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Mathematical Morphology and Fractal Geometry

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MATHEMATICAL MORPHOLOGY AND FRACTAL GEOMETRY

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Mathematical Morphology and Fractal Geometry

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Abstract - Mathematical morphology examines the geometrical structure of an image by probing it with small patterns, called 'structuring elements', of varying size and shape. This procedure results in nonlinear image operators which are suitable for exploring geometrical and topological structures. A series of such operators is applied to an image in order to make certain features more clear. Scale-space is an accepted and often used formalism in image processing and computer vision. Today, this formalism is so important because it makes the choice at what scale visual observations are to be made explicit. Fractal Geometry is a very new branch in Mathematics. An attempt to link Morphological operators and Fractals is made in this paper.
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I. Introduction

Mathematical Morphology is a tool for extracting image components that are useful for representation and description. It provides a quantitative description of geometrical structures. Morphology is useful to provide boundaries of objects, their skeletons, and their convex hulls. It is also useful for many pre- and post-processing techniques, especially in edge thinning and pruning.

Most morphological operations are based on simple expanding and shrinking operations. Morphological operations preserve the main geometric structures of the object. Only features `smaller than' the structuring element are affected by transformations. All other features at `larger scales' are not degraded. (This is not the case with linear transformations, such as convolution).

The primary application of morphology occurs in binary images, though it is also used on grey level images. It can also be useful on range images. (A range image is one where grey levels represent the distance from the sensor to the objects in the scene rather than the intensity of light reflected from them).

a) *Preliminaries*

i. *Notation and Image Definitions*

Types of Images

An image is a mapping denoted as I, from a set, N_P, of pixel coordinates to a set, M, of values such that for every coordinate vector, $p = (p_1, p_2)$ in N_P, there is a value I(p) drawn from M. N_P is also called the image plane.[1]

Under the above defined mapping a real image maps an n-dimensional Euclidean vector space into the real numbers. Pixel coordinates and pixel values are real.

A discrete image maps an n-dimensional grid of points into the set of real numbers. Coordinates are n-tuples of integers, pixel values are real. A digital image maps an n-dimensional grid into a finite set of integers. Pixel coordinates and pixel values are integers. A binary image has only 2 values. That is, M= $\{m_{fg}, m_{bg}\}$, where m_{fg}, is called the foreground value and m_{bg} is called the background value.

The foreground value is $m_{fg} = 0$, and the background is $m_{bg} = -\infty$. Other possibilities are $\{m_{fg}, m_{bg}\}$ = $\{0,\infty\}$, $\{0,1\}$, $\{1,0\}$, $\{0,255\}$, and $\{255,0\}$.

b) *Dilation and Erosion*

Morphology uses 'Set Theory' as the foundation for many functions [1]. The simplest functions to implement are 'Dilation' and 'Erosion'

i. *Definition :* Dilation of the object *A* by the structuring element *B* is given by

$$A \oplus B = \{x : \hat{B}_x \cap A \neq \emptyset\}.$$

Usually A will be the signal or image being operated on A and B will be the Structuring Element'

ii. *Definition Erosion*

The opposite of dilation is known as erosion. Erosion of the object *A* by a structuring element *B* is given by

$$A \ominus B = \{x : B_x \subseteq A\}.$$

Erosion of A by B is the set of points x such that B translated by x is contained in A.

iii. *Definition Opening*

The opening of *A* by *B*, denoted by $A \circ B$, is given by the erosion by *B*, followed by the dilation by *B*, that is

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iv. Closing

The opposite of opening is 'Closing' defined by $A \bullet B = (A \oplus B) \ominus B$.

Closing is the dual operation of opening and is denoted by $A \bullet B$. It is produced by the dilation of A by B, followed by the erosion by B :

c) Morphological Operators defined on a Lattice

i. *Definition Dilation*

Let $(L, \leq)$ be a complete lattice, with infimum and minimum symbolized by $\wedge$ and $\vee$, respectively. [1],[2].[11]

A dilation is any operator $\delta : L \rightarrow L$ that distributes over the supremum and preserves the least element. $\bigvee_i \delta(X_i) = \delta\left(\bigvee_i X_i\right), \delta(\emptyset) = \emptyset$.

ii. *Definition Erosion*

An erosion is any operator $\varepsilon : L \rightarrow L$ that distributes over the infimum $\bigwedge_i \varepsilon(X_i) = \varepsilon\left(\bigwedge_i X_i\right), \varepsilon(U) = U$.

iii. *Galois connections*

Dilations and erosions form Galois connections. That is, for all dilation δ there is one and only one erosion ε that satisfies $X \leq \varepsilon(Y) \Leftrightarrow \delta(X) \leq Y$ for all $X, Y \in L$.

Similarly, for all erosion there is one and only one dilation satisfying the above connection.
Furthermore, if two operators satisfy the connection, then δ must be a dilation, and ε an erosion.

iv. *Definition Adjunctions*

Pairs of erosions and dilations satisfying the above connection are called "adjunctions", and the erosion is said to be the adjoint erosion of the dilation, and vice-versa.

v. *Opening and Closing*

For all adjunction (ε, δ), the morphological opening $\gamma : L \rightarrow L$ and morphological closing $\phi : L \rightarrow L$ are defined as follows:[2]

$\gamma = \delta\varepsilon$, and $\phi = \varepsilon\delta$.

The morphological opening and closing are particular cases of algebraic opening (or simply opening) and algebraic closing (or simply closing). Algebraic openings are operators in *L* that are idempotent, increasing, and anti-extensive. Algebraic closings are operators in *L* that are idempotent, increasing, and extensive.

II. MORPHOLOGICAL OPERATORS DEFINED AS AN ALGEBRAIC STRUCTURE

a) Definition : Morphogenetic field

Let X $\neq \phi$ and W $\subseteq P(X)$ such that i) ϕ, X $\in$ W , ii) If B $\in$ W then its complement $\overline{B} \in$ W , iii) If $_i$B $\in$ W is a sequence of signals defined in X, then $\bigcup_{n\,1}^{\infty} Bi \in$ W.

Let A= $\{\phi : W \rightarrow U / \phi(\cup A_i) = \vee\phi(A_i) \,\&\, \phi(\wedge A_i) = \wedge\phi(A_i)\}$. Then W_U is called Morphogenetic field [16] where the family W_u is the set of all image signals defined on the continuous or discrete image Plane[11],[12] X and taking values in a set U .The pair (W_u, A) is called an operator space where A is the collection of operators defined on X.

i. *Definition : Morphological space*

The triplet (X, W_u, A) consisting of a set X, a morphogenetic field W_u and an operator A(or collection of operators) defined on X is called a Morphological space [16].

Note : If X = Z^2 then it is called Discrete Morphological space

ii. *Definition : Concave morphological space*

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A).

If X is a class of concave functions [14],[15] then (X, W_u, A) is called concave morphological space. If X is a class of convex functions then (X, W_u, A) is called convex morphological space.[16].

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iii. *Definition : Sensitive operator*

Let (X, W_u, A) be a Morphological space[16]. Let B_1 be the neighbourhood of x∈ X i.e., N(x)=B_1⊆ X.Then ∀$x\epsilon X$, x∈B_1, y∈B_1, ∃ B_2 such that B_1⊆ B_2⊆X and α^n(x) ∈B2 andα^n(y) ∉ $\overline{B2}$, nϵZ^+. Then $\alpha \in A$ is called a sensitive operator and the operator space [16] (W_u, A) is called a sensitive space.
Example : Dilation is sensitive. Constant signals f(x)= c are not sensitive.

iv. *Proposition*

Let N: X→P(X) be defined such that $N(x)=\{y\in X / x\rho y\}$ where ρ is the relation, dilation defined between x and y $\forall x, y\in X$. i.e., $x\rho y \Rightarrow y=\delta(x)$ where δ is the dilation [8],[9],[10],[11] and for $\alpha\in A$, α $=\delta$ $\Rightarrow$ δ^n(x) ∈ B_2 =N(x), $\delta^n(y)\notin \overline{B}_2$ $n\in Z^+$, $x, y\in B_1$ 1. Thus δ is sensitive.

v. *Definition : Perfect Set*

Let F$\subseteq X$. Define $S(F)=\{\alpha / \alpha\in (W_u, A)$ *besensitive* $[6] by \alpha\in F\}$. If S(F) ≠ ϕ and (X, W_u, A) is a convex morphological space [16] then F is called Perfect.

vi. *Definition : Stirring Operator*

(Let (X, W_u, A) be a Morphological space and let U, V$\subseteq X$ be two sets. Let α ∈A. Then α is called stirring [6] if given any neighbourhoods N_1 and N_2,of U and V, $\forall x\in U$, y∈V in X, $\exists k\in Z^+$ such that

$$\alpha^k(N_1)\cap\alpha^k(N_2) \neq \phi .$$

α is strongly stirring if $\exists k\in Z^+$ and a set G in X such that $G\subseteq \alpha^k(N_1)\cap\alpha^k(N_2)$.

vii. *Definition : Partial Similarity*

Let (X, Wu, A) be a Morphological space.
Let K ⊆ X. K is called Partial self similar or α *similar* if $\exists K_1, K_2, ...K_t$ such that $K=\bigcup_{i=1}^{t} K_i$ and for each K_i,∃ contraction maps $\varphi(i,j,k)$,for i=1….. t,r=1 t ,j=1,….t and k=1. $w(i,j)$ with $w(i,j)$>0 such that K_i .

viii. *Definition : Scale space*

Let S a scaling on an image space L. The family {T(t)},t> 0 of operators on L is called an (S,+) scale – space [2],[5] if T(t).T(s)=T(t+ s) ,s ,t > 0 and T(t).S(t)=S(t).T(1) ,t > 0

ix. *Proposition*

The erosion $\varepsilon(f) = f \ominus b$ with a convex structuring element b induces an ($S^{1/2}$, + ½) scale space and f is ½ similar.

x. *Definition : Anamorphic Scaling*

A family $S=\{S(t)/t>0\}$ of operators on L is called a scaling if S(1) = identity element.

S(t)S(s)=S(ts) for s,t > 0.Two scalings S and $\acute{S}$ are said to be anamorphic [2].[5]if ∃ an increasing bijection γ on T such that $S(\gamma(t))=S'(t)$ $\forall t\in T$ Also .

xi. *Proposition*

Anamorphic scaling are α – similar

xii. *Proposition*

The erosion $\varepsilon(f) = f \ominus b$ with b∈SP(k) for K>1 induces a $(S^\alpha ;+ v)$scale space if $v=1-\alpha+K^*(2\alpha-1)$ which implies that f is α - similar , b is called the structuring function.

xiii. *Proposition*

Let (X, Wu, A) be a Morphological space. Let f be α similar. Then $\exists\psi\in A$ such that $\psi^\alpha(f)=\alpha\psi(f)$.

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xiv. *Definition*

The cross – section $X_t(f)$ [1],[2],[5] of f at level t is the set obtained by thresholding f at level t. $X_t(f)=\{x/ f(x)\geq t\}$, where $-\infty<t<\infty$

xv. *Proposition*

If f is a fractal then $\exists i\in I$ such that $\forall$ i, $X_{t_i}(f)$ are self similar and $X=\bigcup_{\forall i} X_t(f)$.

III. MORPHOLOGICAL FRACTALS

a) *Surface area of a compact set*

Morphological operators extracts the impact of a particular shape on images [13] using structuring elements. It encodes the primitive shape information. The transformed image is obtained by using a structuring element .Therefore it can be treated as a function of the structuring element.

Dilation of a set X [5],[17] with a structuring element Y is given by the expression $X\oplus Y=\{x/ Y^x\cup X\neq\phi\}$, Y^x denotes the translation of a set Y with x.

Dilation operation can be used to define the surface area of a compact set.

Surface area [19]of a compact set X with respect to a compact convex structuring element Y which is symmetrical with respect to the origin is given by

$$S(X,Y)\quad \lim_{\rho\to 0}\frac{V(\partial X\oplus\rho Y)}{2\rho}$$

Where ∂X is the boundary of set X and $\oplus$ denotes the dilation of the boundary of X by the structuring element Y and ρ is a scaling factor. Volume of a set X is denoted by $V(X)$.

b) *Particular Case – Fractals*

If the object is regular, the surface area will not change with ρ_i .For a fractal object, S is increases exponentially with decreasing ρ.

c) *Definition : Fractal Identification*

An image is segmented into the regions $R_1,R_2,........R_n$ if $\exists$ a relation ρ on Regions such that $R_i\rho R_j$ if $R_i\cap R_j=\phi$ and $\bigcup_i R_i=X$.

Also Image Property of $R_i\cap R_j=\phi$,if $i\neq j$ .If *Image Property of* R_i = *Property of* R_j then each R_i is a fractal.

Note : Converse is not always true. For every Fractal, it is not necessary that Image *Property of* R_i = *Property of* R_j

d) *Definition : Class of Fractal Regions- X(k, t)*

Let $F(p)=\prod_{i\;1}^{m} f_i(p_i)$ ....(1)where ($p=(p_1,p_2,...p_m)$ and f_i, $i=1,2,....m$ is a set of completely defined functions and F is uniquely defined on R.

Define G(F) as $p\in G(F)$ iff F(p)=1.i.e F is a characteristic function of $G(F)$ .The set of graphs which can be generated from (1) by allowing each f_i to vary over all possible logic functions is defined as Class of Fractal Graphs, [18] denoted by $G(k,t)$ where the vectors ($k=(k_1,k_2,...k_m)$) and t= $(t_1,t_2,...t_m)$).

e) *Definition : Compression*

Let (X, Wu, A) be a Morphological space. Let R =X be a rectangular plane and is divided into $2^{n_1}\times 2^{n_2}$ grids represented by R($2^{n_1}\times 2^{n_2}$).X1 is a region on R and χ:R→{0,1} is its characteristic function.

Given two integers r1 and r2,0< r1< n1 ,0< r2<n2,construct a rectangular plane R' regarding its left lower corner as an origin. A function χ' :R' →{0,1} is defined as follows.

$\forall$p'=(x',y')ε R' ,if $\exists$ integers α_1 & α_2 where0<α1< 2^{r1}, 0<α2< 2^{r2} such that χ (p)=1where p=(x,y) εR and then χ'(p')=1,otherwise χ'(p')=0.

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Region X' with χ' as its characteristic function is called a compressed region [18]of X based on (r1,r2) and a compressed region of X is denoted by X' = Comp(r1,r2)(X) and χ'(p') ∀p', p'εR' is given below. χ'(p')=1 ,if there exist 'a' in R such that χ (p)=1, χ'(p') =0 otherwise.

f) Definition : Similarity

Let (X, Wu, A) be a Morphological space. Assume that X1 and X2are two regions on a plane. If ∃ a compression transformation in A, Comp(r1,r2) such that X1is compressed into X3=Comp(r1,r2) (X1) and X2 can coincide with X3 through translating X2,then X1 and X2 are similar, denoted by X1≅X2.

Note: Compression is nonreversible. Therefore Similarity is an asymmetric relation.

g) Definition : Self Similarity

Let (X, Wu, A) be a Morphological space. If ∃ a partition X1,X2,…..X r of X and Xi is a proper sub region of X such that X and each non empty sub region Xi of X are similar.

For any two non empty sub regions Xi and X j ,X j and Xi are similar or Xi and X j are similar,[1],[4]

Then X is said to be a self similar region.

h) The Order of Self Similarity

Let (X, Wu, A) be a Morphological space. X is a self similar region, if any proper sub region among all partitions of X which satisfy the definition of self similarity [18]is not a self similar region, then the order of similarity of X is 1.

If the maximal order of sub regions among all partitions of X which satisfy the definition of self similarity is m, then the order of self –similarity of X is m+1.

i) Definition: Mutual Similarity

Let (X, Wu, A) be a Morphological space. Let X be partitioned into X1,X2,…. of X and Xi is a proper sub region of X such that for any two non empty sub regions Xi and X j,Xi and Xj are similar or Xj and Xi are similar, then X is a mutually similar region[18].

j) Definition: Fractal Regions X(k,t)

Let (X, Wu, A) be a Morphological space. If set X can be partitioned into several sub regions and the sub regions are mutually similar and each sub region can further be partitioned into mutually similar sub regions etc, then G is said to be a mutual- similar region.

Note: If X can be identified as a representation in terms of graphs then X(k, t) is a mutual similar graph.

IV. CONCLUSION

Morphological operators[3],[4],[9] are very useful for gathering informations from images. Most of the operators can also be applied in Medical Imaging [7]. Some results are given in generalized structure [16].The regions can also be taken as graph points. So we can also apply the results from the already developed Graph Theory. We can also reconstruct a fractal image using Dilation and a fractal structuring element. Morphological fractals are useful in Medical imaging and other areas. It is possible to construct soft wares for this particular job.

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