Certain Field of Fractions

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Certain Field of Fractions

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Abstract - An integral domain, \( D^* = (R - ZD), +, \circ \) was defined in [4] as a subset of the set of all real rhotrices \( R \), defined in [1] which is a commutative ring. The definition of a quotient rhotrix \( \frac{R}{S} \), was given in [5], provided \( h(S) \neq 0 \). On the basis of these two definitions, a generalized method for the construction of field of fractions through the use of rhotrices was proposed here.

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1. INTRODUCTION

The concept of a new mathematical array of real numbers in rhomboid form called rhotrix was introduced in [1] as an extension of ideas on matrix-tertions and matrix-noitrets suggested in [2]. In [3] the set of all real rhotrices was considered in the classification of Abstract Group like structures and Boolean algebra as an enrichment exercise through extension to rhotrices. This work was extended in [4] to the theory of rings and fields for further classifications of rhotrices as structures.

Addition and scalar multiplication in rhotrices are like in the case of matrices. But there are two multiplication methods of rhotrices in the literature: the heart based multiplication method as defined in [1] and the row-column multiplication method proposed in [6]. In this note we shall adopt the heart-based multiplication of rhotrices.

Recall that an integral domain was defined in [4] as \( D^* = (R - ZD), +, \circ \) where

\[
R = \left\{ \begin{bmatrix} a & b & c & d \\ e \end{bmatrix} : a, b, c, d, e \in \mathcal{R} \right\}
\]

and

\[
ZD = \left\{ \begin{bmatrix} a & b & 0 & d \\ e \end{bmatrix} : a, b, c, d, 0 \in \mathcal{R} \text{ at least one of } a, b, c, d \neq 0 \right\}.
\]

(1.1)

And quotient rhotrix \( \frac{R}{S} \) was also defined in [5] for any two rhotrices \( R, S \) as follows:

\[
\frac{R}{S} = R \circ S^{-1}, \text{ provided } h(S) \neq 0.
\]

(1.2)

The aim of this paper is to construct the rhotrix field of fraction of integral domain, based on definitions (1.1) and (1.2).

II. FIELD OF FRACTION OF INTEGRAL DOMAIN

Definition 2.1: Let \( D^* \) be as defined above and \( S^* = D^* \setminus \{0\} \), where \( 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). Then a relation \( \sim \) on \( D^* \times S^* \) is defined by cross multiplication as \( (D_1, S_1) \sim (D_2, S_2) \) if \( D_1 \circ S_2 = D_2 \circ S_1, \forall D_1, D_2 \in D^*, S_1, S_2 \in S^* \).

Proposition 2.1: The relation \( \sim \) as defined in Definition (2.1) is an equivalence relation.

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Proof:
Reflexivity and Symmetry of the relation are obvious.

For transitivity, let \( D_1 \circ S_2 = D_3 \circ S_1 \) and \( D_2 \circ S_3 = D_3 \circ S_2 \). To show that \( D_1 \circ S_2 = D_3 \circ S_1 \) we have

\[
(D_1 \circ S_3) \circ S_2 = (D_1 \circ S_2) \circ S_1
= (D_2 \circ S_1) \circ S_3
= (D_2 \circ S_2) \circ S_1
= (D_3 \circ S_2) \circ S_1
= (D_3 \circ S_1) \circ S_2
\]

\[
\therefore (D_1 \circ S_2) = (D_3 \circ S_1) \text{ by cancellation}.
\]

We denote by \( \frac{D}{S} \) the equivalence class of \( (D, S) \) in \( D^* \times S^* \) and define \( D^*_s \) to be the set of all the equivalence classes \( \frac{D}{S} \), where \( D \in D^* \) and \( S \in S^* \).

For all \( \frac{D_1}{S_1}, \frac{D_2}{S_2} \in D^*_s \) we define addition and multiplication on \( D^*_s \) as follows:

\[
\frac{D_1}{S_1} + \frac{D_2}{S_2} = \frac{D_1 \circ S_2 + D_2 \circ S_1}{S_1 \circ S_2} \quad \text{and} \quad \frac{D_1}{S_1} \cdot \frac{D_2}{S_2} = \frac{D_1 \circ D_2}{S_1 \circ S_2}.
\]

**Proposition 2.2:** The operations \((+'), (\circ')\) as defined above are well-defined.

**Proof**
Suppose \( \frac{D_1'}{S_1'} = \frac{D_1}{S_1} \) and \( \frac{D_2'}{S_2'} = \frac{D_2}{S_2} \); then \( D_1' \circ S_1 = D_1 \circ S_1' \) and \( D_2' \circ S_2 = D_2 \circ S_2' \),

So that \( (D_1' \circ S_2' + D_2' \circ S_1')S_1S_2 = D_1' \circ S_1 \circ S_2' + D_2' \circ S_2 \circ S_1' \)

\[
= D_1 \circ S_1' \circ S_1 \circ S_2' + D_2 \circ S_2' \circ S_1 \circ S_1'
= (D_1 \circ S_1 + D_2 \circ S_1)S_1' \circ S_2
\]

implying that \( \frac{D_1'}{S_1'} + \frac{D_2'}{S_2'} = \frac{D_1}{S_1} + \frac{D_2}{S_2} \)

Similarly \( (D_1' \circ D_2')S_1 \circ S_2 = (D_1 \circ D_2)S_1' \circ S_2' \) implies that \( \frac{D_1'}{S_1'} \circ \frac{D_2'}{S_2'} = \frac{D_1 \circ D_2}{S_1 \circ S_2} \).

By definition (1.2) the equivalence class \( \frac{D}{S} = D \circ S^{-1} \) since \( S \neq 0 \in S^* \). Therefore for all \( S \in S^* \), \( 0 \in D^* \),

\[
\frac{0}{S} = 0 \circ S^{-1} = 0 = 0 \circ I = \frac{0}{I}.
\]

Thus \( \frac{0}{I} = \frac{0}{S} \) is the additive identity and \( \frac{-D}{S} = \frac{D}{S} \) is the additive inverse.

Similarly, \( \frac{I}{S} = \frac{S}{I} \) is the multiplicative identity.

**Proposition 2.3:** With the above definitions and the definitions of the operations \((+')\) and \((\circ')\), the set of the equivalence classes \( D^*_s \) is a commutative ring.

**Proof**
One should check that the properties of a ring are fulfilled. But the proof follows from the fact that addition and multiplication are carried the usual way (that of fractions).

**Proposition 2.4:** The function \( \psi : D^* \rightarrow D^*_s \) defined by \( \psi(D) = \frac{D}{I} \) is a ring homomorphism whose kernel is...
The proof follows from propositions 2.3 and 2.4.

Before we state the next proposition it is easy for the reader to verify that the set $\mathbb{R}$ of all real rhotrices is a commutative ring and the set of all diagonal real rhotrices defined in [3], with additive and multiplicative identities is a group and is normal in $\mathbb{R}$ under multiplication.

Proposition 2.6: Let $\mathbb{R}$ be a commutative ring of rhotrices, and let $S^*$ be a submonoid such that $Q \circ S \neq 0$ for every $Q \neq 0 \in \mathbb{R}$ and $S \in S^*$, $Q \circ S \neq 0$ can serve in the above construction for the generalization of proposition 2.3 as stated in the following proposition.

Proof: The proof follows from propositions 2.3 and 2.4.

Recall that from [3] Theorem 3, the set $M = \{n : n \in \mathbb{Z}\}$ where $I$ is the unity element of the commutative ring of rhotrices $\mathbb{R}$ is a subring and submonoid of $\mathbb{R}$ under multiplication ($\circ$). Thus any submonoid, $S^*$ of $\mathbb{R}$ with property that for all $Q \neq 0 \in \mathbb{R}$ and $S \in S^*$, $Q \circ S \neq 0$ can serve in the above construction for the generalization of proposition 2.3 as stated in the following proposition.

Proposition 2.5: $R_{S^*}$ as constructed above is a ring, and there is a homomorphism $\psi: R \rightarrow R_{S^*}$ given by $\psi(Q) = \frac{Q}{I}$.

Proof: The proof follows from propositions 2.3 and 2.4.

Since all the elements of $Q^*_{S^*}$ are also elements of $R_{S^*}$ and $I \in S^*$ then obviously $Q^*_{S^*}$ is an additive subgroup of $R_{S^*}$.

For all $\frac{D}{S} \in R_{S^*}$, $\frac{Q}{S} \in Q^*_{S^*}$, $\frac{D}{S} \circ \frac{Q}{S} = \frac{D \circ Q}{S \circ S} \in Q^*_{S^*}$ since $D \circ Q \in Q^*$, $S \circ S^* \in S^*$.

$Q^*$ is normal in $\mathbb{R}$ implies that

Proposition 2.7: $D^*_{S^*}$ is an integral domain.

Proof: Suppose $\frac{D_1}{S_1} \circ \frac{D_2}{S_2} = 0 \in D^*_{S^*}$, that is $\frac{D_1 \circ D_2}{S_1 \circ S_2} = 0 \in I$.

$\Rightarrow (D_1 \circ D_2, S_1 \circ S_2) \sim (0, I)$ and $D_1 \circ D_2 \circ S = 0$ for some $S \in S^*$.

$D_1 \circ D_2 \circ S = 0 \in D^*$, which is an domain, and $S \neq 0$, thus $D_1 \circ D_2 = 0$.

So either $D_1$ or $D_2$ is 0 and consequently either $\frac{D_1}{S_1}$ or $\frac{D_2}{S_2}$ is 0.
In this piece of note, a somewhat clear but interesting generalization of a method of field of fractions construction using rhotrices was provided. This construction was done step by step, where at each step a particular algebraic property was shown. This work may yield a good understanding of the field of fractions.

**III. CONCLUSION**

In this piece of note, a somewhat clear but interesting generalization of a method of field of fractions construction using rhotrices was provided. This construction was done step by step, where at each step a particular algebraic property was shown. This work may yield a good understanding of the field of fractions.

**REFERENCES Références Referencias**