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Boundary-fixed Homeomorphisms of 2-Manifolds with Boundary

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Abstract - Let X be a closed, orientable 2-manifold and let X_n denote the bounded manifold obtained by removing the interiors of n disjoint closed disks from X. Let $H(X_n)$ denote the group of isotopy classes (rel boundary of X_n) of homeomorphisms of X_n which are the identity on the boundary of X_n. $H(X_n)$ has been determined for all n when X is the 2-sphere (see [8] and [10]). This paper investigates the structure of $H(X_n)$ for X not equal to the 2-sphere. In particular, a relationship between $H(X_n)$ and the homeotopy group (mapping class group) of X (see [4],[5] and [11]) is developed.

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I. Introduction.

Let X be a closed, orientable 2-manifold and let $D_1,\dots,D_n$ be disjoint closed disks in X with p_k a point in the interior of D_k for $1 \le k \le n$. Let $X_n = X - \bigcup_{k=1}^{n} Int(D_k)$ and $F_n = \{p_1,\dots,p_n\}$. This paper is concerned with the group $H(X_n)$ consisting of all isotopy classes (rel ∂X_n) of homeomorphisms of X_n which are the identity on the boundary of X_n. Note that in order for two boundary-fixed homeomorphisms of X_n to represent the same element in $H(X_n)$ not only must these homeomorphisms be isotopic, but also the isotopy between them must be the identity on the boundary of X_n for all values t, $0 \le t \le 1$. Presentations of $H(X_n)$ for all n in the case that X is the 2-sphere are given in [8]. In this paper it will be shown that if X is not the 2-sphere , then $H(X_n)$ can be obtained as part of a short exact sequence involving the free abelian group on n generators, denoted Z^n, and the subhomeotopy group of X consisting of all isotopy classes (rel F_n) of orientation preserving homeomorphisms of X which are the identity on F_n, denoted $H(X,F_n)$. This short exact sequence will then be used to relate $H(x_n)$ to the homeotopy group of X.

II. Boundry-fixed Homeomorphisms

let $f: H(X_n) \to H(X,F_n)$ be the function which sends the isotopy class (rel ∂X_n) of a boundary-fixed homeomorphism h of X_n onto the isotopy class (rel F_n) of the homeomorphism of X which is obtained by extending h by the identity over each disk D_k, $1 \le k \le n$. The function f is clearly well–defined since any isotopy (rel ∂X_n) of X_n can be extended by the identity on $\bigcup_{k=1}^{n} D_k$ to an isotopy (rel F_n) of X. In fact, f is an epimorphism since any orientation preserving homeomorphism of X which is the identity on F_n can be isotope (rel F_n) to a homeomorphism which is the identity on $\bigcup_{k=1}^{n} D_k$ (for details see Part3c of Lemma 3 of [7]). Moreover using Part 4 of Lemma 3 of [ ]we have that a homeomorphism of X_n is in the kernel of f is if and only if it is isotopic to the identity. That is, the nontrivial elements of the kernel of f are represented by boundary-fixed homeomorphisms that are isotopic to the identity, but not by an isotopy that keeps the boundary of X_n fixed. The next two lemmas are concerned with finding representatives of the isotopy classes (rel ∂X_n) of such homeomorphisms.

Let $K(X_n)$ denote the kernel of f and let A_k be a collar neighborhood of ∂D_k for $1 \le k \le n$, with $A_i \cap A_j = \emptyset$ if $i \ne j$.

Lemma 1: Every element of $K(X_n)$ can be represented by a homeomorphism that is the identity on

$$X - \bigcup_{k=1}^{n} Int(A_k)$$

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Proof : Let h be a homeomorphism that represents an element of $K(X_n)$ and let h_t be an isotopy that takes h to the identity. Using the "unwinding " technique of Proposition 3.22 of [6] it is possible to extend h_t^{-1} / X_n to an isotopy g_t of X_n that takes the identity to a homeomorphism that is the identity on $X - \bigcup_{k=1}^{n} Int(A_k)$. The isotopy $g_t h_t$ is then an isotopy (rel ∂X_n) that takes h to a homeomorphism of the type given in the statement of the lemma.

For each *k*, $1 \le k \le n$, let $s_k^r : X_n \to X_n$ be defined by letting s_k^r restricted to the annulus A_k be the homeomorphism given in which spins one component of ∂A_k r-times while holding the other boundary component fixed and by letting s_k^r restricted to $X_n - A_k$ be the identity. s_k^r will be referred to as a"spin homeomorphism " of X_n.

Lemma 2 : If X is not the 2-sphere , then every element of $K(X_n)$ has a unique representation as a product of spin homeomorphisms.

Proof : A consequence of Theorem 7.2 of [3] is that every homeomorphism of A_k which is the identity on ∂A_k is isotopic (rel ∂A_k) to s_k^r / A_k for some *r* . Since A_i is disjoint from A_j for $i \ne j$, this means that any homeomorphism of X_n which is the identity on $X - \bigcup_{k=1}^{n} Int(A_k)$ is isotopic (rel ∂X_n) to a product of homeomorphisms of the form $s_1^{r_1} \cdots s_n^{r_n}$. Thus by Lemma 1, every element of $K(X_n)$ can be represented by a product of spin homeomorphisms.

To show that the representation if unique it suffices to show that if $s_1^{r_1} \cdots s_n^{r_n}$ is isotopic (rel ∂X_n) to the identity, then $r_i = 0$ for $1 \le i \le n$ .On the contrary, assume this product is isotopic (rel ∂X_n) to the identity, but $r_k \ne 0$ for some *k* . Let α be a curve which represents a generator of $\pi_1(X, q)$ where q is in ∂D_k and α is chosen so that $\alpha \cap A_j = \varnothing$ for $i \ne j$ and $\alpha \cap D_k = \{q\}$. Let β_k be a curve based at *q* which wraps once around ∂D_k in the direction of the spin corresponding to $s_k^{r_k}$. In the free group $\pi_1(X_n, q)$, the spin homeomorphism $s_k^{r_k}$ represents the same element as $\beta_k^{-r_k} \alpha \beta_k^{r_k}$. However, since $s_1^{r_1} \cdots s_n^{r_n}$ is isotopic (rel ∂X_n) to the identity and α is outside the support of $s_i^{r_i}$ for $i \ne k$, we have that $s_k^{r_k}(\alpha)$ must also represent the same element as α in the free group $\pi_1(X_n, q)$. This contradiction establishes the lemma.

It should be noted that Lemma 2 is false in the case that *X* is the 2-sphere and $n < 3$. For example, when *X* is the 2-sphere and $n = 1$, then every spin homeomorphism of X_1 is isotopic (rel ∂X_1) to the identity (see [8]).

The nest theorem is an immediate consequence of the fact that since the elements of $K(X_n)$ can be represented uniquely as products of spin homeomorphisms and these spin homeomorphisms all commute, the kernel of the epimorphism *f* is the free group on n generators.

Theorem : If *X* is not the 2-sphere, then the following sequence is exact.

$$1 \to Z^n \to H(X_n) \to H(X, F_n) \to 1$$ where the function from $H(X_n)$ to $H(X, F_n)$ is given by *f* as defined at the beginning of this section.

The above theorem shows that $H(X_n)$ can be obtained as an extension of Z^n by $H(X, F_n)$. In turn $H(X, F_n)$ is part of the short exact sequence

$$1 \to \pi_1(X - F_{n-1}, p_n) \to H(X, F_n) \to H(X, F_{n-1}) \to 1$$

where the function from $H(X, F_n)$ to $H(X, F_{n-1})$ sends the isotopy class (rel F_n) of a homeomorphism of X to the isotopy class (rel F_{n-1}) of this homeomorphism. If we denote this function *d* , then the representation of an element in the kernel of *d* as an element in $\pi_1(X - F_{n-1}, p_n)$ is obtained by taking the curve formed by tracing the path of p_n during the isotopy (rel F_{n-1}) which takes a representative homeomorphism of an element in the kernel of *d* to the identity.

Thus, we can build up $H(X, F_n)$ from the homeotopy group of *X* , $H(X)$, by repeatedly extending $\pi_1(X - F_k)$ by $H(X, F_k)$ for $k = 1, \cdots, n-1$ (see [1] and [2]).

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