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Some Definite Integrals of Gradshteyn-Ryzhik and Other Integrals

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Strictly as per the compliance and regulations of:

Some Definite Integrals of Gradshteyn-Ryzhik and Other Integrals

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I. Introduction

The Pochhammer's symbol or Appell's symbol or shifted factorial or rising factorial or generalized factorial function is defined by

$$(b,\ k) = (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } \ k \in \mathbb{N} \\ 1 \quad ; & \text{if } \ k = 0 \\ k! \quad ; & \text{if } \ b = 1,\ k \in \mathbb{N} \end{cases}$$

Where b is neither zero nor negative integer and the notation Γ stands for Gamma function. Throughout this work we shall employ the following definitions.

Generalized Gaussian Hypergeometric Function

Generalized ordinary hypergeometric function of one variable [4,p.73(2);5,p.42(1)] is defined by

$$ {}_AF_B\left[\begin{array}{lll} (a_j)_{j=1}^A & ; & \\ & & z \\ (b_j)_{j=1}^B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{((a_A))_k z^k}{((b_B))_k k!} \qquad (1.1)$$

Where denominator parameters $b_1, b_2, \ldots, b_B$ are neither zero nor negative integers and A, B are non-negative integers. The symbol $(a_j)_{j=1}^A$ represents the array of A parameters given by $a_1, a_2, \ldots, a_A$ with similar interpretation for others.

Conditions for Convergence of (1.1)

If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z (real or complex).

If A = B + 1, then series ${}_AF_B$ is convergent for $|z| < 1$.

For more convergence conditions we refer [4,pp.73-74;5,p.43].

Kampé de Fériet's General Double Hypergeometric Function

We recall the definition of general double hypergeometric function of Kampé de Fériet in slightly modified notation of H.M.Srivastava and R.Panda [5,pp.63-64(16,17)]:

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$$F^{A:B;D}_{E:G;H}\left[\begin{array}{ccc}(a_j)^A_{j=1}:(b_j)^B_{j=1};(d_j)^D_{j=1} & ; & \\ & & x,\,y \\ (e_j)^E_{j=1}:(g_j)^G_{j=1};(h_j)^H_{j=1} & ; & \end{array}\right] = \sum_{m,n=0}^{\infty} \frac{((a_A))_{m+n}\,((b_B))_m\,((d_D))_n\,x^m\,y^n}{((e_E))_{m+n}\,((g_G))_m\,((h_H))_n\,m!\,n!} \tag{1.2}$$

Conditions for Convergence of (1.2)

(i) $A+B<E+G+1,\ A+D<E+H+1,\ \ |x|<\infty,\ |y|<\infty,$ or

(ii) $A+B=E+G+1,\ A+D=E+H+1,$ and

$$\left\{\begin{array}{ll}|x|^{\frac{1}{(A-E)}}+|y|^{\frac{1}{(A-E)}}<1 & ,\text{if } A>E \\ \max\{|x|,|y|\}<1 & ,\text{if } A\le E\end{array}\right\}$$

Leibnitz Rule for Differentiation Under the Integral Sign[3]

If $F(x,\alpha)$ and $\frac{\partial}{\partial\alpha}F(x,\alpha)$ are continuous functions of x and α, then

$$\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} F(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)}\left\{\frac{\partial}{\partial\alpha}F(x,\alpha)\right\}dx + F(\psi(\alpha),\alpha)\frac{d\psi}{d\alpha} - F(\phi(\alpha),\alpha)\frac{d\phi}{d\alpha} \tag{1.3}$$

provided that $\phi(\alpha)$ and $\psi(\alpha)$ possesses continuous first order derivatives with respect to α.

Wallis' Formula

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\,\cos^n\theta\,d\theta = \frac{\Gamma(\frac{m+1}{2})\,\Gamma(\frac{n+1}{2})}{2\,\Gamma(\frac{m+n+2}{2})};\quad \Re(m)>-1,\ \Re(n)>-1 \tag{1.4}$$

Master Integral

In a paper of Boros and Moll [2,p.972, see also p.974(Th.1)], the following master formula

$$\int_0^{\infty}\left[\frac{x^2}{x^4+2ax^2+1}\right]^r \frac{x^2+1}{x^b+1}\,\frac{dx}{x^2} = 2^{-\frac{1}{2}-r}(1+a)^{\frac{1}{2}-r}\frac{\sqrt{\pi}\,\Gamma(r-\frac{1}{2})}{\Gamma(r)} \tag{1.5}$$

$$\left(a>-1,\ r>\frac{1}{2}\right)$$

Was used to evaluate a large number of definite integrals.

In the continuation of master integral, we evaluated certain definite integrals in sections 4 and 5.

II. Some Integrals of Gradshteyn and Ryzhik

[1,p.20(4);2,p.974(2.1);3,p.346(3.257]

$$\int_0^{\infty}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}dx = \frac{\sqrt{\pi}}{2a(c)^{p+\frac{1}{2}}}\,\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} \tag{2.1}$$

$$\left(a>0,\ b<0,\ c>0,\ \Re(p)+\frac{1}{2}>0\right)$$

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[1,p.20(19);3,p.351(3.276(1))]

$$\int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\mathrm{d}x=\frac{\sqrt{\pi}}{2b(c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} \tag{2.2}$$

$$\left(a<0,\ b>0,\ c>0,\ \Re(p)+\frac{1}{2}>0\right)$$

Under the stated conditions, integrals (2.1) and (2.2) are true. Since these conditions are not given in the table of integrals[3].

[1,p.20(5);3,p.351(3.276(2))]

$$\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\mathrm{d}x=\frac{\sqrt{\pi}}{(c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)};\qquad \Re(p)+\frac{1}{2}>0 \tag{2 .3}$$

Under any condition on $a,\ b,\ c$ and p, the integral (2.3) is not true.

III. Other Forms of Above Integrals

$$\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\mathrm{d}x=\frac{\sqrt{\pi}}{2a(4ab+c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} \tag{3.1}$$

$$\left(a>0;\ b\geq 0;\ c+4ab>0;\ \Re(p)+\frac{1}{2}>0\right)$$

$$\int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\mathrm{d}x=\frac{\sqrt{\pi}}{2b\,(4ab+c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} \tag{3.2}$$

$$\left(a\geq 0;\ b>0;\ c+4ab>0;\ \Re(p)+\frac{1}{2}>0\right)$$

$$\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\mathrm{d}x=\frac{\sqrt{\pi}}{(4ab+c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} \tag{3.3}$$

$$\left(a>0;\ b>0;\ c+4ab>0;\ \Re(p)+\frac{1}{2}>0\right)$$

IV. Proofs of (3.1) - (3.3)

Suppose left hand side of (3.1) is denoted by

$$I(b)=\int_0^\infty \left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\mathrm{d}x;\quad b\geq 0 \tag{4.1}$$

Therefore

$$I(0)=\int_0^\infty \frac{\mathrm{d}x}{(a^2x^2+c)^{p+1}}=\frac{1}{a(c)^{p+\frac{1}{2}}}\int_0^{\frac{\pi}{2}}\cos^{2p}\theta\,\mathrm{d}\theta=\frac{\sqrt{\pi}}{2a(c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} \tag{4.2}$$

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If we denote left hand side of (3.1) by $I_1^*(a)$, then $I_1^*(0)$ can not be calculated due to the divergent nature of resulting integral.

Differentiate (4.1) with respect to b and apply Leibnitz rule (1.3), we get

$$\frac{\mathrm{d}I}{\mathrm{d}b} = -2(p+1)\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-2}\mathrm{d}x$$

$$= -2(p+1)\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax-\frac{b}{x}\right)^2+(c+4ab)\right]^{-p-2}\mathrm{d}x$$

$$= \frac{-4(p+1)}{(4ab+c)^{\frac{2p+3}{2}}}\int_0^{\frac{\pi}{2}}\cos^{2p+2}\theta\,\mathrm{d}\theta$$

Or

$$\mathrm{d}I = \frac{-\sqrt{\pi}(2p+1)\Gamma(p+\frac{1}{2})}{(4ab+c)^{\frac{2p+3}{2}}\Gamma(p+1)}\mathrm{d}b \quad (4.3)$$

Now integrate (4.3), we get

$$I(b) = \frac{\sqrt{\pi}}{2a(4ab+c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} + H \quad (4.4)$$

Where H is constant of integration.

By putting $b=0$ in (4.4) and in view of the result (4.2), we get $H=0$, therefore (4.4) reduces to right hand side of (3.1).

Similarly, if we denote the left hand of (3.2) by

$$I(a) = \int_0^\infty \frac{1}{x^2}\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\mathrm{d}x;\quad a\geq 0 \quad (4.5)$$

Then

$$I(0) = \int_0^\infty \frac{x^{2p}}{(b^2+cx^2)^{p+1}}\mathrm{d}x = \frac{\sqrt{\pi}}{2b(c)^{p+\frac{1}{2}}}\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)} \quad (4.6)$$

If we denote left hand side of (3.2) by $I_2^*(b)$, , then I_2^* (0) can not be calculated due to the divergent nature of resulting integral.

Differentiate (4.5) with respect to a and apply Leibnitz rule (1.3), we get

$$\frac{\mathrm{d}I}{\mathrm{d}a} = -2(p+1)\int_0^\infty \left(a+\frac{b}{x^2}\right)\left[\left(ax-\frac{b}{x}\right)^2+(4ab+c)\right]^{-p-2}\mathrm{d}x = \frac{-(2p+1)\sqrt{\pi}\Gamma(p+\frac{1}{2})}{(4ab+c)^{\frac{2p+3}{2}}\Gamma(p+1)} \quad (4.7)$$

Now integrate (4.7), we get

$$I(a) = \frac{\sqrt{\pi}\,\Gamma(p+\frac{1}{2})}{2b(4ab+c)^{\frac{2p+1}{2}}\Gamma(p+1)} + G \quad (4.8)$$

Where G is constant of integration.

When $a=0$ in (4.8) and in view of the result (4.6), we get $G=0$ therefore (4.8) reduces to right hand side of (3.2).

We can not apply Leibnitz rule (1.3) in the left hand side of (3.3).

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The left hand side of (3.3) is denoted by

$$I=\int_0^\infty\left(a+\frac{b}{x^2}\right)\left[\left(ax+\frac{b}{x}\right)^2+c\right]^{-p-1}\mathrm{d}x$$

$$=\int_0^\infty\left(a+\frac{b}{x^2}\right)\left[\left(ax-\frac{b}{x}\right)^2+(4ab+c)\right]^{-p-1}\mathrm{d}x$$

$$=\frac{2}{(4ab+c)^{p+\frac{1}{2}}}\int_0^{\frac{\pi}{2}}\cos^{2p}\theta\,\mathrm{d}\theta$$

On solving above integral with the help of (1.4), we get the right hand side of (3.3).

Or, if we multiply both sides of (3.1) by a, multiply both sides of (3.2) by b and adding the resulting integrals, we can obtain (3.3).

V. Additional Integrals

Since Pochhammer's symbol is associated with Gamma function and Gamma function is undefined for zero and negative integers, therefore arguments, numerator and denominator parameters are adjusted in such a way that following integrals are completely well defined and meaningful then without any loss of convergence, we have

$$\int_0^\infty e^{-ax-bx^2}\,\mathrm{d}x=\frac{\sqrt{\pi}}{2\sqrt{b}}\,e^{\frac{a^2}{4b}}-\frac{a}{2b}\,{}_1F_1\left[\begin{array}{ccc}1 & ; & \\ & & \frac{a^2}{4b}\\ \frac{3}{2} & ; & \end{array}\right];\ a\geq 0,\ b>0 \qquad (5.1)$$

In view of Leibnitz rule (1.3) and Kummer's first transformation [4,p.125(Th.42)] and using same technique,we can derive (5.1).

Using series expansions and hypergeometric forms [4,p.108(1),p.115(2,4)] of Sine,Cosine functions and ordinary Bessel function of first kind,a reduction formula for the product of two ${}_0F_1[4,\mathrm{p.}105(\mathrm{Q.No.}1)]$,interchanging the order of summation and integration, using series rearrangement technique and some algebraic properties of Pochhammer's symbol, we can derive the integrals (5.2)-(5.5) which are convergent for all finite values of parameters.

$$\int_0^t\cos(ax)\,\mathrm{J}_\nu(bx)\,\mathrm{d}x=\frac{b^\nu\,t^{\nu+1}}{2^\nu\,\Gamma(\nu+2)}\,F^{1:0;0}_{1:1;1}\left[\begin{array}{l}\frac{\nu+1}{2}:\text{——};\text{——};\\ \qquad\qquad\qquad\qquad -\frac{a^2t^2}{4},\ -\frac{b^2t^2}{4}\\ \frac{\nu+3}{2}:\ \frac{1}{2}\ ;\nu+1\ ;\end{array}\right] \qquad (5.2)$$

where $b\neq a$ and $\nu\neq-1$.

$$\int_0^t\sin(ax)\,\mathrm{J}_\nu(bx)\,\mathrm{d}x=\frac{a\,b^\nu\,t^{\nu+2}}{2^\nu\,(\nu+2)\Gamma(\nu+1)}\,F^{1:0;0}_{1:1;1}\left[\begin{array}{l}\frac{\nu+2}{2}:\text{——};\text{——};\\ \qquad\qquad\qquad\qquad -\frac{a^2t^2}{4},\ -\frac{b^2t^2}{4}\\ \frac{\nu+4}{2}:\ \frac{3}{2}\ ;\nu+1\ ;\end{array}\right] \qquad (5.3)$$

where $b\neq a$ and $\nu\neq-2$.

$$\int_0^t\cos(ax)\,\mathrm{J}_\nu(ax)\,\mathrm{d}x=\frac{a^\nu\,t^{\nu+1}}{2^\nu\,\Gamma(\nu+2)}\,{}_3F_4\left[\begin{array}{l}\frac{\nu+1}{2},\ \frac{2\nu+1}{4},\ \frac{2\nu+3}{4}\ ;\\ \qquad\qquad\qquad\qquad -a^2\,t^2\\ \frac{1}{2},\ \nu+1,\ \frac{2\nu+1}{2},\ \frac{\nu+3}{2}\ ;\end{array}\right] \qquad (5.4)$$

where $\nu\neq-1$.

$$\int_0^t\sin(ax)\,\mathrm{J}_\nu(ax)\,\mathrm{d}x=\frac{a^{\nu+1}\,t^{\nu+2}}{2^\nu\,(\nu+2)\,\Gamma(\nu+1)}\,{}_3F_4\left[\begin{array}{l}\frac{\nu+2}{2},\ \frac{2\nu+3}{4},\ \frac{2\nu+5}{4}\ ;\\ \qquad\qquad\qquad\qquad -a^2\,t^2\\ \frac{3}{2},\ \nu+1,\ \frac{2\nu+3}{2},\ \frac{\nu+4}{2}\ ;\end{array}\right] \qquad (5.5)$$

where $\nu\neq-2$.

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