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On Some Transformations Involving Unit And Quarter Arguments

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Abstract - The main object of this paper is to obtain a general theorem on multiple series identity involving bounded sequences. The theorem, in turn, is expressed in terms of Srivastava- Daoust hypergeometric function of three variables. A known result of Joshi and Vyas is deduced as a special case of our reduction formula. Certain results involving unit and quarter arguments associated with hypergeometric polynomials ${}_4F_3$, ${}_5F_4$, ${}_6F_5$, ${}_7F_6$ are also obtained. Further many more known or new results can be obtained by specializing the parameters or the variables or both.

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Global Journal of Science Frontier Research Volume XI Issue V Version I August 2011

29

I. Introduction

The summation theorem concerned with single Gaussian hypergeometric functions play an important role in the study of transformation and reduction formulae of multiple hypergeometric functions.

In 1836, Kümmer [6] gave a list of quadratic transformations for ${}_2F_1$. Making use of Gauss evaluation of ${}_2F_1$, he was able to calculate some ${}_2F_1$ whose arguments were not unity. In 1881, Goursat [3] used the same technique for the third, fourth and sixth degree transformations of certain ${}_2F_1$.

In the monumental work of Prudnikov et al. [7], we find more information concerning reducible cases of hypergeometric functions. In fact, a number of results of interest do exists and in particular this is true for Clausen's ${}_3F_2$ with argument $\frac{1}{4}$.

In the literature of special functions, many hypergeometric transformations for terminating or infinite series of the type ${}_{q+1}F_q$ involving $+1, -1$, and $\frac{1}{2}$ arguments with appropriate parametric restrictions, are available. Some evaluation of ${}_3F_2(\frac{1}{4})$ and ${}_3F_2(\frac{3}{4})$ have already been derived by Karlsson [5;176 - 177, p.178(Table 1)] and Prudnikov et al. [7; p.551].

The results and definitions which we need in our subsequent work are as follows:

In the usual notations, the Pochhammer's symbol $(a)_n$ is defined by

$$(a)_n = \begin{cases} \dfrac{\Gamma(a+n)}{\Gamma(a)} & ; \quad a \neq 0, -1, -2, \ldots \\ 1 & ; \quad n = 0 \\ a(a+1)(a+2)\ldots(a+n-1) & ; \quad n = 1, 2, 3 \ldots \end{cases} \tag{1.1}$$

If m is either a positive integer or zero, then

$$\frac{(c+1)_m}{(c)_m} = 1 + \frac{m}{c} \tag{1.2}$$

$$\begin{aligned} \frac{(c+2)_m}{(c)_m} &= \frac{(c+1+m)(c+m)}{c(c+1)} \\ &= \frac{c(c+1) + (2c+2)m + m(m-1)}{c(c+1)} \\ &= 1 + \frac{2}{c}m + \frac{m(m-1)}{c(c+1)} \end{aligned} \tag{1.3}$$

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$$ {}_3F_2\left[\begin{array}{ll} a,\ b,\ c+1 & ; \\ & 1 \\ d,\ c & ; \end{array}\right] = \sum_{m=0}^{\infty}\frac{(a)_m\,(b)_m}{(d)_m\,m!}\left[1+\frac{m}{c}\right] $$

$$ = {}_2F_1\left[\begin{array}{ll} a,\ b & ; \\ & 1 \\ d & ; \end{array}\right] + \frac{ab}{cd}\,{}_2F_1\left[\begin{array}{ll} a+1,\ b+1 & ; \\ & 1 \\ d+1 & ; \end{array}\right] $$

$$ = \frac{\Gamma(d)\,\Gamma(d-a-b)}{\Gamma(d-a)\,\Gamma(d-b)} + \frac{ab}{c}\,\frac{\Gamma(d)\,\Gamma(d-a-b-1)}{\Gamma(d-a)\,\Gamma(d-b)} \tag{1.4} $$

Where $\mathrm{Re}(d-a-b)>1$.

and

$$ {}_3F_2\left[\begin{array}{ll} a,\ b,\ c+2 & ; \\ & 1 \\ d,\ c & ; \end{array}\right] = {}_2F_1\left[\begin{array}{ll} a,\ b & ; \\ & 1 \\ d & ; \end{array}\right] + \frac{2ab}{cd}\,{}_2F_1\left[\begin{array}{ll} a+1,\ b+1 & ; \\ & 1 \\ d+1 & ; \end{array}\right] $$

$$ + \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)}\,{}_2F_1\left[\begin{array}{ll} a+2,\ b+2 & ; \\ & 1 \\ d+2 & ; \end{array}\right] $$

$$ = \frac{\Gamma(d)\,\Gamma(d-a-b)}{\Gamma(d-a)\,\Gamma(d-b)} + \frac{2ab\,\Gamma(d)\,\Gamma(d-a-b-1)}{c\,\Gamma(d-a)\,\Gamma(d-b)} + \frac{a(a+1)b(b+1)\Gamma(d)\Gamma(d-a-b-2)}{c(c+1)\Gamma(d-a)\Gamma(d-b)} \tag{1.5} $$

Where $\mathrm{Re}(d-a-b)>2$.

Further, in our study, we shall be using the following identity:

$$ \sum_{m,n,p=0}^{\infty} A(m,n,p) = \sum_{n,p=0}^{\infty}\sum_{m=0}^{p} A(m,n,p-m) \tag{1.6} $$

Srivastava and Daoust Function: Srivastava and Daoust function [9; p.454, see also 11; p.37(21,22,23) and 12; pp.64-65(18,19,20)] is defined as follows:

$$ \mathrm{F}^{A:B';B'';\ldots;B^{(n)}}_{C:D';D'';\ldots;D^{(n)}}\left(\begin{array}{lll} [(a_A):\theta',\ldots,\theta^{(n)}] & :[(b'_{B'}):\phi'];\ldots;[(b^{(n)}_{B^{(n)}}):\phi^{(n)}] & ; \\ & & z_1,z_2,\ldots,z_n \\ [(c_C):\psi',\ldots,\psi^{(n)}] & :[(d'_{D'}):\delta'];\ldots;[(d^{(n)}_{D^{(n)}}):\delta^{(n)}] & ; \end{array}\right) $$

$$ = \sum_{m_1,\ldots,m_n=0}^{\infty}\Omega(m_1,\ldots,m_n)\,\frac{z_1^{m_1}\,z_2^{m_2}\ldots z_n^{m_n}}{m_1!\;m_2!\ldots m_n!} \tag{1.7} $$

Where for convenience,

$$ \Omega(m_1,\ldots,m_n) = \frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\ldots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\ldots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}} \tag{1.8} $$

The coefficients

$$\begin{cases} \theta_j^{(k)}, \quad j=1,\ldots,A; \quad \phi_j^{(k)}, \quad j=1,\ldots,B^{(k)}; \; \psi\,{}_j^{(k)}, \quad j=1,\ldots,C \\ \delta_j^k, \quad j=1,\ldots,D^{(k)}; \quad \text{for all} \quad k\in\{1,2,3,\ldots,n\} \end{cases}$$

are real and positive, and (a_A) abbreviates the array of A parameters $a_1, a_2, \ldots, a_A$, $(b_j{}^{(k)}), j=1,\ldots,B^{(k)}$; for all $k\in\{1,2,\ldots,n\}$, with similar interpretations for (c_C) and $(d\,{}_{D^{(k)}}^{(k)}), k=1,\ldots,n$ et cetra.

The convergence conditions of the multiple series occuring in (1.7) is given by Srivastava and Daoust [8 and 10; see also 2].

In present paper, we establish a multiple series identity (2.1). The investigation of this identity is, infact, immediately connected with the transformations of hypergeometric series of two and three variables, which are obtained in Section 3. Further our main result allows a variety of hypergeometric transformation formulas involving unit and quarter arguments which are not available in the literature. For this reason our results (3.4) to (3.9) seem to be of interest.

II. Multiple Series Identity

Theorem : If $\{S_1(\phi n+\theta p)\}, \{S_2(\gamma p)\}$ and $\{S_3(\delta n)\}$ are the bounded sequences of real or complex numbers; $m,n,p\in\{0,1,2,\ldots\}$; $\alpha,\beta,\theta,\phi,\gamma$ and δ are real constants; the values of a and b are adjusted in such a way that the parameters $\frac{6a+2b+1}{2}$, $2b$, $3a-b+1$ and $b-3a$ are neither zero nor negative integers, then

$$\sum_{m,n,p=0}^{\infty} \frac{(6a)_{3m+\alpha n+p}}{4^m \left(\frac{6a+2b+1}{2}\right)_{m+\left(\frac{\alpha+\beta}{2}\right)n} (3a-b+1)_{m+\left(\frac{\alpha-\beta}{2}\right)n}} S_1(\theta m+\phi n+\theta p)\, S_2(\gamma m+\gamma p)\, S_3(\delta n)$$

$$\times \frac{(-1)^m\, x^{m+p}\, y^n}{m!\; n!\; p!} = \sum_{n,p=0}^{\infty} \frac{(6a)_{\alpha n+p}\,(2b)_{\beta n+p}\,(b-3a)_{\left(\frac{\beta-\alpha}{2}\right)n-p}}{(2b)_{\beta n-p}\left(\frac{6a+2b+1}{2}\right)_{\left(\frac{\alpha+\beta}{2}\right)n+p}(3a-b+1)_{\left(\frac{\alpha-\beta}{2}\right)n}}$$

$$\times\; S_1(\phi n+\theta p)\, S_2(\gamma p)\, S_3(\delta n)\; \frac{y^n\left(\frac{x}{4}\right)^p}{(b-3a)_{\left(\frac{\beta-\alpha}{2}\right)n}\, n!\, p!} \tag{2.1}$$

Provided that each of the multiple series involved converges absolutely.

Proof. Let the left hand side of the Theorem 2.1 is denoted by T, then

$$T = \sum_{m,n,p=0}^{\infty} S_1(\theta m+\phi n+\theta p)\, S_2(\gamma m+\gamma p)\, S_3(\delta n)$$

$$\times \frac{(-1)^m\,(6a)_{3m+\alpha n+p}\, x^{m+p}\, y^n}{4^m\left(\frac{6a+2b+1}{2}\right)_{m+\left(\frac{\alpha+\beta}{2}\right)n}(3a-b+1)_{m+\left(\frac{\alpha-\beta}{2}\right)n}\, m!\; n!\; p!}.$$

Replacing p by $p-m$, we get

$$T = \sum_{p=0}^{\infty}\sum_{m=0}^{p}\sum_{n=0}^{\infty}(-p)_m\, S_1(\phi n+\theta p) S_2(\gamma p) S_3(\delta n)$$

$$\times \frac{(6a)_{2m+\alpha n+p}\, x^p\, y^n}{(1)_p\, 4^m\, \left(\frac{6a+2b+1}{2}\right)_{m+\left(\frac{\alpha+\beta}{2}\right)n} (3a-b+1)_{m+\left(\frac{\alpha-\beta}{2}\right)n}\, m!\; n!}$$

$$= \sum_{p,n=0}^{\infty} \frac{S_1(\phi n+\theta p)\, S_2(\gamma p)\, S_3(\delta n)\, (6a)_{\alpha n+p}\, x^p y^n}{p!\, \left(\frac{6a+2b+1}{2}\right)_{\left(\frac{\alpha+\beta}{2}\right)n} (3a-b+1)_{\left(\frac{\alpha-\beta}{2}\right)n}\, n!}$$

$$\times \sum_{m=0}^{p} \frac{(-p)_m \left(\frac{6a+\alpha n+p}{2}\right)_m \left(\frac{6a+\alpha n+p+1}{2}\right)_m}{\left(\frac{6a+2b+1+\alpha n+\beta n}{2}\right)_m \left(3a-b+1+\frac{\alpha n-\beta n}{2}\right)_m m!}$$

$$= \sum_{p,n=0}^{\infty} \frac{S_1(\phi n+\theta p)\, S_2(\gamma p)\, S_3(\delta n)\, (6a)_{\alpha n+p}\, x^p\, y^n}{\left(\frac{6a+2b+1}{2}\right)_{\left(\frac{\alpha+\beta}{2}\right)n} (3a-b+1)_{\left(\frac{\alpha-\beta}{2}\right)n}\, p!\; n!}$$

$$\times\, {}_3F_2\left[\begin{matrix} -p,\ \dfrac{6a+\alpha n+p}{2},\ \dfrac{6a+\alpha n+p+1}{2} & ; \\ & & 1 \\ \dfrac{6a+2b+1+\alpha n+\beta n}{2},\ \dfrac{6a-2b+2+\alpha n-\beta n}{2} & ; \end{matrix}\right]. \tag{2.2}$$

Now using terminating Saalschütz's summation theorem [1; p.9(2.2.1)]:

$${}_3F_2\left[\begin{matrix} -p,\ A,\ B & ; \\ & & 1 \\ C,\ 1+A+B-C-p & ; \end{matrix}\right] = \frac{(C-A)_p\,(C-B)_p}{(C)_p\,(C-A-B)_p} \tag{2.3}$$

Where p is a non - negative integer, we get

$$T = \sum_{p,n=0}^{\infty} \frac{S_1(\phi n+\theta p)\, S_2(\gamma p)\, S_3(\delta n)\, (6a)_{\alpha n+p}\, x^p\, y^n}{\left(\frac{6a+2b+1}{2}\right)_{\left(\frac{\alpha+\beta}{2}\right)n} (3a-b+1)_{\left(\frac{\alpha-\beta}{2}\right)n}\, p!\; n!}$$

$$\times \frac{\left(\frac{2b+\beta n-p}{2}\right)_p \left(\frac{2b+\beta n-p+1}{2}\right)_p}{\left(\frac{6a+2b+1+\alpha n+\beta n}{2}\right)_p \left(b-3a-p+\frac{\beta n-\alpha n}{2}\right)_p}$$

$$= \sum_{p,n=0}^{\infty} \frac{S_1(\phi n+\theta p)\, S_2(\gamma p)\, S_3(\delta n)\, (6a)_{\alpha n+p}\, x^p\, y^n}{\left(\frac{6a+2b+1}{2}\right)_{\left(\frac{\alpha+\beta}{2}\right)n} (3a-b+1)_{\left(\frac{\alpha-\beta}{2}\right)n}\, p!\; n!}$$

$$\times \frac{(2b+\beta n-p)_{2p} \left(\frac{6a+2b+1}{2}\right)_{\frac{\alpha n+\beta n}{2}} (b-3a)_{\frac{\beta n-\alpha n-2p}{2}}}{2^{2p} \left(\frac{6a+2b+1}{2}\right)_{\frac{2p+\alpha n+\beta n}{2}} (b-3a)_{\frac{\beta n-\alpha n}{2}}}$$

$$= \sum_{p,n=0}^{\infty} \frac{S_1(\phi n+\theta p)\, S_2(\gamma p)\, S_3(\delta n)\, (6a)_{\alpha n+p}\, x^p\, y^n}{\left(\frac{6a+2b+1}{2}\right)_{\left(\frac{\alpha+\beta}{2}\right)n} (3a-b+1)_{\left(\frac{\alpha-\beta}{2}\right)n}\, p!\; n!}$$

$$\times \frac{(2b)_{\beta n+p}\left(\frac{6a+2b+1}{2}\right)_{\frac{\alpha n+\beta n}{2}}(b-3a)_{\frac{\beta n-\alpha n-2p}{2}}}{2^{2p}\,(2b)_{\beta n-p}\left(\frac{6a+2b+1}{2}\right)_{\frac{2p+\alpha n+\beta n}{2}}(b-3a)_{\frac{\beta n-\alpha n}{2}}}$$

$$= \sum_{p,n=0}^{\infty} \frac{S_1(\phi n+\theta p)\,S_2(\gamma p)\,S_3(\delta n)\,(6a)_{\alpha n+p}\,x^p\,y^n}{(3a-b+1)_{\frac{\alpha n-\beta n}{2}}\,p!\,n!}$$

$$\times \frac{(2b)_{\beta n+p}\,(b-3a)_{\frac{\beta n-\alpha n-2p}{2}}}{2^{2p}\,(2b)_{\beta n-p}\left(\frac{6a+2b+1}{2}\right)_{\frac{\alpha n+\beta n+2p}{2}}(b-3a)_{\frac{\beta n-\alpha n}{2}}} \tag{2.4}$$

$$= \sum_{n,p=0}^{\infty} S_1(\phi n+\theta p)\,S_2(\gamma p)\,S_3(\delta n)\,\frac{(6a)_{\alpha n+p}\,y^n\left(\frac{x}{4}\right)^p}{(3a-b+1)_{\frac{\alpha n-\beta n}{2}}\,n!\,p!}$$

$$\times \frac{(2b)_{\beta n+p}\,(b-3a)_{\left(\frac{\beta-\alpha}{2}\right)n-p}}{(2b)_{\beta n-p}\left(\frac{6a+2b+1}{2}\right)_{\left(\frac{\alpha+\beta}{2}\right)n+p}(b-3a)_{\left(\frac{\beta-\alpha}{2}\right)n}}$$

Which is the right hand side of the (2.1).

III. Applications

Suppose the notation (a_A) denotes the sequence of A parameters $a_1, a_2, \ldots, a_A$ and $[(a_A)]_m$ stands for the continued product of "A" Pochhammer's symbols given by $[(a_A)]_m = \prod_{i=1}^{A}(a_i)_m$, with similar interpretation for others.

In Theorem 2.1, setting $\theta=\phi=\gamma=\delta=1,\ S_1(q)=\dfrac{[(d_D)]_q}{[(e_E)]_q},\ S_2(q)=\dfrac{[(g_G)]_q}{[(h_H)]_q}$ and $S_3(q)=\dfrac{[(k_K)]_q}{[(r_R)]_q}$, We get

$$\sum_{m,n,p=0}^{\infty} \frac{[(d_D)]_{m+n+p}\,[(g_G)]_{m+p}\,[(k_K)]_n}{[(e_E)]_{m+n+p}\,[(h_H)]_{m+p}\,[(r_R)]_n}\ \frac{(6a)_{3m+\alpha n+p}\left(\frac{-x}{4}\right)^m y^n\,x^p}{\left(\frac{6a+2b+1}{2}\right)_{m+\left(\frac{\alpha+\beta}{2}\right)n}(3a-b+1)_{m+\left(\frac{\alpha-\beta}{2}\right)n}\,m!\,n!\,p!}$$

$$= \sum_{n,p=0}^{\infty} \frac{[(d_D)]_{n+p}\,[(k_K)]_n\,[(g_G)]_p}{[(e_E)]_{n+p}\,[(r_R)]_n\,[(h_H)]_p}$$

$$\times \frac{(6a)_{\alpha n+p}\,(2b)_{\beta n+p}\,(b-3a)_{\left(\frac{\beta-\alpha}{2}\right)n-p}\,y^n\left(\frac{x}{4}\right)^p}{(2b)_{\beta n-p}\left(\frac{6a+2b+1}{2}\right)_{\left(\frac{\alpha+\beta}{2}\right)n+p}(3a-b+1)_{\left(\frac{\alpha-\beta}{2}\right)n}(b-3a)_{\left(\frac{\beta-\alpha}{2}\right)n}\,n!\,p!}$$

Which can be interpretated in the form of Srivastava and Daoust function as follows:

$$\mathrm{F}^{D+G+1:0;K;0}_{E+H+2:0;R;0}\left(\begin{array}{c} [(d_D):1,1,1],\,[(g_G):1,0,1],\,[6a:3,\,\alpha,1] \quad : \\ [(e_E):1,1,1],\,[(h_H):1,0,1],\,\left[\frac{6a+2b+1}{2}:1,\frac{\alpha+\beta}{2},0\right],\,\left[3a-b+1:1,\frac{\alpha-\beta}{2},0\right] \quad : \end{array}\right.$$

$$\left.\begin{matrix} \_\,;\ [(k_K):1];\ \_\ ; \\ \\ \_\,;\ [(r_R):1];\ \_\ ; \end{matrix}\ -\frac{x}{4},\ y,\ x\right)$$

$$= \mathrm{F}^{D+3:K;G}_{E+2:R+2;H}\left(\begin{matrix} [(d_D):1,1],\ [6a:\alpha,1],\ [2b:\beta,1],\ \left[b-3a:\frac{\beta-\alpha}{2},-1\right] & : \\ [(e_E):1,1],\ [2b:\beta,-1],\ \left[\frac{6a+2b+1}{2}:\frac{\alpha+\beta}{2},1\right] & : \end{matrix}\right.$$

$$\left.\begin{matrix} [(k_K):1] & ;[(g_G):1] & ; \\ [(r_R):1],\ \left[3a-b+1:\frac{\alpha-\beta}{2}\right],\ \left[b-3a:\frac{\beta-\alpha}{2}\right] & ;[(h_H):1] & ; \end{matrix}\ y,\ \frac{x}{4}\right) \tag{3.1}$$

Global Journal of Science Frontier Research Volume XI Issue V Version I August 2011

In (3.1), setting $D = E = G = H = 0$ and using binomial theorem, we get a transformation formula:

$$(1-x)^{-6a}\ \mathrm{F}^{1:0;K}_{2:0;R}\left(\begin{matrix} [6a:3,\ \alpha] & :\_; \\ \left[\frac{6a+2b+1}{2}:1,\ \frac{\alpha+\beta}{2}\right],\ \left[3a-b+1:1,\ \frac{\alpha-\beta}{2}\right] & :\_; \end{matrix}\right.$$

$$\left.\begin{matrix} [(k_K):1] & ; \\ [(r_R):1] & ; \end{matrix}\ \frac{-x}{4(1-x)^3},\ \frac{y}{(1-x)^{\alpha}}\right)$$

$$= \mathrm{F}^{3:K;0}_{2:R+2;0}\left(\begin{matrix} [6a:\alpha,1],\ [2b:\beta,1],\ [b-3a:\frac{\beta-\alpha}{2},-1] & : \\ [2b:\beta,-1],\ [\frac{6a+2b+1}{2}:\frac{\alpha+\beta}{2},1] & : \end{matrix}\right.$$

$$\left.\begin{matrix} [(k_K):1] & ;\_ & ; \\ [(r_R):1],[3a-b+1:\frac{\alpha-\beta}{2}],\ [b-3a:\frac{\beta-\alpha}{2}] & ;\_ & ; \end{matrix}\ y,\ \frac{x}{4}\right). \tag{3.2}$$

In (3.1), put $y=0$ and $D = E = 0$, we get

$$\sum_{m=0}^{\infty}\frac{[(g_G)]_m\,(6a)_{3m}\left(-\frac{x}{4}\right)^m}{[(h_H)]_m\left(\frac{6a+2b+1}{2}\right)_m(3a-b+1)_m\ m!}\ {}_{G+1}F_H\left[\begin{matrix} (g_G)+m,\ 6a+3m & ; \\ (h_H)+m & ; \end{matrix}\ x\right]$$

$$= {}_{G+3}F_{H+2}\left[\begin{matrix} 6a,\ 2b,\ 1-2b,\ (g_G) & ; \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ (h_H) & ; \end{matrix}\ \frac{x}{4}\right]. \tag{3.3}$$

In (3.3), setting $G = H = 1,\ h_1 = h,\ g_1 = -k$ (where k is a non- negative integer) and $x = 1$, we get

$$\sum_{m=0}^{k}\frac{(-k)_m\,(6a)_{3m}\left(-\frac{1}{4}\right)^m}{(h)_m\left(\frac{6a+2b+1}{2}\right)_m(3a-b+1)_m\ m!}\ {}_2F_1\left[\begin{matrix} -k+m,\ 6a+3m & ; \\ h+m & ; \end{matrix}\ 1\right]$$

$$= {}_4F_3\left[\begin{matrix} -k,\ 6a,\ 2b,\ 1-2b & ; \\ & & \dfrac{1}{4} \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ h & ; \end{matrix}\right] \tag{3.4}$$

Where $\mathrm{Re}(\ h-6a)>2k$.

Now on using Gauss first summation theorem [1; p.2(1.3.1)], we get

$$\sum_{m=0}^{k} \frac{(-k)_m\,(6a)_{3m}}{\left(\frac{6a+2b+1}{2}\right)_m\,(3a-b+1)_m\;m!}\;\frac{(h-6a)_k\left(\frac{1-h+6a}{2}\right)_m\left(\frac{2-h+6a}{2}\right)_m}{(27)^m\,(h)_k\left(\frac{1-h-k+6a}{3}\right)_m\left(\frac{2-h-k+6a}{3}\right)_m\left(\frac{3-h-k+6a}{3}\right)_m}$$

$$= {}_4F_3\left[\begin{matrix} -k,\ 6a,\ 2b,\ 1-2b & ; \\ & & \dfrac{1}{4} \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ h & ; \end{matrix}\right]$$

Which on little simplification gives the known transformation of Joshi and Vyas [4; p.1915(4.1)] in the form:

$${}_6F_5\left[\begin{matrix} -k,\ \Delta(3;6a),\Delta(2;1-h+6a) & ; \\ & & 1 \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ \Delta(3;\ 1-h-k+6a) & ; \end{matrix}\right]$$

$$= \frac{(h)_k}{(h-6a)_k}\,{}_4F_3\left[\begin{matrix} -k,\ 6a,\ 2b,\ 1-2b & ; \\ & & \dfrac{1}{4} \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ h & ; \end{matrix}\right] \tag{3.5}$$

Where k is a non-negative integer and $\mathrm{Re}(h-6a)>2k$. Also the notation $\Delta(M;b)$ denotes the M parameters $\dfrac{b}{M},\ \dfrac{b+1}{M},\ \dfrac{b+2}{M},\ldots,\ \dfrac{b+M-1}{M};\ M=1,2,3,\ldots$.

In (3.3), setting $G=H=2,\ g_1=-k,\ g_2=c+1,\ h_1=d,\ h_2=c$ and $x=1$, we get

$$\sum_{m=0}^{k} \frac{(-k)_m\,(c+1)_m\,(6a)_{3m}\,(-1)^m}{(d)_m\,(c)_m\left(\frac{6a+2b+1}{2}\right)_m\,(3a-b+1)_m\,4^m\;m!}\,{}_3F_2\left[\begin{matrix} 6a+3m,\ -k+m,\ c+m+1 & ; \\ & & 1 \\ d+m,\ c+m & ; \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix} -k,\ 6a,\ 2b,\ 1-2b,\ c+1 & ; \\ & & \dfrac{1}{4} \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ c,\ d & ; \end{matrix}\right] \tag{3.6}$$

Where $\mathrm{Re}(d-1-6a)>2k$ and k is a non-negative integer.

Now using summation theorem (1.4) for ${}_3F_2$ in (3.6), we get

$${}_5F_4\left[\begin{matrix} -k,\ 6a,\ 2b,\ 1-2b,\ c+1 & ; \\ & & \dfrac{1}{4} \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ c,\ d & ; \end{matrix}\right]$$

$$= \sum_{m=0}^{k} \frac{(-k)_m\,(c+1)_m\,(6a)_{3m}\,(-1)^m\,\Gamma(d+m)\,\Gamma(d-6a+k-3m)}{(d)_m\,(c)_m\,\left(\frac{6a+2b+1}{2}\right)_m\,(3a-b+1)_m\,4^m\,m!\,\Gamma(d+k)\,\Gamma(d-6a-2m)}$$

$$+ \sum_{m=0}^{k} \frac{(-k)_m\,(c+1)_m\,(6a)_{3m}\,(-1)^m\,(6a+3m)\,(-k+m)\,\Gamma(d+m)\,\Gamma(d-6a+k-1-3m)}{(d)_m\,(c)_m\,\left(\frac{6a+2b+1}{2}\right)_m\,(3a-b+1)_m\,4^m\,m!\,(c+m)\,\Gamma(d+k)\,\Gamma(d-6a-2m)}.$$

Further on using algebraic properties of Pochammer's symbols and making little simpli-fication, we get a new hypergeometric transformation formula:

$$\begin{aligned}
{}_5F_4\left[\begin{matrix} -k,\,6a,\,2b,\,1-2b,\,c+1 \;; \\ 3a+b+\frac{1}{2},\,3a-b+1,\,c,\,d \;; \end{matrix}\;\frac{1}{4}\right]
&= \frac{(d-6a)_k}{(d)_k}\,{}_7F_6\left[\begin{matrix} -k,\,c+1,\,\Delta(3;\,6a),\,\Delta(2;\,1-d+6a) \;; \\ c,\,3a+b+\frac{1}{2},\,3a-b+1,\,\Delta(3;\,1-d+6a-k) \;; \end{matrix}\;1\right] \\
&\quad - \frac{6ak(d-6a)_{k-1}}{c(d)_k}\,{}_6F_5\left[\begin{matrix} -k+1,\,\Delta(3;\,6a+1),\,\Delta(2;\,1-d+6a) \;; \\ 3a+b+\frac{1}{2},\,3a-b+1,\,\Delta(3;\,2-d+6a-k) \;; \end{matrix}\;1\right]
\end{aligned} \tag{3.7}$$

Where $\mathrm{Re}(\,d-1-6a) > 2k.$

In (3.3), setting $G=H=2,\ g_1=-k,\ g_2=c+2,\ h_1=d,\ h_2=c$ and $x\,=1,$ we get

$$\begin{aligned}
\sum_{m=0}^{k} \frac{(-k)_m\,(c+2)_m\,(6a)_{3m}\,(-1)^m}{(d)_m\,(c)_m\,\left(\frac{6a+2b+1}{2}\right)_m\,(3a-b+1)_m\,4^m\,m!}\,{}_3F_2\left[\begin{matrix} 6a+3m,\,-k+m,\,c+m+2 \;; \\ d+m,\,c+m \;; \end{matrix}\;1\right] \\
= {}_5F_4\left[\begin{matrix} -k,\,6a,\,2b,\,1-2b,\,c+2 \;; \\ 3a+b+\frac{1}{2},\,3a-b+1,\,c,\,d \;; \end{matrix}\;\frac{1}{4}\right]
\end{aligned} \tag{3.8}$$

Where $\mathrm{Re}(\,d-2-6a) > 2k$ and k is a non-negative integer.

Again on making use of summation theorem (1.5) for ${}_3F_2$ in (3.8), we get

$${}_5F_4\left[\begin{matrix} -k,\,6a,\,2b,\,1-2b,\,c+2 \;; \\ 3a+b+\frac{1}{2},\,3a-b+1,\,c,\,d \;; \end{matrix}\;\frac{1}{4}\right] = \sum_{m=0}^{k} \frac{(-k)_m\,(c+2)_m\,(6a)_{3m}\,(-1)^m}{(d)_m\,(c)_m\,\left(\frac{6a+2b+1}{2}\right)_m\,(3a-b+1)_m\,4^m\,m!}$$

$$\times \left\{ \frac{\Gamma(d+m)\,\Gamma(d-6a-3m+k)}{\Gamma(d-6a-2m)\,\Gamma(d+k)} + \frac{2(6a+3m)\,(-k+m)\,\Gamma(d+m)\,\Gamma(d-6a-3m+k-1)}{(c+m)\,\Gamma(d-6a-2m)\,\Gamma(d+k)} \right.$$

$$\left. + \frac{(6a+3m)\,(6a+3m+1)\,(-k+m)\,(-k+m+1)\,\Gamma(d+m)\;\Gamma(d-6a-3m+k-2)}{(c+m)\,(c+m+1)\,\Gamma(d-6a-2m)\,\Gamma(d+k)} \right\}$$

Which on simplification gives:

$$
{}_5F_4\left[\begin{array}{ll} -k,\ 6a,\ 2b,\ 1-2b,\ c+2 & ; \\ & \frac{1}{4} \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ c,\ d & ; \end{array}\right]
$$

$$
= \frac{(d-6a)_k}{(d)_k}\,{}_7F_6\left[\begin{array}{ll} -k,\ c+2,\ \Delta(3;\ 6a),\ \Delta(2;\ 1-d+6a) & ; \\ & 1 \\ c,\ 3a+b+\frac{1}{2},\ 3a-b+1,\ \Delta(3;\ 1-d+6a-k) & ; \end{array}\right]
$$

$$
-\frac{12ak(d-6a)_{k-1}}{c(d)_k}\,{}_7F_6\left[\begin{array}{ll} -k+1,\ c+2,\ \Delta(3;\ 6a+1),\ \Delta(2;\ 1-d+6a) & ; \\ & 1 \\ c+1,\ 3a+b+\frac{1}{2},\ 3a-b+1,\ \Delta(3;\ 2-d+6a-k) & ; \end{array}\right]
$$

$$
+\frac{k(k-1)\,(6a)\,(6a+1)\,(d-6a)_{k-2}}{c(c+1)\,(d)_k}
$$

$$
\times\,{}_6F_5\left[\begin{array}{ll} -k+2,\ \Delta(3;\ 6a+2),\ \Delta(2;\ 1-d+6a) & ; \\ & 1 \\ 3a+b+\frac{1}{2},\ 3a-b+1,\ \Delta(3;\ 3-d+6a-k) & ; \end{array}\right] \tag{3.9}
$$

Where $\mathrm{Re}(d-2-6a) > 2k$ and k is a non-negative integer.

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