Fuzzy ideals in $\Gamma$–semiring

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Keywords: $\Gamma$–semiring, left(right) operator semiring, fuzzy left(right) ideal, fuzzy ideal, fuzzy $k$–ideal, fuzzy $h$–ideal.

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Fuzzy ideals in $\Gamma-$semiring

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Abstract: In this paper we have studied the relation between the fuzzy left (respectively right) ideals of $\Gamma-$semiring and that of operator semiring. Thereafter, we have established that the Lattices of all fuzzy left (respectively right) ideal of $\Gamma-$semiring is equivalent to that of Left operator semiring. Also we have established few properties relating the $k-$ideals and $h-$ideals of $\Gamma-$semiring with that of operator semiring.

Keywords: $\Gamma-$semiring, left(right) operator semiring,fuzzy left(right) ideal,fuzzy ideal,fuzzy $k-$ideal,fuzzy $h-$ideal.

I. INTRODUCTION

The notion of $\Gamma-$ring in algebra was first introduced by N. Naobuswa$^5$ in 1964 and also he defined the $\Gamma-$ring. In 1969, J. Luh$^9$ introduced the concept of left operator ring and right operator ring of $\Gamma-$ring. In 1995, M. M. K. Rao$^4$ introduced the concept of $\Gamma-$semiring as a generalization of semiring and $\Gamma-$ring. Thereafter S. K. Sardar and T. K. Dutta$^6$ modified the definition of $\Gamma-$semiring of Rao$^4$ and then they redefined the left operator semiring and right operator semiring of a $\Gamma-$semiring and obtained a few interesting properties. Later on, much has been developed on this concepts by different researchers.

Fuzzy set theory was first introduced by Zadeh$^2$ in 1965 and thereafter several authors contributed different articles on this concept and applied it on different branches of pure and applied mathematics. In 1971, Rosenfeld$^3$ defined fuzzy subgroups, fuzzy ideals and studied its important properties. Thereafter in 1982, Liu$^{10}$ introduced and developed basic results of fuzzy subrings and fuzzy ideals of a ring. In 1992, Jun and Lee$^7$ introduced the notion of fuzzy ideal in $\Gamma-$ring and studied a few properties. In 2005, Dutta and Chanda$^8$ studied the structures of fuzzy ideals of $\Gamma-$ring via operator rings of $\Gamma-$ring.

In this paper, we have established a few results in respect of fuzzy left (respectively right) ideals, fuzzy ideals of a $\Gamma-$semiring and its operator semirings. If $\sigma$ is a fuzzy ideal of a $\Gamma-$semiring then we have proved $\sigma^+$ is a fuzzy ideal of the corresponding left operator semiring. Also, if $\mu$ is a fuzzy ideal of left operator semiring then $\mu^-$ is a fuzzy ideal of the corresponding $\Gamma-$semiring. Then it is shown that there exist an inclusion preserving bijection $\sigma \rightarrow \sigma^+$ between the Lattices of all fuzzy right ideals (respectively fuzzy ideals) of a $\Gamma-$semiring and the Lattices of all fuzzy right ideals (respectively fuzzy ideals) of the corresponding left operator semiring. Similarly the above results hold for right operator semiring of a $\Gamma-$semiring. Also we have studied similar results relative to fuzzy k–ideals, fuzzy $h-$ideals of a $\Gamma-$semiring and its operator semirings.

II. PRELIMINARIES

This section contain some basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1$^6$ Let $S$ and $\Gamma$ be two additive commutative semigroups. Then $S$ is called a $\Gamma-$semiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ (image to be denoted by $a \alpha b$ where $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

1. $a \alpha (b + c) = a \alpha b + a \alpha c$
2. $(a + b) \alpha c = a \alpha c + b \alpha c$
3. $a \alpha (a + \beta) c = a \alpha c + a \beta c$
4. $a \alpha (b \beta c) = (a \alpha b) \beta c$

for all $a,b,c \in S$ and for all $\alpha, \beta \in \Gamma$.

Definition 2.2 Let $S$ be a $\Gamma-$semiring and $\mu$ be a fuzzy subset of $S$. Then $\mu$ is called a fuzzy left ideal of $S$ if

1. $\mu(a + b) \geq \min(\mu(a), \mu(b))$
Definition 2.3 Let $S$ be a $\Gamma$—semiring and $\mu$ be a fuzzy subset of $S$. Then $\mu$ is called a fuzzy right ideal of $S$ if

1. $\mu(a + b) \geq \min(\mu(a), \mu(b))$
2. $\mu(ab) \geq \mu(a)$

for all $a, b \in S$ and for all $a \in \Gamma$.

Note: If a fuzzy subset $\mu$ of $\Gamma$—semiring $S$ is both fuzzy left and fuzzy right ideal of $S$ then $\mu$ is called a fuzzy two-sided ideal or simply fuzzy ideal of $S$.

Definition 2.4 Let $S$ be a $\Gamma$—semiring. A fuzzy ideal $\mu$ of $S$ is called fuzzy $k$—ideal of $S$ if

$$\mu(x) \geq \min(\mu(x + y), \mu(y))$$

for all $x, y \in S$ and for all $a \in \Gamma$.

Definition 2.5 Let $S$ be a $\Gamma$—semiring. A fuzzy ideal $\mu$ of $S$ is called fuzzy $h$—ideal of $S$ if for all $x, z, y_1, y_2 \in S$ such that $x + y_1 + z = y_2 + z$ implies $\mu(x) \geq \min(\mu(y_1), \mu(y_2))$.

Similarly we define fuzzy one-sided $k$—ideal and one-sided $h$—ideal of $S$.

Definition 2.6 [6] Left operator semiring and Right operatorsemiring of a $\Gamma$—semiring.

Let $S$ be a $\Gamma$—semiring and $F$ be the free additive commutative semigroup generated by $S \times \Gamma$. Then the relation $\rho$ on $F$ defined by $\Sigma_i(x_i, \alpha_i) \rho \Sigma_j(y_j, \beta_j)$ iff $\Sigma_i(x_i, \alpha_i) = \Sigma_j(y_j, \beta_j)$ for all $a \in S$, is congruence on $F$. We denote the congruence class containing $\Sigma_i(x_i, \alpha_i)$ by $\Sigma_i[x_i, \alpha_i]$. Then $F/\rho$ is an additive commutative semigroup. Now we define a multiplication on $F/\rho$ by $\Sigma_i[x_i, \alpha_i](\Sigma_j[y_j, \beta_j]) = \Sigma_{ij}[x_i x_j, \alpha_i \beta_j]$. Then $F/\rho$ forms a semiring with multiplication defined above. We denote this semiring by $L$ and call it the left operator semiring of the $\Gamma$—semiring $S$.

Dually we define the right operator semiring $R$ of the $\Gamma$—semiring $S$ where $R = \{\Sigma_i[\alpha_i, x_i] : \alpha_i \in \Gamma, x_i \in S\}$ and the multiplication on $R$ is defined as $(\Sigma_i[\alpha_i, x_i])(\Sigma_j[\beta_j, y_j]) = \Sigma_{ij}[\alpha_i x_j, \beta_j y_j]$.

We also note here that for $[x + y, \alpha], [x, \alpha + \beta] \in L$,

$[x + y, \alpha] = [x, \alpha] + [y, \alpha]$ and $[x, \alpha + \beta] = [x, \alpha] + [\alpha, \beta]$.

Similarly for $[\alpha, x + y], [\alpha + \beta, x] \in R$,

$\lbrack \alpha, x + y \rbrack = [\alpha, x] + [\alpha, \beta]$ and $[\alpha + \beta, x] = [\alpha, x] + [\beta, x]$.

Definition 2.7 Let $S$ be a $\Gamma$—semiring and $L$ be its left operator semiring. Let $\mu$ be a fuzzy subset of $L$.

Then $\mu$ is called a fuzzy left ideal of $L$ if

1. $\mu(\Sigma_i[x_i, \alpha_i] + \Sigma_j[y_j, \beta_j]) \geq \min(\mu(\Sigma_i[x_i, \alpha_i]), \mu(\Sigma_j[y_j, \beta_j]))$
2. $\mu(\Sigma_i[x_i, \alpha_i] \cdot \Sigma_j[y_j, \beta_j]) \geq \mu(\Sigma_j[y_j, \beta_j])$

for all $\Sigma_i[x_i, \alpha_i], \Sigma_j[y_j, \beta_j] \in L$.

Definition 2.8 Let $S$ be a $\Gamma$—semiring and $L$ be its left operator semiring. Let $\mu$ be a fuzzy subset of $L$.

Then $\mu$ is called a fuzzy right ideal of $L$ if

1. $\mu(\Sigma_i[x_i, \alpha_i] + \Sigma_j[y_j, \beta_j]) \geq \min(\mu(\Sigma_i[x_i, \alpha_i]), \mu(\Sigma_j[y_j, \beta_j]))$
2. $\mu(\Sigma_i[x_i, \alpha_i] \cdot \Sigma_j[y_j, \beta_j]) \geq \mu(\Sigma_i[x_i, \alpha_i])$

for all $\Sigma_i[x_i, \alpha_i], \Sigma_j[y_j, \beta_j] \in L$.

Note: If a fuzzy subset $\mu$ of $L$ is both fuzzy left and fuzzy right ideal of $L$ then $\mu$ is called fuzzy two-sided ideal or simply fuzzy ideal of $L$.

Definition 2.9 Let $S$ be a $\Gamma$—semiring and $L$ be its left operator semiring. A fuzzy ideal $\mu$ of $L$ is called fuzzy $k$—ideal if for all $\Sigma_i[x_i, \alpha_i], \Sigma_j[y_j, \beta_j] \in L$, $\mu(\Sigma_i[x_i, \alpha_i] + \Sigma_j[y_j, \beta_j]) \geq \min(\mu(\Sigma_i[x_i, \alpha_i]), \mu(\Sigma_j[y_j, \beta_j]))$.

Definition 2.10 Let $S$ be a $\Gamma$—semiring and $L$ be its left operator semiring. A fuzzy ideal $\mu$ of $L$ is called fuzzy $h$—ideal if for all $\Sigma_i[x_i, \alpha_i], \Sigma_j[y_j, \beta_j], \Sigma_k[z_k, \gamma_k] \in L$ s.t. $\Sigma_i[x_i, \alpha_i] + \Sigma_j[y_j, \beta_j] + \Sigma_k[z_k, \gamma_k] = \Sigma_i[u_i, \delta_i] + \Sigma_k[z_k, y_k]$ implies $\mu(\Sigma_i[x_i, \alpha_i]) \geq \min(\mu(\Sigma_j[y_j, \beta_j]), \mu(\Sigma_k[u_i, \delta_i]))$.

Similarly we define fuzzy left ideal, fuzzy right ideal, fuzzy ideal, fuzzy $k$—ideal, fuzzy $h$—ideal of the right operator semiring $R$. 
Definition 2.11 [8] Let $S$ be a $\Gamma$ -semiring and $L$ be its left operator semiring. For a fuzzy subset $\mu$ of $L$, a fuzzy subset 
$\mu^+$ of $S$ is defined by $\mu^+(a) = \inf_{a \in \Gamma} \mu([a, a])$ where $a \in S$.

For a fuzzy subset $\sigma$ of $S$, a fuzzy subset $\sigma^+$ of $L$ is defined by $\sigma^+(\Sigma_i x_i a_i) = \inf_{a \in \Gamma} \sigma(\Sigma_i x_i a_i)$ where $\Sigma_i [x_i a_i] \in L$.

Definition 2.12 [8] Let $S$ be a $\Gamma$ -semiring and $R$ be its right operator semiring. For a fuzzy subset $\mu$ of $R$, a fuzzy subset 
$\mu^+$ of $S$ is defined by $\mu^+(a) = \inf_{a \in \Gamma} \mu([a, a])$ where $a \in S$.

For a fuzzy subset $\sigma$ of $S$, a fuzzy subset $\sigma^+$ of $R$ is defined by $\sigma^+(\Sigma_i a_i x_i) = \inf_{a \in \Gamma} \sigma(\Sigma_i a_i x_i)$ where $\Sigma_i [a_i x_i] \in R$.

Definition 2.13 [6] Let $S$ be a $\Gamma$ -semiring and $L$ be its left operator semiring and $R$ be its right operator semiring. If there exist an element $\Sigma_i [e_i, \delta_i] \in L$ (respectively $\Sigma_i [y_j, f_j] \in R$) such that $\Sigma_i e_i \delta_i a = a$ (respectively $\Sigma_i a_j f_j = a$) for all $a \in S$ then $S$ is said to have left unity $\Sigma_i [e_i, \delta_i]$ (respectively the right unity $\Sigma_i [y_j, f_j]$).

Proposition 2.14 [6] Let $S$ be a $\Gamma$ -semiring and $L$ be the left operator semiring of $S$. If $\Sigma_i [e_i, \delta_i]$ is the left unity of $S$, then it is the identity of $L$.

Proposition 2.15 [6] Let $S$ be a $\Gamma$ -semiring and $R$ be the right operator semiring of $S$. If $\Sigma_i [y_j, f_j]$ is the right unity of $S$, then it is the identity of $R$.

Throughout the text, unless otherwise stated explicitly, we consider a $\Gamma$ -semiring $S$ which has the left unity, the right unity which implies that the left operator semiring $L$ and the right operator semiring $R$ of $S$ has identity.

III. CORRESPONDING FUZZY IDEALS

Proposition 3.1 Let $S$ be a $\Gamma$ -semiring and $L$ be its left operator semiring. Then
(1) if $\mu$ is a fuzzy left ideal of $L$ then $\mu^+$ is a fuzzy left ideal of $S$.
(2) if $\mu$ is a fuzzy right ideal of $L$ then $\mu^+$ is a fuzzy right ideal of $S$.
(3) if $\mu$ is a fuzzy ideal of $L$ then $\mu^+$ is a fuzzy ideal of $S$.

Proof. (1) Let $\mu$ be a fuzzy left ideal of $L$. Let $a, b \in S$ and $\gamma \in \Gamma$ then
$\mu^+(a + b)$
$= \inf_{a \in \Gamma} \mu([a + b, \alpha])$
$\geq \inf_{a \in \Gamma} \{ \min \{ \mu([a, \alpha]), \mu([b, \alpha]) \} \}$
$= \min \{ \inf_{a \in \Gamma} \mu([a, \alpha]) , \inf_{a \in \Gamma} \mu([b, \alpha]) \}$
$\geq \inf_{a \in \Gamma} \mu([\gamma, \alpha])$
$\geq \inf_{a \in \Gamma} \mu([b, \alpha])$
$= \mu^+(b)$.
So, $\mu^+$ is a fuzzy left ideal of $S$.

(2) Let $\mu$ be a fuzzy right ideal of $L$. Let $a, b \in S$ and $\gamma \in \Gamma$.
Then by (1) we have $\mu^+(a + b) \geq \min \{ \mu^+(a), \mu^+(b) \}$
Now,
$\mu^+(a \gamma b)$
$= \inf_{a \in \Gamma} \mu([a \gamma b, \alpha])$
$\geq \inf_{a \in \Gamma} \mu([b, \alpha])$
$\geq \inf_{a \in \Gamma} \mu([b, \alpha])$
$= \mu^+(b)$.
So, $\mu^+$ is a fuzzy right ideal of $S$.

(3) Follows from (1) and (2).

Proposition 3.2 Let $S$ be a $\Gamma$ -semiring and $L$ be its left operator semiring. Then
(1) if $\sigma$ is a fuzzy left ideal of $S$ then $\sigma^+$ is a fuzzy left ideal of $L$.
(2) if $\sigma$ is a fuzzy right ideal of $S$ then $\sigma^+$ is a fuzzy right ideal of $L$.
(3) if $\sigma$ is a fuzzy ideal of $S$ then $\sigma^+$ is a fuzzy ideal of $L$.
Proof. (1) Let $\sigma$ be a fuzzy left ideal of $S$. Let $\sum_{i}[x_i, \alpha_i], \sum_{j}[y_j, \beta_j] \in L$.

Then $\sigma^+(\sum_{i}[x_i, \alpha_i] + \sum_{j}[y_j, \beta_j])$

$= \inf_{\text{aes}} \{ \sigma(\sum_{i}[x_i, \alpha_i]) + \sigma(\sum_{j}[y_j, \beta_j]) \}$

$\geq \inf_{\text{aes}} \{ \min \{ \sigma(\sum_{i}[x_i, \alpha_i]), \sigma(\sum_{j}[y_j, \beta_j]) \} \}$

$= \min \{ \sigma^+(\sum_{i}[x_i, \alpha_i]), \sigma^+(\sum_{j}[y_j, \beta_j]) \}$.

Now $\sigma^+(\sum_{i}[x_i, \alpha_i], \sum_{j}[y_j, \beta_j])$

$= \inf_{\text{aes}} \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j) a_j$

$= \inf_{\text{aes}} \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j)$

$\geq \inf_{\text{aes}} \{ \min \{ \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j) \} \}$

Since $\sigma$ is a fuzzy left ideal of $S$.

Then by (1), $\sigma^+(\sum_{i}[x_i, \alpha_i], \sum_{j}[y_j, \beta_j]) \geq \min \{ \sigma^+(\sum_{i}[x_i, \alpha_i]), \sigma^+(\sum_{j}[y_j, \beta_j]) \}$.

Now $\sigma^+(\sum_{i}[x_i, \alpha_i], \sum_{j}[y_j, \beta_j])$

$= \sigma^+(\sum_{i}[x_i, \alpha_i] y_j, \beta_j) a_j$

$= \inf_{\text{aes}} \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j)$

$\geq \inf_{\text{aes}} \{ \min \{ \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j) \} \}$

Since $\sigma$ is a fuzzy right ideal of $S$.

Therefore, $\sigma(a) = \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j) a_j \geq \inf_{\text{aes}} \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j)$

So, by (1) we have $\sigma^+(a) \leq \sigma(a)$.

Again, since $\sigma$ is a fuzzy right ideal of $S$, therefore $\sigma(a) = \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j) a_j \geq \inf_{\text{aes}} \sigma(\sum_{i}[x_i, \alpha_i] y_j, \beta_j)$

Now let $\mu$ be a fuzzy right ideal of $L$. Then for all $\sum_{i}[x_i, \alpha_i] \in L$,

$(\mu^+)^+ \{ \sum_{i}[x_i, \alpha_i] \} = \inf_{\text{aes}} \mu^+ \{ \sum_{i}[x_i, \alpha_i] \} = \inf_{\text{aes}} \inf_{\text{aes}} \mu(\sum_{i}[x_i, \alpha_i], \sum_{j}[y_j, \beta_j])$

$= \inf_{\text{aes}} \inf_{\text{aes}} \mu(\sum_{i}[x_i, \alpha_i], \sum_{j}[y_j, \beta_j])$ since $\{x + y, \alpha\} = [x, \alpha] + [y, \alpha]$. 

Hence the mapping $\sigma \rightarrow \sigma^+$ is one-to-one.
Hence the mapping.

Now let

\[ \mu(\Sigma[x_i, a_i]) = \inf_{a \in S} \min(\mu(\Sigma[x_i, a_i] + \Sigma[y_j, \beta_j]), 1) \]  

Since \( L \) has the identity say, \( \Sigma[e_j, \delta_j] \)

So, \( \Sigma[x_i, a_i] \cdot \Sigma[e_j, \delta_j] = \Sigma[x_i, a_i] \)

Therefore, \( \mu(\Sigma[x_i, a_i]) = \mu(\Sigma[x_i, a_i] \cdot \Sigma[e_j, \delta_j]) = \mu(\Sigma[x_i, a_i, e_j, \delta_j]) \)

\[ \geq \min(\mu(\Sigma[x_i, a_i]), \mu(\Sigma[e_j, \delta_j])) \quad \text{since } \mu \text{ is a fuzzy right ideal of } L \]

\[ \geq \inf_{a \in S} \inf_{a \in S} \mu(\Sigma[x_i, a_i]) \]

So, by (4) we have \( (\mu^+)^+ \Sigma[x_i, a_i] \) \( \leq \mu(\Sigma[x_i, a_i]) \)  

By (5) and (6) we have \( (\mu^+)^+ \Sigma[x_i, a_i] = \mu(\Sigma[x_i, a_i]) \).

Hence the mapping \( \sigma \rightarrow \sigma^{+} \) is onto. Thus the mapping is bijective.

Now let \( \eta, \nu \) be two fuzzy right ideals of \( S \) s.t. \( \eta \leq \nu \). Then for all \( \Sigma[x_i, a_i] \in L \),

\[ \sigma^{+}(\Sigma[x_i, a_i]) = \inf_{a \in S} \sigma(\Sigma[x_i, a_i]) \leq \inf_{a \in S} \nu(\Sigma[x_i, a_i]) = \sigma^{+}(\Sigma[x_i, a_i]) \]

Hence the said mapping is order preserving.

Also let \( \eta, \nu \) be two fuzzy right ideals of \( L \) s.t. \( \eta \leq \nu \). Then for all \( a \in S \)

\[ \eta^+(a) = \inf_{a \in S} \eta([a, a]) \leq \inf_{a \in S} \nu([a, a]) = \nu^+(a) \]

So, inverse of the said mapping is also order preserving. Hence for the lattices of all fuzzy right ideals the theorem is proved.

Similar proof for the case for the lattices of all fuzzy two-sided ideals or simply fuzzy ideals.

Hence the theorem is proved.

Now we obtain analogous results of Proposition 3.1 and 3.2 and Theorem 3.3 for the right operator semiring \( R \) of a \( \Gamma \) — semiring \( S \). We only state the proof is similar as in case of the left operator semiring \( L \).

**Proposition 3.4** Let \( S \) be a \( \Gamma \) — semiring and \( R \) be its right operator semiring. Then

(1) if \( \mu \) is a fuzzy left (respectively right) ideal of \( R \) then \( \mu^+ \) is a fuzzy left (respectively right) ideal of \( S \).

(2) if \( \mu \) is a fuzzy ideal of \( R \) then \( \mu^+ \) is a fuzzy ideal of \( S \).

**Proposition 3.5** Let \( S \) be a \( \Gamma \) — semiring and \( R \) be its right operator semiring. Then

(1) if \( \sigma \) is a fuzzy left (respectively right) ideal of \( S \) then \( \sigma^+ \) is a fuzzy left (respectively right) ideal of \( R \).

(2) if \( \sigma \) is a fuzzy ideal of \( S \) then \( \sigma^+ \) is a fuzzy ideal of \( R \).

**Theorem 3.6** Let \( S \) be a \( \Gamma \) — semiring and \( R \) be its right operator semiring. Then there exist an inclusion preserving bijection \( \sigma \rightarrow \sigma^+ \) between the lattices of all fuzzy left ideals (respectively fuzzy ideals) of \( S \) and the lattices of all fuzzy left ideals (respectively fuzzy ideals) of \( R \). Where \( \sigma \) denotes a fuzzy left ideal (respectively fuzzy ideal) of \( S \).

**Proposition 3.7** Let \( S \) be a \( \Gamma \) — semiring and \( L \) be its left operator semiring. Then

(1) if \( \sigma \) is a fuzzy \( k \) — ideal of \( S \) then \( \sigma^+ \) is a fuzzy \( k \) — ideal of \( L \).

(2) if \( \mu \) is a fuzzy \( k \) — ideal of \( S \) then \( \mu^+ \) is a fuzzy \( k \) — ideal of \( S \).

**Proof:** (1) Let \( \sigma \) be a fuzzy \( k \) — ideal of \( S \). Then by proposition 3.2 \( \sigma^{+} \) is a fuzzy ideal of \( L \).

Now, let \( \Sigma[x_i, a_i], \Sigma[y_j, \beta_j] \in L \), then \( \Sigma[x_i, a_i] \cdot \Sigma[y_j, \beta_j] \in S \) for all \( a \in S \).

As \( \sigma \) is a fuzzy \( k \) — ideal of \( S \) then \( \sigma(\Sigma[x_i, a_i]) \geq \min(\sigma(\Sigma[x_i, a_i] + \Sigma[y_j, \beta_j]), 1) \).

So, \( \inf_{a \in S} \sigma(\Sigma[x_i, a_i]) \geq \min(\inf_{a \in S} \sigma(\Sigma[x_i, a_i] + \Sigma[y_j, \beta_j]), \inf_{a \in S} \sigma(\Sigma[y_j, \beta_j])) \).

This implies that \( \sigma^{+}(\Sigma[x_i, a_i]) \geq \min(\sigma^{+}(\Sigma[x_i, a_i] + \Sigma[y_j, \beta_j]), \sigma^{+}(\Sigma[y_j, \beta_j])) \).

Hence \( \sigma^{+} \) is a fuzzy \( k \) — ideal of \( L \).

(2) Let \( \mu \) be a fuzzy \( k \) — ideal of \( S \). Then by proposition 3.1 \( \mu^{+} \) is a fuzzy ideal of \( S \).

Now, let \( x, y \in S \), then \( [x, a], [y, a] \in L \) for all \( a \in L \).

As \( \mu \) is a fuzzy \( k \) — ideal of \( L \), then \( \mu(\Sigma[x, a]) \geq \min(\mu(\Sigma[x, a] + [a, a]), \mu([a, a])) \).

So, \( \inf_{a \in S} \mu(\Sigma[x, a]) \geq \min(\inf_{a \in S} \mu(\Sigma[x, a] + [a, a]), \inf_{a \in S} \mu([a, a])) \).

This implies that \( \mu^{+}(x) \geq \min(\mu^{+}(x + y), \mu^{+}(y)) \).

Hence \( \mu^{+} \) is a fuzzy \( k \) — ideal of \( S \).
Proposition 3.8 Let $S$ be a $\Gamma$-semiring and $L$ be its left operator semiring. Then
(1) if $\sigma$ is a fuzzy $h$-ideal of $S$ then $\sigma^{+}$ is a fuzzy $h$-ideal of $L$.
(2) if $\mu$ is a fuzzy $h$-ideal of $L$ then $\mu^{+}$ is a fuzzy $h$-ideal of $S$.

Proof: (1) Let $\sigma$ be a fuzzy $h$-ideal of $S$. Then by proposition 3.2 $\sigma^{+}$ is a fuzzy ideal of $L$.
Now, let $\Sigma[x_{a}, \alpha_{a}], \Sigma[y_{a}, \beta_{a}], \Sigma[u_{a}, \eta_{a}] \in L$ such that
$\Sigma[x_{a}, \alpha_{a}] + \Sigma[y_{a}, \beta_{a}] + \Sigma[u_{a}, \eta_{a}] = \Sigma[v_{a}, \gamma_{a}]$.
Then $\Sigma[x_{a}, \alpha_{a}] + \Sigma[y_{a}, \beta_{a}] + \Sigma[k_{a}, \kappa_{a}] + \Sigma[u_{a}, \eta_{a}]$ for all $a \in S$ and
$\Sigma[k_{a}, \kappa_{a}] = \Sigma[v_{a}, \gamma_{a}]$.
As $\sigma$ is a fuzzy $h$-ideal of $S$ then
$\sigma(\Sigma[x_{a}, \alpha_{a}]) \geq \min(\sigma(\Sigma[y_{a}, \beta_{a}]), \sigma(\Sigma[u_{a}, \eta_{a}]))$ for all $a \in S$.
So, $\inf_{a \in S} \sigma(\Sigma[x_{a}, \alpha_{a}]) \geq \min(\inf_{a \in S} \sigma(\Sigma[y_{a}, \beta_{a}]), \inf_{a \in S} \sigma(\Sigma[u_{a}, \eta_{a}]))$.
This implies that $\sigma^{+}(\Sigma[x_{a}, \alpha_{a}]) \geq \min(\sigma^{+}(\Sigma[y_{a}, \beta_{a}]), \sigma^{+}(\Sigma[u_{a}, \eta_{a}]))$.
Hence $\sigma^{+}$ is a fuzzy $h$-ideal of $S$.

(2) Let $\mu$ be a fuzzy $h$-ideal of $L$. Then by proposition 3.1 $\mu^{+}$ is a fuzzy ideal of $S$.
Now, let $x, u, y, z \in S$ such that $x + y + z = y_{1} + z$.
Then for all $a \in \Gamma$, we have $[x, a], [y_{1}, a], [z, a] \in L$ and
$[x + y + z, a] = [y_{1}, a] + [z, a]$ i.e. $[x, a] + [y_{1}, a] + [z, a] = [y_{1}, a] + [z, a]$.
As $\mu$ is a fuzzy $h$-ideal of $L$ then $\mu([x, a]) \geq \min(\mu([y_{1}, a]), \mu([z, a]))$ for all $a \in \Gamma$.
So, $\inf_{a \in \Gamma} \mu([x, a]) \geq \min(\inf_{a \in \Gamma} \mu([y_{1}, a]), \inf_{a \in \Gamma} \mu([z, a]))$.
This implies that $\mu^{+}(x) \geq \min(\mu^{+}(y_{1}), \mu^{+}(z))$.
Hence $\mu^{+}$ is a fuzzy $h$-ideal of $S$.

Now we obtain analogous results of Propositions 3.7, 3.8 for the right operator semiring $R$ of a $\Gamma$-semiring $S$. We only state the results as the proof is similar as in case of the left operator semiring $L$.

Proposition 3.9 Let $S$ be a $\Gamma$-semiring and $R$ be its right operator semiring. Then
(1) if $\sigma$ is a fuzzy $k$-ideal of $S$ then $\sigma^{r}$ is a fuzzy $k$-ideal of $R$.
(2) if $\mu$ is a fuzzy $k$-ideal of $R$ then $\mu^{r}$ is a fuzzy $k$-ideal of $S$.

Proposition 3.10 Let $S$ be a $\Gamma$-semiring and $R$ be its right operator semiring. Then
(1) if $\sigma$ is a fuzzy $h$-ideal of $S$ then $\sigma^{r}$ is a fuzzy $h$-ideal of $R$.
(2) if $\mu$ is a fuzzy $h$-ideal of $R$ then $\mu^{r}$ is a fuzzy $h$-ideal of $S$.

Remark 3.11 Propositions 3.7, 3.8, 3.9 and 3.10 are also valid for fuzzy one-sided $k$-ideals and $h$-ideals.

We now only state the following theorems as the proof is analogous to that of the Theorem 3.3.

Theorem 3.12 Let $S$ be a $\Gamma$-semiring and $L$ be its left operator semiring. Then there exist an inclusion preserving bijection $\sigma \to \sigma^{+}$ between the set of all fuzzy $k$-ideals (respectively fuzzy $h$-ideals) of $S$ and the set of all fuzzy $k$-ideals (respectively fuzzy $h$-ideals) of $L$. Where $\sigma$ denotes a fuzzy $k$-ideal (respectively fuzzy $h$-ideal) of $S$.

Theorem 3.13 Let $S$ be a $\Gamma$-semiring and $R$ be its right operator semiring. Then there exist an inclusion preserving bijection $\sigma \to \sigma^{r}$ between the set of all fuzzy $k$-ideals (respectively fuzzy $h$-ideals) of $S$ and the set of all fuzzy $k$-ideals (respectively fuzzy $h$-ideals) of $R$. Where $\sigma$ denotes a fuzzy $k$-ideal (respectively fuzzy $h$-ideal) of $S$.

Remark 3.14 Theorem 3.12 is also valid for fuzzy right $k$-ideals and fuzzy right $h$-ideals and Theorem 3.13 is also valid for fuzzy left $k$-ideals and fuzzy left $h$-ideals.

IV. References

2) Zadeh L A. Fuzzy sets, Information and control (1965); 8: pp. 338-353.