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Two Ordinary Hypergeometric Definite Integrals Involving Ramanujan's Formula

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TWO ORDINARY HYPERGEOMETRIC DEFINITE INTEGRALS INVOLVING RAMANUJANS FORMULA

Strictly as per the compliance and regulations of :

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I. Introduction

The Pochhammer's symbol is defined by

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k = 1,2,3,\cdots \\ 1 \qquad ; & \text{if } k = 0 \\ k! \qquad ; & \text{if } b = 1,\ k = 1,2,3,\cdots \end{cases}$$

where b is neither zero nor negative integer and the notation Γ stands for Gamma function.

$$(\lambda)_{mn} = m^{mn}\left(\frac{\lambda}{m}\right)_n\left(\frac{\lambda+1}{m}\right)_n\left(\frac{\lambda+1}{m}\right)_n\cdots\left(\frac{\lambda+m-1}{m}\right)_n$$

where m is a positive integer and n is a non negative integer.

The notation $\Delta(N;b)$ is used to denote the set of N parameters given by $\frac{b}{N}, \frac{b+1}{N}, \ldots, \frac{b+N-1}{N}$.

Generalized Gaussian hypergeometric function of one variable is defined by

$${}_AF_B\left[\begin{matrix} a_1, a_2, \cdots, a_A & ; \\ b_1, b_2, \cdots, b_B & ; \end{matrix}\quad z\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_k z^k}{(b_1)_k(b_2)_k\cdots(b_B)_k k!}$$

or

$${}_AF_B\left[\begin{matrix} (a_A) & ; \\ (b_B) & ; \end{matrix}\quad z\right] = \sum_{k=0}^{\infty}\frac{[(a_A)]_k z^k}{[(b_B)]_k k!}$$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and A, B are non-negative integers.

If A _ B, then series ${}_AF_B$ is always convergent for all finite values of z(real or complex).

In 1933, E. M. Wright defined a more interesting generalized hypergeometric function of one variable[12, p.287; 13 , p.11(1.7.8)] in the following forms:

$${}_p\Psi_q\left[\begin{matrix} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) ; \\ \\ (\beta_1, B_1), \cdots, (\beta_q, B_q) ; \end{matrix}\quad z\right] = \sum_{n=0}^{\infty}\frac{\Gamma(n\alpha_1 + A_1)\ \Gamma(n\alpha_2 + A_2)\cdots\Gamma(n\alpha_p + A_p)\ z^n}{\Gamma(n\beta_1 + B_1)\ \Gamma(n\beta_2 + B_2)\cdots\Gamma(n\beta_q + B_q)\ n!}$$

$$= H^{1,\ p}_{p,\ q+1}\left[-z\ \middle|\ \begin{matrix} (1-\alpha_1, A_1), \ldots\ldots\ldots\ldots, (1-\alpha_p, A_p) \\ \\ (0,1), (1-\beta_1, B_1), \ldots\ldots, (1-\beta_q, B_q) \end{matrix}\right]$$

where the coefficients $A_1,\ A_2,\ A_3, \cdots,\ A_p,\ B_1,\ B_2,\ B_3\ \cdots,\ B_q$ are positive rational numbers and $\alpha_1,\ \alpha_2,\ \alpha_3, \cdots, \alpha_p,\ \beta_1,\ \beta_2,\ \beta_3, \cdots, \beta_q$ are complex parameters .

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$$ {}_p\Psi_q^* \left[\begin{matrix} (\alpha_1, A_1), \cdots, (\alpha_p, A_p) \ ; \\ \\ (\beta_1, B_1), \cdots, (\beta_q, B_q) \ ; \end{matrix} \ z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{nA_1} \, (\alpha_2)_{nA_2} \cdots (\alpha_p)_{nA_p} \ z^n}{(\beta_1)_{nB_1} \, (\beta_2)_{nB_2} \cdots (\beta_q)_{nB_q} \ n!} $$

The Fox H-function makes sense when either

$$ \delta \ \equiv \ (1 + B_1 + B_2 + \cdots + B_q) - (A_1 + A_2 + \cdots + A_p) > 0 $$

and $\quad 0 < |z| < \infty \quad ; \quad z \neq 0$

The equality holds only for suitably constrained values of |z| or appropriately bounded $|z|$ i.e. $\delta = 0$ and $0 < |z| < R \equiv {A_1}^{-A_1} {A_2}^{-A_2} \cdots {A_p}^{-A_p} {B_1}^{B_1} {B_2}^{B_2} \cdots {B_q}^{B_q}$.

$$ {}_p\Psi_q \left[\begin{matrix} (\alpha_1, 1), \cdots, (\alpha_p, 1) \ ; \\ \\ (\beta_1, 1), \cdots, (\beta_q, 1) \ ; \end{matrix} \ z \right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \ {}_pF_q \left[\begin{matrix} \alpha_1, \cdots, \alpha_p \ ; \\ \\ \beta_1, \cdots, \beta_q \ ; \end{matrix} \ z \right] $$

The function $H_{p,q}^{m,n}$ was given by C. F. Fox in 1961.

$$ H_{p,q}^{m,n} \left[z \ \middle| \ \begin{matrix} (a_1, A_1), \, (a_2, A_2), \cdots, (a_n, A_n), \, (a_{n+1}, \, A_{n+1}), \cdots, (a_p, A_p) \\ \\ (b_1, B_1), \, (b_2, B_2), \cdots, (b_m, B_m), \, (b_{m+1}, B_{m+1}), \cdots, (b_q, B_q) \end{matrix} \right] $$

$$ = H_{p,q}^{m,n} \left[z \ \middle| \ \begin{matrix} ((a_p, A_p)) \\ \\ ((b_q, B_q)) \end{matrix} \right] = H_{q,p}^{n,m} \left[\frac{1}{z} \ \middle| \ \begin{matrix} ((1 - b_q, B_q)) \\ \\ ((1 - a_p, A_p)) \end{matrix} \right] $$

where $m, \ n, \ p, \ q$ are non-negative integers such that $p \geq n \geq 0, \ \ q \geq m \geq 1. \, a_j, \ \ b_j$ are complex numbers and are positive numbers.

The H-function is an analytic function of z and makes sense if the following existence conditions are satisfied.

For all $z \neq 0$ and $(B_1 + B_2 + \cdots + B_q) - (A_1 + A_2 + \cdots + A_p) > 0$. For $0 < |z| < {A_1}^{-A_1} {A_2}^{-A_2} \cdots {A_p}^{-A_p} {B_1}^{B_1} {B_2}^{B_2} \cdots {B_q}^{B_q}$ and $(B_1 + B_2 + \cdots + B_q) - (A_1 + A_2 + \cdots + A_p) = 0$.

The H-function is symmetric in the set of pairs $(a_1, A_1), \cdots, (a_n, A_n)$; in the set of pairs $(a_{n+1}, A_{n+1}), \cdots, (a_p, A_p)$ in the set of pairs $(b_1, B_1), \cdots, (b_m, B_m)$ in the set of pairs $(b_{m+1}, B_{m+1}), \cdots, (b_q, B_q)$.

Ramanujan's Formula is defined as

Ramanujan[1,p.191(2.20,2.21)]

$$ \int_0^\infty x^{P-1} \left(\frac{2}{1 + \sqrt{1 + 4x}} \right)^N dx = \frac{N \ \Gamma(P) \Gamma(N - 2P)}{\Gamma(N - P + 1)} $$

where, $N > 0$ and $0 < P < \frac{N}{2}$.

II. Main Formulae of The Integrals

$$ \int_0^\infty \frac{x^{c-1}}{\left(x + \sqrt{1 + x^2} \right)^p} \ {}_AF_B \left[\begin{matrix} (a_A) \ ; \\ \\ (b_B) \ ; \end{matrix} \ \ y x^g \left(\frac{1}{x + \sqrt{1 + x^2}} \right)^h \right] dx $$

$$= \frac{\prod_{i=1}^{B}\Gamma(b_i)}{2^{c+1}\prod_{j=1}^{A}\Gamma(a_j)}\, {}_{A+3}\Psi_{B+2}\left[\begin{matrix}(a_1,1),\,\ldots,\,(a_A,1),\,(p+1,h),\,(\frac{p-c}{2},\frac{h-g}{2}),\,(c,g)\ ;\\ (b_1,1),\,\ldots,\,(b_B,1),\,(p,h),\,(\frac{p+c+2}{2},\frac{h+g}{2})\ ;\end{matrix}\ \frac{y}{2^g}\right] \tag{2.1a}$$

$$= \frac{p\,\Gamma(c)\,\Gamma(\frac{p-c}{2})}{2^{c+1}\,\Gamma(\frac{p+c+2}{2})}\, {}_{A+3}\Psi^{*}_{B+2}\left[\begin{matrix}(a_1,1),\,\ldots,\,(a_A,1),\,(p+1,h),\,(\frac{p-c}{2},\frac{h-g}{2}),\,(c,g)\ ;\\ (b_1,1),\ldots,\,(b_B,1),\,(p,h),\,(\frac{p+c+2}{2},\frac{h+g}{2})\ ;\end{matrix}\ \frac{y}{2^g}\right] \tag{2.1b}$$

$$\int_0^\infty \frac{x^{c-1}}{\left(x+\sqrt{1+x^2}\right)^k}\, {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\,\cdots,\,(\alpha_p,A_p)\ ;\\ (\delta_1,B_1),\,\cdots,\,(\delta_q,B_q)\ ;\end{matrix}\ yx^g\left(\frac{1}{x+\sqrt{1+x^2}}\right)^h\right]dx$$

$$= \frac{\prod_{i=1}^{p}\Gamma(\delta_i)}{2^{c+1}\prod_{j=1}^{q}\Gamma(\alpha_j)}\, {}_{p+3}\Psi_{q+2}\left[\begin{matrix}(\alpha_1,A_1),\,\cdots,\,(\alpha_p,A_p),\,(k+1,h),\,(\frac{k-c}{2},\frac{h-g}{2}),\,(c,g)\ ;\\ (\delta_1,B_1),\,\cdots,\,(\delta_q,B_q),\,(k,h),\,(\frac{k+c+2}{2},\frac{h+g}{2})\ ;\end{matrix}\ \frac{y}{2^g}\right] \tag{2.2a}$$

$$= \frac{k\,\Gamma(c)\,\Gamma(\frac{k-c}{2})}{2^{c+1}\Gamma(\frac{k+c+2}{2})}\, {}_{p+3}\Psi^{*}_{q+2}\left[\begin{matrix}(\alpha_1,A_1),\,\cdots,\,(\alpha_p,A_p),\,(k+1,h),\,(\frac{k-c}{2},\frac{h-g}{2}),\,(c,g)\ ;\\ (\delta_1,B_1),\,\cdots,\,(\delta_q,B_q),\,(k,h),\,(\frac{k+c+2}{2},\frac{h+g}{2})\ ;\end{matrix}\ \frac{y}{2^g}\right] \tag{2.2b}$$

III. Derivations of the Main Formulae

First of all we shall derive the integrals (2.1a) and (2.1b). Let the integral in left hand side is denoted by Υ

$$\Upsilon = \int_0^\infty \frac{x^{c-1}}{\left(x+\sqrt{1+x^2}\,\right)^p}\, {}_AF_B\left[\begin{matrix}(a_A)\ ;\\ (b_B)\ ;\end{matrix}\ yx^g\left(\frac{1}{x+\sqrt{1+x^2}}\right)^h\right]dx$$

$$= \int_0^\infty \frac{x^{c-1}}{\left(x+\sqrt{1+x^2}\,\right)^p}\left\{\sum_{m=0}^{\infty}\frac{[(a_A)]_m\,y^m\,x^{gm}}{[(b_B)]_m\,m!}\left(\frac{1}{(x+\sqrt{1+x^2})^{hm}}\right)\right\}dx$$

$$= \sum_{m=0}^{\infty}\frac{[(a_A)]_m\,(p+hm)\,\Gamma(gm+c)\,\Gamma(\frac{p+hm-c-gm}{2})\,y^m}{[(b_B)]_m\,2^{c+gm+1}\,\Gamma(\frac{p+hm+c+gm+2}{2})\,m!}$$

$$= \sum_{m=0}^{\infty}\frac{[(a_A)]_m\,\Gamma(p+hm+1)\,\Gamma(gm+c)\,\Gamma((\frac{p-c}{2}+(\frac{h-g}{2})m)\,y^m}{[(b_B)]_m\,2^{c+1}\,2^{gm}\,\Gamma(p+hm)\Gamma((\frac{p+c+2}{2}+(\frac{h+g}{2})m)\,m!}$$

$$= \frac{\prod_{i=1}^{B}\Gamma(b_i)}{2^{c+1}\prod_{j=1}^{A}\Gamma(a_j)}\, {}_{A+3}\Psi_{B+2}\left[\begin{matrix}(a_1,1),\,\ldots,\,(a_A,1),\,(p+1,h),\,(\frac{p-c}{2},\frac{h-g}{2}),\,(c,g)\ ;\\ (b_1,1),\,\ldots,\,(b_B,1),\,(p,h),\,(\frac{p+c+2}{2},\frac{h+g}{2})\ ;\end{matrix}\ \frac{y}{2^g}\right] \tag{3.1a}$$

On the same way ,it is proved that

$$\Upsilon = \frac{p\,\Gamma(c)\,\Gamma(\frac{p-c}{2})}{2^{c+1}\,\Gamma(\frac{p+c+2}{2})}\,{}_{A+3}\Psi^{*}_{B+2}\left[\begin{array}{ll}(a_1,1),\,\ldots,\,(a_A,1),\,(p+1,h),\,(\frac{p-c}{2},\frac{h-g}{2}),\,(c,g) & ; \\ & \\ (b_1,1),\ldots,\,(b_B,1),\,(p,h),\,(\frac{p+c+2}{2},\frac{h+g}{2}) & ;\end{array}\;\frac{y}{2^g}\right] \quad \text{(3.1b)}$$

Similarly we can obtain remaining integrals on the same parallel lines of the derivation of (2.1a) and (2.1b).

The integrals established here are quite general in nature due to the presence of general quadruple hypergeometric function of Saigo.

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