N)]}{[(\frac{a}{2}-b,N)(-2b,N)(1+a-b,N)]}\times$$

$$\times {}_5F_4\left[\begin{matrix} \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}, & \frac{a}{2}, & -\frac{N}{2}, & \frac{1}{2}-\frac{N}{2} & ; \\ \frac{1}{2}+\frac{a}{2}-\frac{b}{2}, 1+\frac{a}{2}-\frac{b}{2}, \frac{1}{2}+\frac{a}{2}+\frac{N}{2}, 1+\frac{a}{2}+\frac{N}{2}; \end{matrix} -1\right]$$

$$= {}_5F_4\left[\begin{matrix} \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}, & \frac{a}{2}, & -\frac{N}{2}, & \frac{1}{2}-\frac{N}{2} & ; \\ \frac{1}{2}+\frac{a}{2}-\frac{b}{2}, 1+\frac{a}{2}-\frac{b}{2}, \frac{1}{2}+\frac{a}{2}+\frac{N}{2}, 1+\frac{a}{2}+\frac{N}{2}; \end{matrix} -1\right] \tag{16}$$

We verified this result using computer programming languages Octave, Matlab and Mathematica, and confirmed that it is a erroneous result.

In [3, p.274 (3.8)], a result is recorded, as

$$\frac{[(a-2b,N)(-b,N)]}{[(1+a-b,N)(-2b,N)]}\times$$

$$\times {}_6F_5\left[\begin{matrix} 1+\frac{a}{2}, & b, & \frac{1}{3}+\frac{2b}{3}-\frac{N}{3}, & \frac{2}{3}+\frac{2b}{3}-\frac{N}{3}, & 1+\frac{2b}{3}-\frac{N}{3}, & -N; \\ 1+2b+N, 1-a+2b-N, \frac{1}{2}+2b-\frac{N}{2}, 1+\frac{b}{2}-\frac{N}{2}, \frac{1}{2}+b; \end{matrix} \frac{27}{8}\right]$$

$$= {}_6F_5\left[\begin{matrix} 1+\frac{a}{4}, & \frac{b}{2}, & \frac{1}{2}+\frac{b}{2}, & \frac{a}{2}, & -\frac{N}{2}, & \frac{1}{2}-\frac{N}{2}; \\ \frac{a}{4}, \frac{1}{2}+\frac{a}{2}-\frac{b}{2}, 1+\frac{a}{2}-\frac{b}{2}, \frac{1}{2}+b+\frac{N}{2}, 1+b+\frac{N}{2}; \end{matrix} -1\right] \tag{17}$$

We verified this result using computer programming languages Octave, Matlab and Mathematica, and confirmed that it is a erroneous result.

In [3, p.274 (3.9)], a result is recorded, as

$$\frac{[(a-2b,N)(\frac{1}{2}+\frac{a}{2}-b,N)(-b-1,N)]}{[(1+a-b,N)(\frac{a}{2}-\frac{1}{2}-b,N)(-2b-1,N)]}\times$$

$$\times {}_8F_7\left[\begin{matrix} 1+\frac{a}{2}, & b, & 2+b, & \frac{3}{2}-\frac{a}{2}+b-N, & \frac{2}{3}+\frac{2b}{3}-\frac{N}{3}, & 1+\frac{2b}{3}-\frac{N}{3}, & \frac{4}{3}+\frac{2b}{3}-\frac{N}{3}, -N; \\ 3+2b, & 2+2b-a-N, \frac{1}{2}+b-\frac{a}{2}-N, 1+\frac{b}{2}-\frac{N}{2}, & \frac{3}{2}+\frac{b}{2}-\frac{N}{2}, & 1+b, \frac{3}{2}+b; \end{matrix} \frac{27}{8}\right]$$

$$= {}_6F_5\left[\begin{matrix} 1+\frac{a}{4}, & \frac{b}{2}, & \frac{b}{2}+\frac{1}{2}, & \frac{a}{2}, & -\frac{N}{2}, & \frac{1}{2}-\frac{N}{2}; \\ \frac{a}{4}, \frac{1}{2}+\frac{a}{2}-\frac{b}{2}, 1+\frac{a}{2}-\frac{b}{2}, \frac{1}{2}+\frac{a}{2}+\frac{N}{2}, 1+\frac{a}{2}+\frac{N}{2}; \end{matrix} -1\right] \tag{18}$$

We verified this result using computer programming languages Octave, Matlab and Mathematica, and confirmed that it is a erroneous result.

III. Acknowledgement

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