Bivariate Laplace Transform Involving Certain Product of Special Functions

By V.B.L. Chaurasia, R.C. Meghwal
University of Rajasthan, Jaipur, INDIA.

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V.B.L. Chaurasia\(^{\alpha}\), R.C. Meghwal\(^{\beta}\)

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I. INTRODUCTION

The integral equation

\[
\phi(p,q) = pq \int_{0}^{\infty} \int_{0}^{\infty} e^{-px-xy} f(x, y) \, dx \, dy, \quad \text{Re}(p) > 0, \text{Re}(q) > 0 \quad \text{(1.1)}
\]

represents the classical Laplace transform in two variables and \(\phi(p,q)\) and \(f(x,y)\) related by (1.1) are said to be operationally related to each other. Symbolically we can write

\[
\phi(p,q) \quad f(x,y) \quad \text{or} \quad f(x,y) \div \phi(p,q) \quad \text{(1.2)}
\]

and the symbol is called operational.

The multivariable H-function has been introduced by Srivastava and Panda [10] and is defined and represented in the following manner (see also [11, p.251]):

\[
H^{0,u: (M^f,N^f); \ldots; (M^{f'},N^{f'})}_{v,w: (P^f,Q^f); \ldots; (P^{f'},Q^{f'})} \left[ \left[ (a): A^{f'}, \ldots, A^{f'}; \ldots; [b^{f'},B^{f'}]; \ldots; [b^{f'},B^{f'}]; \left[ \left[ (c): C^{f'}, \ldots, C^{f'}; \ldots; [d^{f'},D^{f'}]; \ldots; [d^{f'},D^{f'}]; Z_{1}, \ldots, Z_{r} \right] \right] \right]
\]

\[
= (2\pi)^{-r} \int_{L_{1}} \int_{L_{r}} U_{1}(s_{1}) \ldots U_{r}(s_{r}) V(s_{1}, \ldots, s_{r}) Z_{1}^{s_{1}} \ldots Z_{r}^{s_{r}} \, ds_{1} \ldots ds_{r}, \quad i = \sqrt{-1}. \quad \text{(1.3)}
\]

where

\[
U_{i}(s_{i}) = \prod_{j=1}^{M^{(i)}} \Gamma(d^{(i)} - D^{(j)} s_{i}) \prod_{j=1}^{N^{(i)}} \Gamma(1 - b^{(i)} + B^{(j)} s_{i})
\]

\[
. \left\{ \prod_{j=M^{(i)}+1}^{Q^{(i)}} \Gamma(1 - d^{(i)} + D^{(j)} s_{i}) \prod_{j=N^{(i)}+1}^{p^{(i)}} \Gamma(b^{(i)} - B^{(j)} s_{i}) \right\}^{-1}. \quad \text{(1.4)}
\]

About\(^{\alpha}\): Department of Mathematics, University of Rajasthan Jaipur-302055 (Rajasthan), India.

About\(^{\beta}\): Department of Mathematics, Government Post-Graduate College Neemuch-458441 (MP), India.
\[ V(s_1,\ldots,s_r) = \prod_{j=1}^{u} \Gamma \left( \frac{1}{\Gamma} \sum_{i=1}^{r} A_{j}^{(i)} s_i \right) \]

\[ \prod_{j=u+1}^{v} \Gamma \left( a_j - \sum_{i=1}^{r} A_{j}^{(i)} s_i \right) \prod_{j=1}^{w} \Gamma \left( 1 - c_j + \sum_{i=1}^{r} C_{j}^{(i)} s_i \right) \]

\[ \cdot \left( \prod_{j=1}^{r} \Gamma \left( a_j - \sum_{i=1}^{r} A_{j}^{(i)} s_i \right) \right) \]

\[ \cdot \prod_{j=1}^{u} \Gamma \left( 1 - c_j + \sum_{i=1}^{r} C_{j}^{(i)} s_i \right) \] \[ \cdot \] \[ (1.5) \]

The multiple integral in (1.3) converges absolutely if \( |\text{arg}(z_i)| < \Delta \left( \frac{\pi}{2} \right), i = 1,\ldots,r. \)

when

\[ \Delta_i = - \sum_{j=u+1}^{v} A_{j}^{(i)} + \sum_{j=1}^{N(i)} B_{j}^{(i)} - \sum_{j=N(i)+1}^{p(i)} B_{j}^{(i)} - \sum_{j=1}^{w} C_{j}^{(i)} + \sum_{j=1}^{M(i)} D_{j}^{(i)} - \sum_{j=M(i)+1}^{Q(i)} D_{j}^{(i)} > 0, (i = 1,\ldots,r), \]

\[ (1.6) \]

For other details of this function see [11]. The series representation of I-function is as follows ([8], p.305-306):

\[ I_{u,d}^{m,n} = I(z) = \sum_{r'=0}^{\infty} \sum_{h=1}^{\infty} \prod_{j=1}^{m} \left\{ \Gamma(1 - a_j + \alpha_j \xi_{h,r'}) \right\} A_{j}^{(i)} \]

\[ \prod_{j=n+1}^{u} \left\{ \Gamma(a_j - \alpha_j \xi_{h,r'}) \right\} A_{j}^{(i)} \]

\[ \prod_{j=1}^{m} \left\{ \Gamma(b_j - \beta_j \xi_{h,r'}) \right\} B_j^{(i)} (-1)^{r'} \xi_{h,r'} \]

\[ \cdot \prod_{j=m+1}^{d} \left\{ \Gamma(1 - b_j + \beta_j \xi_{h,r'}) \right\} B_j^{(i)} r'! \beta_h \]

\[ \text{for } |z| < 1, \]

\[ (1.7) \]

where

\[ \xi_{h,r'} = \frac{b_h + r'}{\beta_h}. \]

\[ (1.8) \]

For the sake of brevity,

\[ \Delta = \sum_{j=1}^{m} B_{j} \beta_j - \sum_{j=m+1}^{d} B_{j} \beta_j + \sum_{j=1}^{n} A_{j} \alpha_j - \sum_{j=n+1}^{u} A_{j} \alpha_j. \]

\[ (1.9) \]

\[ \theta = \frac{b_j}{\beta_j}, \quad j = 1,\ldots,m \]

\[ \phi = \frac{a_j - 1}{\alpha_j}, \quad j = 1,\ldots,n \]

\[ (1.10) \]
In this paper we shall obtain correspondences, involving a product of I-function and the multivariable H-function, between the original and the image in two variables.

In what follows we shall denote the original variable by \( x \) and \( y \) and the transformed variable by \( p \) and \( q \). The notations employed are those of Ditkin and Prudnikov’s [5] operational calculus.

\[
\theta_i = \frac{d^{(i)}}{D_j^{(i)}}, \quad j = 1, \ldots, M^{(i)}
\]
\[
\phi_i = \frac{1-b^{(i)}}{B_j^{(i)}}, \quad j = 1, \ldots, N^{(i)} \quad i = 1, \ldots, r
\]

…(1.11)

II. Theorem 1. With \( \Delta_i, \Delta, \theta, \phi, \theta \), and \( \phi \), given by (1.6), (1.9), (1.10) and (1.11) respectively, let

\[
\Delta_i > 0, \Delta > 0, |\arg (z_i)| < \Delta_i \left( \frac{\pi}{2} \right), |\arg z| < \Delta \left( \frac{\pi}{2} \right), k > 0, h_i > 0, i = 1, \ldots, r
\]

and

(i) \( \text{Re} \left\{ \sigma + k\theta + \sum_{i=1}^{r} h_i \theta_i \right\} > 0 \),

(ii) \( \text{Re} \left\{ \rho - \sigma - k\phi - \sum_{i=1}^{r} h_i \phi_i \right\} < \frac{3}{4} \).

Also let \( 0 \leq n \leq u, 0 \leq m \leq d, \) and

(iii) \( \text{Re} \ (p) > 0 \).

\[
p^{-\frac{1}{2}}(pq)^{\frac{1}{2}} \sum_{r'=0}^{\infty} \sum_{h=1}^{m} \prod_{j=1}^{n} \left\{ \Gamma(1-a_j + \alpha_j \xi_{h,r'}) \right\}^{A_j} \prod_{j=n+1}^{u} \left\{ \Gamma(a_j - \alpha_j \xi_{h,r'}) \right\}^{A_j} \\
\prod_{j=1}^{m} \left\{ \Gamma(b_j - \beta_j \xi_{h,r'}) \right\}^{B_j} \frac{k_{\xi_{h,r'}}}{(pq)^{\frac{1}{2}}} \\
\prod_{j=m+1}^{d} \left\{ \Gamma(1-b_j + \beta_j \xi_{h,r'}) \right\}^{B_j} \frac{r! \beta_h}{(pq)^{\frac{1}{2}}} \\
\cdot \ H_{0,0; (M', N'); \ldots; (M', N') \ldots}^{v, w; (P', Q'); \ldots; (P', Q') \ldots} \left( z_1 (\sqrt{pq})^{h_1}, \ldots, z_r (\sqrt{pq})^{h_r} \right)
\]
The Laplace transform of a product of I-function and multivariable H-function is given by

\[
\frac{(4xy)^{\frac{\alpha}{2}-\frac{1}{2}}}{\sqrt{\pi y}} \sum_{r=0}^{\infty} \sum_{h=1}^{m} \prod_{j=1}^{n} \{\Gamma(1-a_j + \alpha_j \xi_{h,r})\}^{A_j} \prod_{j=n+1}^{m} \{\Gamma(1-a_j + \alpha_j \xi_{h,r})\}^{A_j} \frac{\prod_{j=1}^{m} \{\Gamma(b_j - \beta_j \xi_{h,r})\}^{B_j} (-1)^{r'} Z_{h,r'}}{(4xy)^r} = \sum_{d} \frac{\prod_{j=1}^{m} \{\Gamma(b_j - \beta_j \xi_{h,r})\}^{B_j} r'! \beta_h}{(4xy)^r} \]

Prove. The Laplace transform of a product of I-function and multivariable H-function is given by

\[
\int_{0}^{\infty} e^{-pt} \sigma t^{-\frac{1}{2}} \sum_{m,n}^{\infty} \prod_{h=1}^{n} \{\Gamma(1-a_j + \alpha_j \xi_{h,r})\}^{A_j} \prod_{j=n+1}^{m} \{\Gamma(1-a_j + \alpha_j \xi_{h,r})\}^{A_j} \frac{\prod_{j=1}^{m} \{\Gamma(b_j - \beta_j \xi_{h,r})\}^{B_j} (-1)^{r'} Z_{h,r'}}{(4xy)^r} dt \]

The result in (2.2) can be established by substituting the series (1.7) for I-function and changing the order of integration and summation (which is justified due to absolute convergence of the integral involved in the process under the conditions mentioned), then evaluating the inner integral and using the definition (1.1), we arrive at the required result. On writing \((pq)^{-\frac{1}{2}}\) for \(p\), multiplying both sides of (2.2) by \(p^{-\frac{1}{2}}(pq)^{1-p}\) and then interpreting it with the help of a known result ([5], p. 144, eqn. 3.26), we get...
\[
\frac{(4xy)^{\rho-\frac{\sigma}{2}}}{\sqrt{\pi y}} \sum_{r'=0}^{\infty} \sum_{h=1}^{m} \prod_{j=1}^{n} \{\Gamma(1-a_j + \alpha_j \xi_{h,r'})\}^{A_j} \prod_{j \neq h} \{\Gamma(a_j - \alpha_j \xi_{h,r'})\} \frac{k \xi_{h,r'}}{\sqrt{pq}} \prod_{j=m+1}^{d} \{\Gamma(1-b_j + \beta_j \xi_{h,r'})\}^{B_j} \frac{r! \beta_h}{r!} \]

\[
\cdot \frac{(4xy)^{\rho-\frac{\sigma}{2}}}{\sqrt{\pi y}} \sum_{r'=0}^{\infty} \sum_{h=1}^{m} \prod_{j=1}^{n} \{\Gamma(1-a_j + \alpha_j \xi_{h,r'})\}^{A_j} \prod_{j \neq h} \{\Gamma(a_j - \alpha_j \xi_{h,r'})\} \frac{k \xi_{h,r'}}{\sqrt{pq}} \prod_{j=m+1}^{d} \{\Gamma(1-b_j + \beta_j \xi_{h,r'})\}^{B_j} \frac{r! \beta_h}{r!} \]

where \(\text{Re}(\rho) > 0\).

Now evaluating the integral on the left hand side of (2.3) by the process mentioned in (2.2) to obtain the desired result. Hence (2.1) is proved.

### III. Special Cases

(I) We have the following result from 2.1) by tacitly giving some values to parameters

\[
\frac{(4xy)^{\rho-\frac{\sigma}{2}}}{\sqrt{\pi y}} \sum_{r'=0}^{\infty} \sum_{h=1}^{m} \prod_{j=1}^{n} \{\Gamma(1-a_j + \alpha_j \xi_{h,r'})\}^{A_j} \prod_{j \neq h} \{\Gamma(a_j - \alpha_j \xi_{h,r'})\} \frac{k \xi_{h,r'}}{\sqrt{pq}} \prod_{j=m+1}^{d} \{\Gamma(1-b_j + \beta_j \xi_{h,r'})\}^{B_j} \frac{r! \beta_h}{r!} \]

\[
\cdot \frac{(4xy)^{\rho-\frac{\sigma}{2}}}{\sqrt{\pi y}} \sum_{r'=0}^{\infty} \sum_{h=1}^{m} \prod_{j=1}^{n} \{\Gamma(1-a_j + \alpha_j \xi_{h,r'})\}^{A_j} \prod_{j \neq h} \{\Gamma(a_j - \alpha_j \xi_{h,r'})\} \frac{k \xi_{h,r'}}{\sqrt{pq}} \prod_{j=m+1}^{d} \{\Gamma(1-b_j + \beta_j \xi_{h,r'})\}^{B_j} \frac{r! \beta_h}{r!} \]

... (2.3)
II) Taking $A_j = B_j = l = \alpha_j = \beta_j, \forall j$ and $k = l$ in (3.1), we get the following result involving $M$-series [9] and multivariable $H$-function as follows.

$$
\prod_{j=1}^{m} \left\{ \Gamma(b_j - \beta_j, \xi_{h,r}) \right\} \left( -1 \right)^{r'} z^{\xi_{h,r}} \prod_{j \neq h}^{d} \left\{ \Gamma(1 - b_j + \beta_j, \xi_{h,r}) \right\} B_j r! \beta_h
$$

\[
\prod_{j=m+1}^{d} \{ \Gamma(1 - b_j + \beta_j, \xi_{h,r}) \} B_j r! \beta_h
\]

\[\frac{k_{\xi_{h,r}}}{(4xy)^{\frac{2}{2}}}
\]

\[
\sum_{r=0}^{\infty} \left( \frac{a_1 (r') ... a_u (r')}{(b_1 (r') ... b_d (r')} \right) \frac{z^{r'}}{(4xy)^{\frac{r'}{r'}}}
\]

\[
H_{v,w:(P,Q):...(P^r,Q^r)}^{0,0:(M',N')....;(M^r,N^r)} \left[ [(a):A',...,A^r],[2\rho - \sigma - \xi_{h,r};h_1,...,h_r]:
\right]
\[
[z_1 (\sqrt{pq})^{h_1},...,z_r (\sqrt{pq})^{h_r}]
\]

\[
H_{v,w:(P,Q):...(P^r,Q^r)}^{0,0:(M',N')....;(M^r,N^r)} \left[ [(a):A',...,A^r],[2\rho - \sigma - \xi_{h,r};h_1,...,h_r]:
\right]
\[
[z_1 (2 \sqrt{xy})^{-h_1},...,z_r (2 \sqrt{xy})^{-h_r}]
\]

\[\frac{\phi(\xi_{h,r})}{r! \beta_h} \left( \frac{kpq}{2} \right)
\]

\[
H_{v,w:(P,Q):...(P^r,Q^r)}^{0,0:(M',N')....;(M^r,N^r)} \left[ \frac{k_{\xi_{h,r}}}{(4xy)^{\frac{2}{2}}} \right]
\]

III) Putting $A_j = B_j = 1$, the result in (2.1) reduces to a known result derived by Chaurasia [1].
(IV) Letting $B_j = 1$ in eqn. (2.1) with $k \to 0$, we obtain (after a little simplification) the Laplace transform for the multivariable H-function derived by Chaurasia [1].

(V) On taking $B_j = 1$ in eqn. (2.1), we get a known result of Chaurasia and Patni[2] with $n = 0 = n'$. 

(VI) Putting $A_j = B_j = 1$ in eqn. (2.1), we find a known result of Chaurasia and Godika [3] with $m_1 = \ldots = m_R = 0 = M_{r+1} = \ldots = M_r$.

(VII) On giving suitable value to parameters in our results, we have the results recently obtained by Chaurasia and Lata [4].

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