Global Journal of Science Frontier Research
Volume 11 Issue 6 Version 1.0 September 2011
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN : 2249-4626 & Print ISSN: 0975-5896

Sumudu Homotopy Perturbation Technique

By Devendra Kumar, Jagdev Singh, Sushila
Jagan Nath Gupta Institute of Engineering and Technology, Jaipur, Rajasthan, India

Abstract - In this paper, a combinatory method of the sumudu transform and the homotopy perturbation method is proposed for solving one dimensional non-homogeneous partial differential equations with a variable coefficient. This method presents an accurate methodology to solve nonhomogeneous partial differential equations with a variable coefficient. The obtained approximate solutions are compared with exact solutions and those obtained by other analytical methods, showing reliability of the present method. The comparison shows a precise agreement between the results, and introduces this new method as an applicable one which it needs fewer computations and is much easier and more convenient than others, so it can be widely used in science and engineering.

Keywords and Phrases : Sumudu transform, Homotopy perturbation method, Non-homogenous partial differential equations.

Sumudu Homotopy Perturbation Technique

Devendra Kumar[α], Jagdev Singh[Ω], Sushila[β]

Abstract - In this paper, a combinatory method of the sumudu transform and the homotopy perturbation method is proposed for solving one dimensional non-homogeneous partial differential equations with a variable coefficient. This method presents an accurate methodology to solve non-homogeneous partial differential equations with a variable coefficient. The obtained approximate solutions are compared with exact solutions and those obtained by other analytical methods, showing reliability of the present method. The comparison shows a precise agreement between the results, and introduces this new method as an applicable one which it needs fewer computations and is much easier and more convenient than others, so it can be widely used in science and engineering.

Keywords and Phrases : Sumudu transform, Homotopy perturbation method, Non-homogenous partial differential equations.

I. Introduction

Partial differential equations are obtained in modeling of real-life science and engineering phenomena that are inherently nonlinear with variable coefficients. Most of these types of equations do not have an analytical solution. Therefore, these problems should be solved by using numerical or semi-analytical techniques. In numeric methods, computer codes and more powerful processors are required to achieve accurate results. Acceptable results are obtained via semi-analytical methods which are more convenient than numerical methods. The main advantage of semi-analytical methods, compared with other methods, is based on the fact that they can be conveniently applied to solve various complicated problems. In the semi-analytical methods such as the homotopy perturbation method, the variational iteration method, and the Adomian method, we can always obtain conveniently acceptable results in analytical forms instead of numerical ones for partial differential equations. These methods have simple solution procedures to solve various complicated problems [1-3]. The non-homogeneous partial differential equations with variable coefficients can be solved by the above said methods, however, with less accurate approximations [4-6] which might not satisfy initial/boundary conditions. To overcome this deficiency, this paper suggests a new method which is a combination of sumudu transform and homotopy perturbation method (SHPM), so that the obtained solutions satisfy the initial/boundary conditions. In early 90's, Watugala [7] introduced a new integral transform, named the sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The sumudu transform is defined over the set of functions.

$$A = \{f(t) \mid \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j}, \text{if } t \in (-1)^j \times [0, \infty)\}$$

by the following formula

$$\bar{f}(u) = S[f(t)] = \int_0^\infty f(ut)\, e^{-t}\, dt,\ u \in (-\tau_1, \tau_2) \tag{1}$$

For further detail and properties of this transform, see [8-10].

II. Sumudu Homotopy Perturbation Method (SHPM)

To illustrate the basic idea of this method, we consider a general nonlinear form of one-dimension non-homogenous partial differential equation with a variable coefficient of the form:

$$\frac{\partial y}{\partial t} = \mu(x)\frac{\partial^2 y}{\partial x^2} + \phi(x,t), \tag{2}$$

with subject to the boundary conditions

$$y(0,t) = g_0(t),\ \ y(1,t) = g_1(t). \tag{3}$$

Author [α] : Department of Mathematics, Jagan Nath Gupta Institute of Engineering and Technology, Jaipur-302022, Rajasthan, India.
E-mail : dev.ku15@gmail.com.
Author [Ω] : Department of Mathematics, Jagan Nath University,Village-Rampura, Tehsil-Chaksu, Jaipur-303901, Rajasthan, India.
E-mail : jagdevsinghrathore@gmail.com.
Author [β] : Department of Physics, Jagan Nath University,Village-Rampura, Tehsil-Chaksu, Jaipur-303901, Rajasthan, India.
E-mail : sushila.jag@gmail.com.

And the initial condition

$$y(x,0)=f(x). \tag{4}$$

Taking the sumudu transform on equations (2) and (3), we get

$$\frac{d^2\bar{y}}{dx^2}-\frac{\bar{y}(x,u)}{u\mu(x)}+\frac{\bar{\phi}(x,u)+f(x)/u}{\mu(x)}=0, \tag{5}$$

$$\bar{y}(0,u)=\overline{g_0}(u),\ \ \bar{y}(1,u)=\overline{g_1}(u), \tag{6}$$

which is second order boundary value problem. According to HPM, we construct a homotopy in the form

$$H(\nu,p)=(1-p)\left[\frac{d^2\nu}{dx^2}-\frac{d^2\overline{y_0}}{dx^2}\right]+p\left[\frac{d^2\nu}{dx^2}-\frac{\nu}{u\mu(x)}+\frac{\bar{\phi}(x,u)+f(x)/u}{\mu(x)}\right]=0, \tag{7}$$

where $\overline{y_0}$ is the arbitrary function that satisfies boundary conditions (6), therefore

$$\nu(x,u)=\sum_{i=0}^{\infty}p^i\nu_i(x,u)=\nu_0(x,u)+p^1\nu_1(x,u)+p^2\nu_2(x,u)+\cdots \qquad \cdots(8)$$

Taking the inverse sumudu transform from both sides of (10), one obtains

$$\nu(x,t)=\sum_{i=0}^{\infty}p^i\nu_i(x,t)=\nu_0(x,t)+p^1\nu_1(x,t)+p^2\nu_2(x,t)+\cdots \qquad \cdots(9)$$

Setting $p=1$ results in the approximate solutions of eq. (2)

$$y(x,t)=y_0(x,t)+y_1(x,t)+y_2(x,t)+\cdots. \tag{10}$$

In this section, we use sumudu homotopy perturbation method (SHPM) in solving the one-dimension non-homogenous partial differential equations.

Example 4.1 : Consider the problem

$$\frac{\partial y}{\partial t}=\frac{\partial^2 y}{\partial x^2}+e^{-x}(\cos(t)-\sin(t)), \tag{11}$$

subject to the initial condition

$$y(x,0)=x. \tag{12}$$

And the boundary conditions

$$y(0,t)=\sin(t),\ \ y(1,t)=\frac{1+\sin(t)}{e}. \tag{13}$$

This problem has an exact solution that is

$$y(x,t)=x+e^{-x}\sin(t). \tag{14}$$

Taking the sumudu transform of eq. (11) and its boundary conditions with respect to t, and considering the initial condition, we have

$$\frac{d^2\bar{y}}{dx^2}-\frac{\bar{y}}{u}+\frac{x}{u}+e^{-x}\left(\frac{1-u}{1+u^2}\right)=0, \tag{15}$$

$$\bar{y}(0,u)=\frac{u}{1+u^2},\ \ \bar{y}(1,u)=1+\frac{u}{e(1+u^2)}. \tag{16}$$

To solve eq. (15) by means of HPM, a homotopy equation can be readily constructed as follows

$$H(\nu,p)=(1-p)\left[\frac{d^2\nu}{dx^2}-\frac{d^2\overline{y_0}}{dx^2}\right]+p\left[\frac{d^2\nu}{dx^2}-\frac{\nu}{u}+\frac{x}{u}+\frac{e^{-x}(1-u)}{1+u^2}\right]=0,\ p\in[0,1] \qquad (17)$$

Now, we obtain a solution of eq. (17) in the form $\nu(x,u)=\sum_{i\ 0}^{\infty}p^i\nu_i(x,u).$ After substituting it into eq. (17) and rearranging the resultant equation based on powers of p-terms, following sets of linear differential equations can be obtained:

$$p^0:\frac{d^2\nu_0}{dx^2}-\frac{d^2\overline{y_0}}{dx^2}=0,\quad \nu_0(0,u)=0,\ \nu_0(1,u)=0 \qquad (18.a)$$

$$p^1:\frac{d^2\nu_1}{dx^2}-\frac{\nu_0}{u}+\frac{x}{u}-e^{-x}\left(\frac{1-u}{1+u^2}\right)=0,\quad \nu_1(0,u)=0,\ \nu_{1x}(1,u)=0 \qquad (18.b)$$

$$\vdots \quad p^i:\frac{d^2\nu_i}{dx^2}-\frac{\nu_{i-1}}{u}=0,\quad \nu_i(0,u)=0,\ \nu_i(1,u)=0,\ i=2,3,4,\ldots \qquad (18.c)$$

The initial approximation $\nu_0(x,u)$ can be freely chosen, Here we set

$$\nu_0(x,u)=\frac{u(1-x)}{1+u^2}+x+\frac{ux}{e(1+u^2)},$$

which satisfies boundary conditions (16).

Using some mathematical software to solve eq. (18b-18c), and taking inverse sumudu transform, we get the following result

$$y_1(x,t)=x+e^{-x}\sin(t)+\frac{1}{6}(6+3x^2-6e^{-x}+x^3(e^{-1}-1)+x(4e^{-1}-8)\cos(t)). \qquad (19)$$

Comparison of the obtained result with those obtained by other methods is shown in Table 1. As can be seen from Table 1, SHPM leads to more accurate solution.

Table 1 : Comparison between the results and those in open literature

X=0.1 t	u(x,t) exact	u(x,t) SHPM (one iteration)	u(x,t)LHPM[11] (one iteration)	u (x,t) HPM [5] (five iteration)	u(x,t)VIM[6] (five iteration)
0.1	0.190333011	0.187726613	0.187726613	0.19033301	0.19033301
0.3	0.367397741	0.364895251	0.364895251	0.367396826	0.367396826
0.5	0.533802166	0.531503352	0.531503352	0.533782618	0.533782618
0.9	0.808783498	0.807155201	0.807155201	0.8081252	0.8081252
1.5	1.002570788	1.002385493	1.002385493	0.988816989	0.988816989
3	0.227690664	0.230283934	0.230283934	-0.554986914	-0.554986914
4.5	-0.784505828	-0.783953651	-0.783953651	-8.178595887	-8.178595887
7	0.694466058	0.692491222	0.692491222	-67.88113901	-67.88113901
X=0.9					
0.1	0.940589238	0.938138815	0.938138815	0.940589238	0.940589238
0.3	1.02014955	1.017796817	1.017796817	1.020149139	1.020149139
0.5	1.094919878	1.092758632	1.092758632	1.094911094	1.094911094
0.9	1.218476955	1.2169461	1.2169461	1.218181163	1.218181163
1.5	1.305551197	1.305376991	1.305376991	1.299371217	1.299371217
3	0.957375114	0.959813195	0.959813195	0.605695409	0.605695409
4.5	0.502565913	0.503085045	-0.503085045	-2.819812914	2.819812914
7	1.167110818	1.165254163	1.165254163	-29.64589477	-29.64589477

Example 4.2 : Now, consider the problem

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + e^x(\cosh(t) - \sinh(t)), \tag{20}$$

subject to the initial condition

$$y(x,0) = \frac{x^3}{6}. \tag{21}$$

And the boundary conditions

$$y(0,t) = \sinh(t),\ \ y(1,t) = \sinh(t) + t + \frac{1}{6}. \tag{22}$$

This problem has an exact solution that is

$$y(x,t) = e^x \sinh(t) + \frac{x^3}{6} + xt. \tag{23}$$

Taking the sumudu transform of eq. (20) and its boundary conditions with respect to t, and considering the initial condition, we have

$$\frac{d^2\bar{y}}{dx^2} - \frac{\bar{y}}{u} + \frac{x^3}{6u} + e^x\left(\frac{1}{1+u}\right) = 0, \tag{24}$$

$$\bar{y}(0,u) = \frac{u}{1-u^2},\ \ \bar{y}(1,u) = \frac{u}{1-u^2} + u + \frac{1}{6}. \tag{25}$$

To solve eq. (24) by means of HPM, a homotopy equation can be readily constructed as follow

$$H(\nu,p) = (1-p)\left[\frac{d^2\nu}{dx^2} - \frac{d^2\bar{y}_0}{dx^2}\right] + p\left[\frac{d^2\nu}{dx^2} - \frac{\bar{y}}{u} + \frac{x^3}{6u} + e^x\left(\frac{1}{1+u}\right)\right] = 0,\ p \in [0,1] \tag{26}$$

Now, following the same procedure as example 4.1, we assume the solution of equation (26) has a form $\nu(x,u) = \sum_{i=0}^{\infty} p^i \nu_i(x,u)$, and choose an initial solution in the form

$$\nu_0(x,u) = \frac{u(1-x)}{1-u^2} + x\left(\frac{1}{6} + u + \frac{eu}{1-u^2}\right),$$

which satisfies boundary conditions (26). Finally solving sets of linear differential equations that obtained from substituting $\nu(x,u)$ in eq. (26) and taking inverse sumudu transform, we get the following result

$$y_1(x,t) = e^x \sin(t) + xt + \frac{1}{6}(6 + 3x^2 + x^3 - 6e^{-x} + x^3(e-1) + x(5e-8)\cosh(t)). \tag{27}$$

Comparison of the obtained result with those obtained by other methods is shown in Table 2. As it can seen it is so close to the exact solution.

Table 2 : Comparison between the results and those in open literature

X=0.1 t	u(x,t) exact	u(x,t) SHPM (one iteration)	u(x,t)LHPM[11] (one iteration)	u(x,t) HPM[5] (five iteration)	u(x,t)VIM[6] (five iteration)
0.1	0.120868046	0.114140161	0.114140161	0.131708767	0.120868044
0.3	0.366713639	0.35971574	0.35971574	0.458331767	0.366712518
0.5	0.626066044	0.618517281	0.618517281	0.870229167	0.626041953
0.9	1.224643099	1.215049466	1.215049466	1.951824767	1.223815452
1.5	2.503384397	2.48763646	2.48763646	4.222104167	2.485179894
3	11.3716307	11.30423389	11.30423389	13.25116667	10.05353534
4.5	50.18618582	49.88484395	49.88484395	26.71272917	31.61579702
7	606.6832	603.0125545	603.0125545	56.44116667	139.4441579

X=0.9					
0.1	0.45787045	0.329380987	0.329380987	0.4992421	0.457870447
0.3	1.140499061	1.011722397	1.011722397	1.3590651	1.140496567
0.5	1.853187635	1.723825043	1.723825043	2.3665625	1.853134019
0.9	3.456323732	3.324786111	3.324786111	4.8649581	3.454481771
1.5	6.708682372	6.570598712	6.570598712	9.9684375	6.668167506
3	27.46149634	27.26847624	27.26847624	31.372500	24.52802118
4.5	114.8610462	114.41919	114.41919	68.0090625	73.53188592
7	1355.061543	1351.035919	1351.035919	171.36250	315.201931

IV. Conclusions

In this paper, a new modified HPM, namely the sumudu homotopy perturbation method (SHPM) is introduced and the obtained results are compared with those obtained by LHPM, HPM, VIM and exact solutions for non-homogeneous partial differential equations with a variable coefficient. The results reveal that SHPM is an efficient and has good agreement with the exact solutions. In conclusion, the SHPM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

References Références Referencias

1. M. Fathizadeh, F. Rashidi, Boundary layer convective heat transfer with pressure gradient using Homotopy Perturbation Method (HPM) over a flat plate, Chaos Solitons and Fractals 42 (2009) 2413–2419.
2. J-Huan He, Homotopy perturbation technique, Computer methods in applied mechanics and engineering. 178 (1999) 257–262.
3. J-Huan He, A coupling method of homotopy technique and a perturbation technique for nonlinear problems, Int. J. Non linear Mech. 35 (2000) 37–43.
4. Bongsoo Jang, Exact solutions to one dimensional non-homogeneous parabolic problem by the homogeneous Adomian decomposition method, Applied Mathematics and Computation 186 (2007) 969–979.
5. A.Barari, Abdoul R.Ghotbi, T. Barari and D.D.Ganji, Application of He's Homotopy Perturbation Method to Non-Homogeneous Parabolic Problems, International Journal of Applied Mathematics and Computation 1 (2009) 59–66.
6. Abdoul R. Ghotbi, A. Barari, M. Omidvar, D.D. Ganji, Application of Variational Iteration Method to Parabolic Problems, Applied Mathematical Sciences 3 (2009) 927–934.
7. G.K. Watugala, Sumudu transform- a new integral transform to solve differential equations and control engineering problems, Math. Engg. Indust. 6 (4) (1998) 319-329.
8. M.A. Asiru, Sumudu transform and the solution of integral equation of convolution type, International Journal of Mathematical Education in Science and Technology 32 (2001) 906-910.
9. F.B.M. Belgacem, A.A. Karaballi and S.L Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, Mathematical problems in Engineering 3 (2003) 103-118.
10. F.B.M. Belgacem and A.A. Karaballi, Sumudu transform fundamental properties investigations and applications, International J. Appl. Math. Stoch. Anal. (2005) 1-23.
11. M. Madani and M. Fathizadeh, Homotopy perturbation algorithm using Laplace transformation, Nonl. Sci. Lett. A 1(3) (2010) 263-267.

This page is intentionally left blank