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On Ramanujan's Generating Relation For Tau Function

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On Ramanujan's Generating Relation For Tau Function

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Abstracts - **Present paper concerns mainly with verification and extension of the table for $\tau(1)$, $\tau(2)$, $\tau(3)$, ..., $\tau(30)$ of Ramanujan. Our extended table for $\tau(31)$, $\tau(32)$, $\tau(33)$, ..., $\tau(211)$ is obtained without using certain arithmetical functions defined by Ramanujan and also the theory of elliptic functions.**

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I. Introduction

In this paper, we obtain the values of $\tau(1)$, $\tau(2)$, ..., $\tau(211)$, where $\tau(n)$ is Tau function of Ramanujan, defined as follows:

$$\sum_{n=1}^{\infty} \tau(n)\, x^n = x\left\{\prod_{n=1}^{\infty}(1-x^n)\right\}^{24} \qquad (1.1)$$

The right hand side of (1.1) is called generating function for $\tau(n)$. Ramanujan[3,p.196, Table(V); see also 1;2] calculated the values of $\tau(1)$, $\tau(2)$, ..., $\tau(30)$, by means of the theory of elliptic functions and certain arithmetical functions such as $F_{r,s}(x)$, $\Phi_{r,s}(x)$, $E_{r,s}(n)$, $\sigma_s(n)$, Riemann's Zeta function $\zeta(n)$, greatest integer function $[x]$, theory of symbols o, O, continued fraction, asymptotic expansion, some trigonometrical identities, inequalities, Gamma function, theory of order of error terms, number theory, convergence and divergence of infinite series.

II. Verification and Extension

Consider the expanded form of (1.1), we have

$$\sum_{n=1}^{\infty} \tau(n)\; x^n = \; x\{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots(1-x^{210})\cdots\}^{24} \qquad (2.1)$$

$$= x\{(1-x)^3(1-x^2)^3(1-x^3)^3(1-x^4)^3\cdots(1-x^{210})^3\cdots\cdots\}^8 = x\mathrm{T}^8 = x\{(\mathrm{T}^2)^2\}^2 \qquad (2.2)$$

where

$$\mathrm{T} = (1-x)^3(1-x^2)^3(1-x^3)^3(1-x^4)^3\cdots(1-x^{210})^3\cdots\cdots \qquad (2.3)$$

Now consider the product of first two hundred ten polynomials in (2.3) and collecting the terms upto x^{210}, we get

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$$\mathrm{T} = +1 - 3x + 5x^3 - 7x^6 + 9x^{10} - 11x^{15} + 13x^{21} - 15x^{28} + 17x^{36} - 19x^{45} + 21x^{55} -$$

$$-23x^{66} + 25x^{78} - 27x^{91} + 29x^{105} - 31x^{120} + 33x^{136} - 35x^{153} + 37x^{171} - 39x^{190} + 41x^{210} + \cdots \quad (2.4)$$

It is to be noted that the coefficients in (2.4) are alternatively positive and negative such that the sequence $1, 3, 5, 7, 9, \ldots$ form arithmetic progression. Suppose the powers of x (*i.e.* the sequence $0, 1, 3, 6, 10, 15, \ldots$) are generated by the function $F(k)$, therefore

$$\mathrm{T} = \sum_{k=1}^{\infty} (-1)^{k-1}(2k-1)x^{F(k)} \quad (2.5)$$

Now we shall find the function $F(k)$ using the following ordinary finite difference table:

k	$F(k)$	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	$\cdots$
1	0															
		1														
2	1		1													
		2		0												
3	3		1		0											
		3		0		0										
4	6		1		0		0									
		4		0		0		0								
5	10		1		0		0		0							
		5		0		0		0		0						
6	15		1		0		0		0		0					
		6		0		0		0		0		0				
7	21		1		0		0		0		0		0			
		7		0		0		0		0		0		0		
8	28		1		0		0		0		0		0		0	
		8		0		0		0		0		0		0		
9	36		1		0		0		0		0		0			
		9		0		0		0		0		0				
10	45		1		0		0		0		0					
		10		0		0		0		0						
11	55		1		0		0		0							
		11		0		0		0								
12	66		1		0		0									
		12		0		0										
13	78		1		0											
		13		0												
14	91		1													
		14														
15	105															
$\vdots$																

ORDINARY FINITE DIFFERENCE TABLE

Since second order ordinary differences are equal, therefore third and higher order differences will be zero and so $F(k)$ will be a polynomial of second degree (by means of fundamental theorem of finite difference calculus). Thus:

$$F(k) \;=\; A + Bk + Ck^2 \tag{2.6}$$

where the unknowns A, B and C are to be calculated.

Now selecting any three values of k and also their corresponding values of $F(k)$ from above table and putting them in (2.6), we get a system of three linear equations which on simplification gives $A = 0$, $B = -\frac{1}{2}$ and $C = \frac{1}{2}$.

Therefore suitable $F(k)$ is given by

$$F(k) = \; -\frac{1}{2}k + \frac{1}{2}k^2 = \frac{k(k-1)}{2}$$

Consequently (2.5) reduces to:

$$\mathrm{T} = \sum_{k=1}^{\infty} (-1)^{k-1}(2k-1)x^{\frac{k(k-1)}{2}} \tag{2.7}$$

Now squaring the expansion in (2.4) and collecting the terms upto x^{210}, we have

$$\begin{aligned}\mathrm{T}^2 = {}&+1-6x+9x^2+10x^3-30x^4+11x^6+42x^7-70x^9+18x^{10}-54x^{11}+49x^{12}+90x^{13}-22x^{15}-\\ &-60x^{16}-110x^{18}+81x^{20}+180x^{21}-78x^{22}+130x^{24}-198x^{25}-182x^{27}-30x^{28}+90x^{29}+121x^{30}+\\ &+84x^{31}+210x^{34}-252x^{36}-102x^{37}-270x^{38}+170x^{39}-69x^{42}+330x^{43}-38x^{45}+420x^{46}-190x^{48}-\\ &-390x^{49}-108x^{51}-300x^{55}+99x^{56}+442x^{57}+210x^{58}+418x^{60}-294x^{61}-510x^{64}+378x^{65}-\\ &-540x^{66}+138x^{67}-230x^{69}-462x^{70}+611x^{72}+570x^{73}+132x^{76}+50x^{78}-150x^{79}+110x^{81}-\\ &-630x^{83}-350x^{84}-598x^{87}+450x^{88}+361x^{90}+660x^{91}+162x^{92}-550x^{93}+420x^{94}+378x^{97}+\\ &+650x^{99}-798x^{100}-486x^{101}-782x^{102}+58x^{105}-330x^{106}+290x^{108}+441x^{110}+468x^{111}-702x^{112}+\\ &+850x^{114}+522x^{115}+810x^{119}-700x^{120}-780x^{121}-1260x^{123}+1188x^{126}-918x^{127}-558x^{130}+\\ &+529x^{132}+180x^{133}+682x^{135}+1092x^{136}-198x^{137}+330x^{139}+180x^{141}-462x^{142}-1150x^{144}-\\ &-540x^{146}+930x^{148}-1102x^{150}-726x^{151}-70x^{153}+210x^{154}-779x^{156}+2100x^{157}+490x^{159}+\\ &+1218x^{160}-630x^{163}-990x^{164}+1178x^{165}+770x^{168}-1350x^{169}-1260x^{171}+900x^{172}-540x^{174}-\\ &-1302x^{175}-518x^{177}+462x^{181}+729x^{182}+1450x^{183}+612x^{186}-1190x^{189}-78x^{190}+1620x^{191}+\\ &+962x^{192}-390x^{193}-1020x^{196}-220x^{198}-1110x^{199}-702x^{200}-1518x^{202}+858x^{205}+1258x^{207}-\\ &-1470x^{208}+923x^{210}+\cdots\end{aligned} \tag{2.8}$$

Further repeating the same process for $(\mathrm{T}^2)^2$, we get

$$\mathrm{T}^4 = +1-12x+54x^2-88x^3-99x^4+540x^5-418x^6-648x^7+594x^8+836x^9+1056x^{10}-4104x^{11}-$$

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$$
\begin{aligned}
&-209x^{12}+4104x^{13}-594x^{14}+4256x^{15}-6480x^{16}-4752x^{17}-298x^{18}+5016x^{19}+17226x^{20}-\\
&-12100x^{21}-5346x^{22}-1296x^{23}-9063x^{24}-7128x^{25}+19494x^{26}+29160x^{27}-10032x^{28}-\\
&-7668x^{29}-34738x^{30}+8712x^{31}-22572x^{32}+21812x^{33}+49248x^{34}-46872x^{35}+67562x^{36}+2508x^{37}-\\
&-47520x^{38}-76912x^{39}-25191x^{40}+67716x^{41}+32076x^{42}+7128x^{43}+29754x^{44}+36784x^{45}-\\
&-51072x^{46}+45144x^{47}-122398x^{48}-53460x^{49}+11286x^{50}-27256x^{51}+57024x^{52}+122364x^{53}+\\
&+99902x^{54}+3576x^{55}-29646x^{56}-221616x^{57}+41382x^{58}-52272x^{59}+130549x^{60}-206712x^{61}-\\
&-180036x^{62}+336512x^{63}+145200x^{64}+100980x^{65}-73568x^{66}+221616x^{67}-317142x^{68}-148324x^{69}+\\
&+15552x^{70}-225720x^{71}-32076x^{72}+108756x^{73}+196614x^{74}+74360x^{75}-58806x^{76}+229824x^{77}+\\
&+120878x^{78}-233928x^{79}+361152x^{80}-111340x^{81}-349920x^{82}-491832x^{83}-196569x^{84}-\\
&-82764x^{85}+707454x^{86}+18392x^{87}+92016x^{88}+493668x^{89}-559450x^{90}+416856x^{91}-16092x^{92}+\\
&+320760x^{93}-361152x^{94}-724032x^{95}+7106x^{96}+270864x^{97}-530442x^{98}+56168x^{99}-261744x^{100}+\\
&+52272x^{101}+930204x^{102}+406296x^{103}+451440x^{104}-339196x^{105}+562464x^{106}-653400x^{107}-\\
&-374528x^{108}-810744x^{109}-248292x^{110}+779360x^{111}+20691x^{112}-744876x^{113}-272746x^{114}+\\
&+570240x^{115}-153846x^{116}-69984x^{117}+922944x^{118}+1154736x^{119}+657074x^{120}-694980x^{121}-\\
&-489402x^{122}-349448x^{123}-812592x^{124}+1341900x^{125}-2216160x^{126}-384912x^{127}+132354x^{128}+\\
&+26224x^{129}+58806x^{130}+943272x^{131}+1052676x^{132}-357048x^{133}+967518x^{134}-518320x^{135}-\\
&-441408x^{136}-112860x^{137}+2222726x^{138}-421344x^{139}-196614x^{140}-1552276x^{141}-541728x^{142}-\\
&-1515888x^{143}-1067021x^{144}+1468776x^{145}-1072170x^{146}-414072x^{147}+2216160x^{148}+\\
&+1715472x^{149}+1064800x^{150}-135432x^{151}-1875852x^{152}+1585892x^{153}+327072x^{154}-730728x^{155}+\\
&+584858x^{156}+470448x^{157}-2482866x^{158}-320760x^{159}-1468368x^{160}+496584x^{161}+87362x^{162}-\\
&-1198824x^{163}+114048x^{164}+377948x^{165}+29502x^{166}+1177848x^{167}+639122x^{168}+355752x^{169}+\\
&+2298240x^{170}+2276560x^{171}+2659392x^{172}-2904660x^{173}-3991570x^{174}-1715472x^{175}+\\
&+1429218x^{176}-2531088x^{177}+627264x^{178}+1161864x^{179}-1777203x^{180}-1566588x^{181}+\\
&+3648348x^{182}-1089232x^{183}-1705374x^{184}-1715472x^{185}+3505766x^{186}+2160432x^{187}+\\
&+248292x^{188}+4043852x^{189}-4038144x^{190}+5187456x^{191}-2566080x^{192}+1197900x^{193}-\\
&-950346x^{194}-2437776x^{195}-1211760x^{196}-4153248x^{197}-520738x^{198}+882816x^{199}+764370x^{200}-\\
&-1779008x^{201}-1360314x^{202}-160920x^{203}+2640506x^{204}+3805704x^{205}+674784x^{206}+\\
&+3656664x^{207}+1779888x^{208}-4980204x^{209}-237994x^{210}+\cdots \qquad (2.9)
\end{aligned}
$$

Finally adopting the same procedure for $(\mathrm{T}^4)^2$ and multiplying $(\mathrm{T}^4)^2$ by x and comparing the coefficients of $x,\ x^2,\ x^3,\ x^4,\cdots,x^{210},\ x^{211}$ with the coefficients of left hand side of (2.2), we get the values of $\tau(1),\tau(2),\tau(3),\tau(4),\cdots,\tau(210),\tau(211)$ and are given in tabular form as follows:

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III. EXTENDED TABLE FOR $\tau(n)$; $n \in \{1,2,3,4,5,\ldots,211\}$

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	+1	37	−182213314	73	+1463791322
2	−24	38	−255874080	74	+4373119536
3	+252	39	−145589976	75	−6425804700
4	−1472	40	+408038400	76	−15693610240
5	+4830	41	+308120442	77	−8951543328
6	−6048	42	+101267712	78	+3494159424
7	−16744	43	−17125708	79	+38116845680
8	+84480	44	−786948864	80	+4767866880
9	−113643	45	−548895690	81	+1665188361
10	−115920	46	−447438528	82	−7394890608
11	+534612	47	+2687348496	83	−29335099668
12	−370944	48	+248758272	84	+6211086336
13	−577738	49	−1696965207	85	−33355661220
14	+401856	50	+611981400	86	+411016992
15	+1217160	51	−1740295368	87	+32358470760
16	+987136	52	+850430336	88	+45164021760
17	−6905934	53	−1596055698	89	−24992917110
18	+2727432	54	+1758697920	90	+13173496560
19	+10661420	55	+2582175960	91	+9673645072
20	−7109760	56	−1414533120	92	−27442896384
21	−4219488	57	+2686677840	93	−13316478336
22	−12830688	58	−3081759120	94	−64496363904
23	+18643272	59	−5189203740	95	+51494658600
24	+21288960	60	−1791659520	96	−49569988608
25	−25499225	61	+6956478662	97	+75013568546
26	+13865712	62	+1268236032	98	+40727164968
27	−73279080	63	+1902838392	99	−60754911516
28	+24647168	64	+2699296768	100	+37534859200
29	+128406630	65	−2790474540	101	+81742959102
30	−29211840	66	−3233333376	102	+41767088832
31	−52843168	67	−15481826884	103	−225755128648
32	−196706304	68	+10165534848	104	−48807306240
33	+134722224	69	+4698104544	105	−20380127040
34	+165742416	70	+1940964480	106	+38305336752
35	−80873520	71	+9791485272	107	+90241258356
36	+167282496	72	−9600560640	108	+107866805760

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n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
109	+73482676310	144	−112181096448	178	+599830010640
110	−61972223040	145	+620204022900	179	+1681384224780
111	−45917755128	146	−35130991728	180	+807974455680
112	−16528605184	147	−427635232164	181	−996774496018
113	−85146862638	148	+268217998208	182	−232167481728
114	−64480268160	149	−1115433620850	183	+1753032622824
115	+90047003760	150	+154219312800	184	+1574983618560
116	−189014559360	151	−824447297848	185	−880090306620
117	+65655879534	152	+900676761600	186	+319595480064
118	+124540889760	153	+784811057562	187	−3691995187608
119	+115632958896	154	+214837039872	188	−3955776986112
120	+102825676800	155	−255232501440	189	+1226984915520
121	+498319933	156	+214308444672	190	−1235871806400
122	−166955487888	157	+1315116754406	191	+2762403350592
123	+77646351384	158	−914804296320	192	+680222785536
124	+77785143296	159	−402206035896	193	+5442387685442
125	−359001100500	160	−950091448320	194	−1800325645104
126	−45668121408	161	−312162946368	195	−703199584080
127	−262717201024	162	−39964520664	196	+2497932784704
128	+338071388160	163	−357832759588	197	−2876091504354
129	−4315678416	164	−453553290624	198	+1458117876384
130	+66971388960	165	+650708341920	199	+728391402200
131	+631528759932	166	+704042392032	200	−2154174528000
132	−198311113728	167	+2754833892216	201	−3901420374768
133	−178514816480	168	−356462346240	202	−1961831018448
134	+371563845216	169	−1458379197393	203	−2150040612720
135	−353937956400	170	+800535869280	204	+2561714781696
136	−583413304320	171	−1211595753060	205	+1488221734860
137	−297198746214	172	+25209042176	206	+5418123087552
138	−112754509056	173	−950387449578	207	−2118677359896
139	+596793577940	174	−776603298240	208	−570305978368
140	+119045821440	175	+426959023400	209	+5699723069040
141	+677211820992	176	+527734751232	210	+489123048960
142	−234995646528	177	−1307679342480	211	−6793168439188
143	−308865667656				

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