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A Class of Convolution Integral Equations and Special Functions

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A CLASS OF CONVOLUTION INTEGRAL EQUATIONS AND SPECIAL FUNCTIONS

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A Class of Convolution Integral Equations and Special Functions

V.B.L.Chaurasia[α] , Amber Srivastava[Ω]

Abstract - In the present paper a convolution integral equation of Fredholm type whose kernel involves a product of generalized polynomial set, general multivariable polynomials, Fox's H-function and $\overline{\text{H}}$ -function, has been solved by using the theory of Mellin transforms. Our main result is believed to be general and unified in nature. A number of (known and new) results follow as special cases by specializing the coefficients and parameters involved in the kernel.

Keywords and phrases : Convolution integral equation , Generalized polynomial set, General multivariable polynomials, Fox's H-function, $\overline{\text{H}}$ *-function, Mellin transforms.*

I. Introduction

In the present paper, we have used the following special case of Raizada's generalized polynomial set defined as [9] :

$$S_n^{\alpha,\beta,0}[x;\, r,q,A,B,k,\ell] = (Ax+B)^{-\alpha}\exp(\beta x^r)\, T_{k,\ell}^{n}[(Ax+B)^{\alpha+qn}\exp(-\beta x^r)] \qquad \ldots(1.1)$$

whose explicit form is

$$S_n^{\alpha,\beta,0}[x;\, r,q,A,B,k,\ell] = \sum_{e,p,u,v} \varphi(e,p,u,v)\, x^L , \qquad \ldots (1.2)$$

where

$$\varphi(e,p,u,v) = B^{qn-p}\,\ell^n \frac{(-1)^p(-v)_u(-p)_e(\alpha)_p}{u!\; v!\; e!\; p!}\, \frac{(-\alpha-qn)_e}{(1-\alpha-p)_e}\left(\frac{e+k+ru}{\ell}\right)_n A^p\beta^v , \qquad \ldots (1.3)$$

$$L = \ell n + p + rv , \ (p,v = 0,1,\ldots,n), \qquad \ldots (1.4)$$

and

$$\sum_{e,p,u,v} \equiv \sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{p=0}^{n}\sum_{e=0}^{p} \qquad \ldots (1.5)$$

The polynomial set defined by (1.2) is very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [3] , [2], Gould and Hopper [5] , Krall and Frink [8] , Srivastava and Singhal [13] etc. Some of its special cases are given by Raizada [9] in tabular form.

The main object of the present paper is to derive an exact solution of the following convolution integral equation of Fredholm type

$$\int_0^\infty y^{-1} u\left(\frac{x}{y}\right) f(y)\, dy = g(x) , \qquad \ldots (1.6)$$

where g is a prescribed function, f is an unknown function to be determined and the kernel u(x) is given by

$$u(x) = S_n^{\alpha,\beta,0}[z\,x^\rho; r,q,A,B,k,\ell]\; S_{q_1,\ldots,q_m}^{p_1,\ldots,p_m}[w_1 x^{v_1},\ldots,w_m x^{v_m}]\; H_{P,Q}^{M,N}\left[t\,x^\lambda \left| \begin{matrix}(a_j,A_j)_{1,P}\\ (b_j,B_j)_{1,Q}\end{matrix}\right.\right]$$

$$\times\, \overline{H}_{P_1,Q_1}^{M_1,N_1}\left[\tau x^\gamma \left| \begin{matrix}(e_j,\alpha_j:E_j)_{1,N_1},(e_j,\alpha_j)_{N_1+1,P_1}\\ (f_j,\beta_j)_{1,M_1},(f_j,\beta_j:F_j)_{M_1+1,Q_1}\end{matrix}\right.\right], \qquad \ldots (1.7)$$

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where $S_{q_1,\ldots,q_m}^{p_1,\ldots,p_m}[x_1,\ldots,x_m]$ is the general multivariable polynomials introduced and defined by Srivastava [11] in the following manner:

$$S_{q_1,\ldots,q_m}^{p_1,\ldots,p_m}[x_1,\ldots,x_m]=\sum_{k_1=0}^{[q_1/p_1]}\cdots\sum_{k_m=0}^{[q_m/p_m]}\frac{(-q_1)_{p_1k_1}}{k_1!}\cdots\frac{(-q_m)_{p_mk_m}}{k_m!}$$

$$\times A[q_1,k_1;\ldots;q_m,k_m]x_1^{k_1}\ldots x_m^{k_m}, \qquad \ldots (1.8)$$

where $p_1,\ldots,p_m$ are arbitrary positive integers and the coefficients $A[q_1,k_1;\ldots;q_m,k_m]$ are arbitrary constants, real or complex.

Also, $H_{P,Q}^{M,N}[y]$ is the well known Fox's H-function whose series representation due to Braaksma [1] and Skibiñski [70] is given by

$$H_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]=\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G}{G!\,B_g}\varphi(\eta_G)\,z^{\eta_G}, \qquad \ldots (1.9)$$

where

$$\varphi(\eta_G)=\frac{\prod_{j=1}^{M}\Gamma(b_j-B_j\eta_G)\prod_{j=1}^{N}\Gamma(1-a_j+A_j\eta_G)}{\prod_{j=M+1}^{Q}\Gamma(1-b_j+B_j\eta_G)\prod_{j=N+1}^{P}\Gamma(a_j-A_j\eta_G)}, \qquad \ldots (1.10)$$

and

$$\eta_G=(b_g+G)/B_g \qquad \ldots (1.11)$$

$$\Omega=\sum_{i=1}^{N}A_i-\sum_{i=N+1}^{P}A_i+\sum_{i=1}^{M}B_i-\sum_{i=M+1}^{Q}B_i>0 \qquad \ldots (1.12)$$

and the $\bar{H}$-function, introduced by Inayat-Hussain (1987) in terms of Mellin-Barnes type contour integral, is defined by

$$\bar{H}_{P_1,Q_1}^{M_1,N_1}\left[z\left|\begin{matrix}(e_j,\alpha_j:E_j)_{1,N_1},(e_j,\alpha_j)_{N_1+1,P_1}\\(f_j,\beta_j)_{1,M_1},(f_j,\beta_j:F_j)_{M_1+1,Q_1}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\psi(s)\,z^s\,ds, \qquad \ldots (1.13)$$

where

$$\psi(s)=\frac{\prod_{j=1}^{M_1}\Gamma(f_j-\beta_js)\prod_{j=1}^{N_1}\{\Gamma(1-e_j+\alpha_js)\}^{E_j}}{\prod_{j=M_1+1}^{Q_1}\{\Gamma(1-f_j+\beta_js)\}^{F_j}\prod_{j=N_1+1}^{P_1}\Gamma(e_j-\alpha_js)}, \qquad \ldots (1.14)$$

which contains fractional powers of some of the Γ-functions.

Here and throughout the paper $e_j(j=1,\ldots,P_1)$ and $f_j(j=1,\ldots,Q_1)$ are complex parameters, $\alpha_j\geq 0(j=1,\ldots,P_1)$, $\beta_j\geq 0(j=1,\ldots,Q_1)$, (not all zero simultaneously) and the exponents $E_j(j=1,\ldots,N_1)$ and $F_j(j=M_1+1,\ldots,Q_1)$ can take non-integer values.

For the sake of brevity

$$T = \sum_{j=1}^{M_1} |\beta_j| + \sum_{j=1}^{N_1} E_j\alpha_j - \sum_{j=M_1+1}^{Q_1} |F_j\beta_j| - \sum_{j=N_1+1}^{P_1} \alpha_j > 0 \qquad \ldots (1.15)$$

Our method of solution of the integral equation (1.6) with kernel u(x) given by (1.7) would depends on the theory of Mellin transform defined by

$$F(s) = M\{f(x); s\} = \int_0^\infty x^{s-1} f(x)\, dx, \qquad \ldots (1.16)$$

provided that the integral exists.

II. Mellin Transform of U(x)

In order to solve the integral equation given by (1.6) we shall require the following result contained in

LEMMA. Let $U(s) = M\{u(x); s\}$, where u(x) is defined by (1.7), then

$$U(s) = \sum_{e,p,u,v} \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_m=0}^{[q_m/p_m]} \sum_{G=0}^{\infty} \sum_{g=1}^{M} \varphi(e,p,u,v)$$

$$\times \frac{(-q_1)_{p_1k_1} \cdots (-q_m)_{p_mk_m} A[q_1,k_1;\ldots;q_m,k_m] w_1^{k_1} \cdots w_m^{k_m} z^L}{k_1! \ldots k_m!\ \gamma}$$

$$\times \frac{(-1)^G \varphi(\eta_G) t^{\eta_G}}{G!\, B_g} \tau^{-(s+\rho L+\lambda\eta_G+v_1k_1+\ldots+v_mk_m)/\gamma}$$

$$\times \psi\left(-\frac{s+\rho L+\lambda\eta_G+v_1k_1+\ldots+v_mk_m}{\gamma}\right), \qquad \ldots (2.1)$$

where $L = \ell n + p + rv$ and $\rho > 0$, $v_i > 0$ $(i = 1,\ldots,m)$, $\gamma > 0$, $\lambda > 0\, |\arg t| < \frac{1}{2}\Omega\pi$,

$$\Omega = \sum_{i=1}^{N} A_i - \sum_{i=N+1}^{P} A_i + \sum_{i=1}^{M} B_i - \sum_{i=M+1}^{Q} B_i > 0,$$

$$T = \sum_{j=1}^{M_1} |\beta_j| + \sum_{j=1}^{N_1} E_j\alpha_j - \sum_{j=M_1+1}^{Q_1} |F_j\beta_j| - \sum_{j=N_1+1}^{P_1} \alpha_j > 0,$$

$$|\arg\tau| < \frac{1}{2} T\pi,$$

$$-\min_{1\le j\le M_1} \mathrm{Re}\left[\frac{f_j}{\beta_j}\right] < \mathrm{Re}\left[\frac{s+\rho L+\lambda\eta_G+v_1k_1+\ldots+v_mk_m}{\gamma}\right] < \min_{1\le j\le N_1} \mathrm{Re}\left[\frac{1-e_j}{\alpha_j}\right].$$

Proof. We have

$$U(s) = \int_0^\infty x^{s-1} S_n^{\alpha,\beta,0}[z x^\rho; r,q,A,B,k,\ell]$$

$$\times S_{q_1,\ldots,q_m}^{p_1,\ldots,p_m}[w_1 x^{v_1},\ldots,w_m x^{v_m}].\, H_{P,Q}^{M,N}\left[t x^\lambda \,\middle|\, \begin{matrix}(a_j,A_j)_{1,P}\\ (b_j,B_j)_{1,Q}\end{matrix}\right]$$

$$\times \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[\tau x^\gamma \,\middle|\, \begin{matrix}(e_j,\alpha_j:E_j)_{1,N_1},(e_j,\alpha_j)_{N_1+1,P_1}\\ (f_j,\beta_j)_{1,M_1},(f_j,\beta_j:F_j)_{M_1+1,Q_1}\end{matrix}\right] dx. \qquad \ldots (2.2)$$

Making use of the series expansions for the generalized polynomial set, the general multivariable polynomials and the Fox's H-function given by (1.2), (1.8) and (1.9) respectively and then changing the order of summation and integration, we get

$$U(s) = \sum_{e,p,u,v} \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_m=0}^{[q_m/p_m]} \sum_{G=0}^{\infty} \sum_{g=1}^{M} \varphi(e,p,u,v)$$

$$\times (-q_1)_{p_1 k_1} \cdots (-q_m)_{p_m k_m} A[q_1,k_1;\dots;q_m,k_m] \frac{w_1^{k_1} \cdots w_m^{k_m}}{k_1! \dots k_m!} \cdot \frac{(-1)^G \varphi(\eta_G) t^{\eta_G} z^L}{G!\ B_g}$$

$$\times M\left\{ x^{\rho L + \lambda \eta_G + v_1 k_1 + \dots + v_m k_m} \bar{H}_{P_1,Q_1}^{M_1,N_1}[\tau x^{\gamma}]; s \right\} \quad \dots (2.3)$$

Now applying the following known formula due to Erdélyi (1954)

$$M\left\{ x^{\mu} f(z x^{h}); s \right\} = h^{-1} z^{-(s+\mu)/h} \left(\frac{s+\mu}{h} \right) \quad \dots (2.4)$$

and

$$M\left\{ \bar{H}_{P_1,Q_1}^{M_1,N_1}[x]; s \right\} = \psi(-s), \quad \dots (2.5)$$

provided that $T > 0, |\arg \tau| < \frac{1}{2} T \pi$, and

$$-\min_{1 \le j \le M_1} Re\left[\frac{f_j}{\beta_j} \right] < Re(s) < \min_{1 \le j \le N_1} Re\left[\left(\frac{1-e_j}{\alpha_j} \right) \right]$$

in (2.3), we arrive at the required result (2.1).

III. SOLUTION OF THE INTEGRAL EQUATION (1.6)

The solution of the convolution integral equation (1.6) is contained in the following theorem.

Theroem. Let the Mellin transform F(s), G(s) and U(s) $\neq 0$ of the functions f(x), g(x) and u(x) (defined by (1.7)) respectively exist and are analytic in some infinite strip $s_1 < Re\ s < s_2$ of the complex s-plane. Also suppose that for a fixed $c \in (s_1,s_2)$, $u^*(x)$ is defined by

$$u^*(x) = M^{-1}\{U^*(s); x\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} U^*(s)\, ds \,, \quad \dots (3.1)$$

where

$$U^*(s) = \left\{ \frac{\mu^k \Gamma(-s/\mu)}{\Gamma[-(s+\mu k)/\mu]} \sum_{e,p,u,v} \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_m=0}^{[q_m/p_m]} \sum_{G=0}^{\infty} \sum_{g=1}^{M} \right.$$

$$\times \varphi(e,p,u,v) (-q_1)_{p_1 k_1} \cdots (-q_m)_{p_m k_m} A[q_1,k_1;\dots;q_m,k_m]$$

$$\times \frac{w_1^{k_1} \cdots w_m^{k_m} z^L (-1)^G \varphi(\eta_G) t^{\eta_G}}{\gamma\, k_1! \dots k_m!\, G!\ B_g} \tau^{-(s+\mu k+\xi+\rho L+\lambda \eta_G + v_1 k_1 + \dots + v_m k_m)/\gamma}$$

$$\times\psi\left(-\frac{s+\mu k+\xi+\rho L+\lambda\eta_G+\nu_1 k_1+...+\nu_m k_m}{\gamma}\right)\Bigg\}^{-1}, \qquad \ldots (3.2)$$

provided that the following set of conditions hold true

i. $|\arg x| < \frac{1}{2} T\pi$ where $T > 0$

ii. $\mu \neq 0$ and k is non-negative integer; $\rho > 0$ and $\nu_i > 0$ ($i = 1,\ldots,m$)

iii. $-\min_{1\le j\le M_1} \mathrm{Re}\left[\frac{f_j}{\beta_j}\right] < \mathrm{Re}\left[\frac{s+\mu k+\xi+\rho L+\lambda\eta_G+\nu_1 k_1+...+\nu_m k_m}{\gamma}\right] < \min_{1\le j\le N_1} \mathrm{Re}\left[\left(\frac{1-e_j}{\alpha_j}\right)\right]$

iv. $p_1,\ldots,p_m$ a arbitrary positive integers and the coefficients $A[q_1,k_1;\ldots;q_m,k_m]$ are arbitrary (real or complex) constants.

Then the integral equation (1.6) has its solution given by

$$f(x) = x^{-\mu k-\xi}\int_0^\infty y^{-1} u^*\left(\frac{x}{y}\right)\left(y^{\mu+1}D_y\right)^k [y^\xi g(y)]\, dy\,, \qquad \ldots (3.3)$$

provided that the integral on the right hand side of (3.3) exists.

Proof. Applying the convolution theorem for Mellin transform due to Erdélyi [4] we find from (1.6) that

$$U(s)\,F(s) = G(s)\,, \qquad \ldots (3.4)$$

where U(s) is given by (2.1), and F(s) and G(s) are Mellin transforms of f(x) and g(x) respectively.

Replacing s by $s+\mu k+\xi$ in (3.4), we get

$$F(s+\mu k+\xi) = U^*(s)\,\mu^k\left(-\frac{s+\mu k}{\mu}\right)_k G(s+\mu k+\xi) \qquad \ldots (3.5)$$

Now using the formula due to Srivastava [14]

$$M\left\{(x^{\ell+1}D_x)^n\; f(x)\,;s\right\} = \ell^n\left(-\frac{s+\ell n}{\ell}\right)_n F(s+\ell n) \qquad \ldots (3.6)$$

($\ell \neq 0$, n is a non-negative integer)

in (3.5), we get

$$F(s+\mu k+\xi) = U^*(s)\,M\left\{(y^{\mu+1}D_y)^k\;[y^\xi g(y)];s\right\} \qquad \ldots (3.7)$$

Again using elementary result

$$M\left\{x^\mu\; f(x)\,;s\right\} = F(s+\mu) \qquad \ldots (3.8)$$

and well known convolution theorem for Mellin transform in (3.7), we obtain

$$M\left\{x^{\mu k+\xi}\; f(x);s\right\} = M\left\{\int_0^\infty y^{-1}u^*\left(\frac{x}{y}\right)\left(y^{\mu+1}D_y\right)^k [y^\xi g(y)]dy\,;s\right\}. \qquad \ldots (3.9)$$

On inverting both sides of (3.9) by using the well-known Mellin inversion theorem, we arrive at the desired solution (3.3).

IV. Special Cases

If we set m = 1 in the theorem and denote $A[q_1, k_1]$ thus obtained by A_{q_1,k_1}, then the general multivariable polynomials reduces to the general class of polynomials introduced by Srivastava [12] and we get

Corollary 1 : Under the hypothesis of the theorem, the integral equation

$$\int_0^{\infty} y^{-1} u_1\left(\frac{x}{y}\right) f(y)\, dy = g(x) \quad , \qquad \ldots (4.1)$$

where the kernel

$$u_1(x) = S_n^{\alpha,\beta,0}[z x^{\rho}; r,q,A,B,k,\ell] S_{q_1}^{p_1}[w_1 x^{v_1}] H_{P,Q}^{M,N}\left[t x^{\lambda}\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]$$

$$\times \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[\tau x^{\gamma}\left|\begin{matrix}(e_j,\alpha_j:E_j)_{1,N_1},(e_j,\alpha_j)_{N_1+1,P_1}\\(f_j,\beta_j)_{1,M_1},(f_j,\beta_j:F_j)_{M_1+1,Q_1}\end{matrix}\right.\right] \qquad \ldots (4.2)$$

has its solution given by

$$f(x) = x^{-\mu k-\xi}\int_0^{\infty} y^{-1} u_1^*\left(\frac{x}{y}\right)\left(y^{\mu+1}D_y\right)^k [y^{\xi} g(y)]\, dy \quad , \qquad \ldots (4.3)$$

provided that the integral in (4.3) exists, and $u_1^*(x)$ is the Mellin transform of

$$U_1^*(s) = \left\{\frac{\mu^k \Gamma(-s/\mu)}{\Gamma[-(s+\mu k)/\mu]}\sum_{e,p,u,v}\sum_{k_1=0}^{[q_1/p_1]}\sum_{G=0}^{\infty}\sum_{g=1}^{M}\varphi(e,p,u,v)\right.$$

$$\times \frac{(-q_1)_{p_1 k_1} A_{q_1,k_1} w_1^{k_1} z^L (-1)^G \varphi(\eta_G) t^{\eta_G}}{\gamma \quad k_1!\ G!\ B_g}\tau^{-(s+\mu k+\xi+\rho L+\lambda\eta_G+v_1 k_1)/\gamma}$$

$$\left.\times \psi\left(-\frac{s+\mu k+\xi+\rho L+\lambda\eta_G+v_1 k_1}{\gamma}\right)\right\}^{-1}, \qquad \ldots (4.4)$$

provided the conditions mentioned with the Theorem are satisfied.

If we set $w_1 = 1 = v_1, q_1 = 0$ and $A_{0,0} = 1$ in (4.2) then the general class of polynomials reduces to unity and we get

Corollary 2 : Under the hypothesis of the Corollary 1, the integral equation

$$\int_0^{\infty} y^{-1} u_2\left(\frac{x}{y}\right) f(y)\, dy = g(x) \quad , \qquad \ldots (4.5)$$

where the kernel

$$u_2(x) = S_n^{\alpha,\beta,0}[z x^{\rho}; r,q,A,B,k,\ell]\, H_{P,Q}^{M,N}\left[t x^{\lambda}\left|\begin{matrix}(a_j,A_j)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right] \qquad \ldots (4.6)$$

$$\times \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[\tau x^{\gamma}\left|\begin{matrix}(e_j,\alpha_j:E_j)_{1,N_1},(e_j,\alpha_j)_{N_1+1,P_1}\\(f_j,\beta_j)_{1,M_1},(f_j,\beta_j:F_j)_{M_1+1,Q_1}\end{matrix}\right.\right]$$

has its solution given by

$$f(x) = x^{-\mu k-\xi}\int_0^{\infty} y^{-1} u_2^*\left(\frac{x}{y}\right)\left(y^{\mu+1}D_y\right)^k [y^{\xi} g(y)]\, dy \quad , \qquad \ldots (4.7)$$

provided that the integral in (4.7) exists, and $u_2^*(x)$ is the Mellin transform of

$$U_2^*(s) = \left\{\frac{\mu^k \Gamma(-s/\mu)}{\Gamma[-(s+\mu k)/\mu]}\sum_{e,p,u,v}\sum_{G=0}^{\infty}\sum_{g=1}^{M}\varphi(e,p,u,v) z^L\right.$$

$$\times \frac{(-1)^G \varphi(\eta_G) t^{\eta_G}}{\gamma\, G!\, B_g} \tau^{-(s+\mu k+\xi+\rho L+\lambda \eta_G)/\gamma} \psi\left(-\frac{s+\mu k+\xi+\rho L+\lambda\eta_G}{\gamma}\right)\Bigg\}^{-1}, \quad \ldots (4.8)$$

provided the conditions mentioned with the Corollary 1 are satisfied.

If we set n = q = k = B = 0, ℓ = r = - 1 and A = 1 in the theorem, the generalized polynomial set $S_n^{\alpha,\beta,0}[x; r, q, A, B, k, \ell]$ red

Corollary 3 : Under the hypothesis of the theorem, the integral equation

$$\int_0^\infty y^{-1} u_3\left(\frac{x}{y}\right) f(y)\, dy = g(x) \quad , \quad \ldots (4.9)$$

where the kernel

$$u_3(x) = S_{q_1,\ldots,q_m}^{p_1,\ldots,p_m}[w_1 x^{v_1},\ldots, w_m x^{v_m}]\, H_{P,Q}^{M,N}\left[t x^\lambda \middle| \begin{matrix} (a_j, A_j)_{1,P} \\ (b_j, B_j)_{1,Q} \end{matrix}\right]$$

$$\times \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[\tau x^\gamma \middle| \begin{matrix} (e_j,\alpha_j:E_j)_{1,N_1}, (e_j,\alpha_j)_{N_1+1,P_1} \\ (f_j,\beta_j)_{1,M_1}, (f_j,\beta_j:F_j)_{M_1+1,Q_1} \end{matrix}\right] , \quad \ldots (4.10)$$

has its solution given by

$$f(x) = x^{-\mu k-\xi} \int_0^\infty y^{-1} u_3^*\left(\frac{x}{y}\right)\left(y^{\mu+1} D_y\right)^k [y^\xi g(y)]\, dy \quad , \quad \ldots (4.11)$$

provided that the integral in (4.11) exists, and $u_3^*(x)$ is the Mellin transform of

$$U_3^*(s) = \Bigg\{ \frac{\mu^k \Gamma(-s/\mu)}{\Gamma[-(s+\mu k)/\mu]} \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_m=0}^{[q_m/p_m]} \sum_{G=0}^{\infty} \sum_{g=1}^{M}$$

$$\times (-q_1)_{p_1 k_1} \cdots (-q_m)_{p_m k_m} A[q_1,k_1;\ldots;q_m,k_m]$$

$$\times \frac{w_1^{k_1} \ldots w_m^{k_m} (-1)^G \varphi(\eta_G) t^{\eta_G}}{\gamma\, k_1! \ldots k_m!\, G!\, B_g} \tau^{-(s+\mu k+\xi+\lambda\eta_G+v_1 k_1+\ldots+v_m k_m)/\gamma}$$

$$\times \psi\left(-\frac{s+\mu k+\xi+\lambda\eta_G+v_1 k_1+\ldots+v_m k_m}{\gamma}\right)\Bigg\}^{-1}, \quad \ldots (4.12)$$

provided the conditions mentioned with the Theorem are satisfied.

If we set M = Q = 1, N = P = 0, b_1 = 0, B_1 = 1 and t → 0 in the theorem, the Fox's H-function reduces to unity and we get

Corollary 4 : Under the hypothesis of the theorem, the integral equation

$$\int_0^\infty y^{-1} u_4\left(\frac{x}{y}\right) f(y)\, dy = g(x) \quad , \quad \ldots (4.13)$$

where the kernel

$$u_4(x) = S_n^{\alpha,\beta,0}[z x^\rho; r,q,A,B,k,\ell]\, S_{q_1,\ldots,q_m}^{p_1,\ldots,p_m}[w_1 x^{v_1},\ldots, w_m x^{v_m}]$$

$$\times \bar{H}_{P_1,Q_1}^{M_1,N_1}\left[\tau x^\gamma \middle| \begin{matrix} (e_j,\alpha_j:E_j)_{1,N_1}, (e_j,\alpha_j)_{N_1+1,P_1} \\ (f_j,\beta_j)_{1,M_1}, (f_j,\beta_j:F_j)_{M_1+1,Q_1} \end{matrix}\right] \quad \ldots (4.14)$$

has its solution given by

$$f(x) = x^{-\mu k-\xi} \int_0^{\infty} y^{-1} u_4^*\left(\frac{x}{y}\right)\left(y^{\mu+1} D_y\right)^k [y^{\xi} g(y)]\, dy \,, \qquad \ldots (4.15)$$

provided that the integral in (4.15) exists, and $u_4^*(x)$ is the Mellin transform of

$$U_4^*(s) = \left\{ \frac{\mu^k \Gamma(-s/\mu)}{\Gamma[-(s+\mu k)/\mu]} \sum_{e,p,u,v} \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_m=0}^{[q_m/p_m]} \varphi(e,p,u,v) \right.$$

$$\times \frac{(-q_1)_{p_1 k_1} \ldots (-q_m)_{p_m k_m} \; A\,[q_1,k_1;\ldots;q_m,k_m]\, w_1^{k_1} \ldots w_m^{k_m}\, z^L}{\gamma \quad k_1! \ldots k_m!}$$

$$\left. \times \tau^{-(s+\mu k+\xi+\rho L+\nu_1 k_1+\ldots+\nu_m k_m)/\gamma} \; \psi\left(-\frac{s+\mu k+\xi+\rho L+\nu_1 k_1+\ldots+\nu_m k_m}{\gamma}\right)\right\}^{-1}, \qquad \ldots (4.16)$$

provided the conditions mentioned with the theorem are satisfied.

Further, if we set $E_j(j=1,\ldots,N_1) = F_j(j=M_1+1,\ldots,Q_1) = 1, m = 1$ in (4.14),then $\bar{H}$ -function reduces to the Fox's H-function and the general multivariable polynomials reduces to the general class of polynomials and we get the solution of the integral equation considered by Goyal and Mukherjee [6].

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