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A Generalization of Fifth and Seventh Order Theta Functions and Their Partial Sums

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A Generalization of Fifth and Seventh Order Theta Functions and Their Partial Sums

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I. INTRODUCTION

In his last letter to Hardy [6, p 354-355], Ramanujan defined seventeen functions $f(q)$, where $|q| < 1$.

He called them mock theta functions for as q radially approaches to the unit circle there is a theta function $f_r(q)$, $f(q) - f_r(q) = O(1)$. Ramanujan included in his letter four separate classes of mock theta functions: one class of third order, two of fifth order, and one of seventh order. Watson[9] made extensive study of the third order mock theta functions, because he was able to find representations of these third order mock theta functions under the fundamental transformations of the modular group. However the fifth and the seventh order mock theta functions have been more of a problem. Watson states "I have failed to construct a complete and exact transformation theory of the

functions on the lines of the transformation theory of the functions of the third order." Andrews[2] provided for the fifth and the seventh order mock theta functions certain identities for modular forms.

In this paper we consider the generalized fifth and the seventh order mock theta functions defined by Srivastava. He made an extensive study of these generalized functions in his paper [7], [8]. We define and consider partial generalized fifth and seventh order mock theta functions and give expansions of generalized fifth order mock theta functions in terms of the partial generalized mock theta functions. A study of these sums and expansions has been made by Andrews[2]. Earlier Agarwal[1] had showed that one could find a number of partial theta relations by making a systematic study. Andrews has shown that some of these partial theta function identities have some interesting number theoretic interpretations. The above considerations and the interest shown by a number of mathematicians in the study of partial theta function identities motivated us to define "partial" mock theta functions and study their behaviour.

The fifth order mock theta functions of Ramanujan are:

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n}, \quad \phi_0(q) = \sum_{n=0}^{\infty} q^{n^2} (-q; q^2)_n,$$

$$\psi_0(q) = \sum_{n=0}^{\infty} q^{\frac{(n+1)(n+2)}{2}} (-q; q)_n, \quad F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q; q^2)_n},$$

$$\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n (q; q)_n}{(q; q)_{2n}},$$

And

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q; q)_n}, \quad \phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2} (-q; q^2)_n,$$

$$\psi_1(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (-q; q)_n, \quad F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}},$$

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$$\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n(q; q)_n}{(q; q)_{2n+1}}.$$

And his seventh mock theta functions are:

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(q; q)_n}{(q; q)_{2n}}, \quad F_1(q) = 1 + \sum_{n=0}^{\infty} \frac{q^{n^2}(q; q)_{n-1}}{(q; q)_{2n-1}}$$

$$F_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(q; q)_n}{(q; q)_{2n+1}}.$$

Srivastava generalized these fifth and seventh order mock theta functions. The generalized fifth order mock theta functions are:

$$f_0(t, z, \alpha; q) = \frac{1}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-3n+\alpha} z^{2n}}{(-z; q)_n}, \tag{1.1}$$

$$\phi_0(t, z, \alpha; q) = \frac{1}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_n q^{n^2+n+\alpha} \left(-\frac{q^3}{z^2}; q^2\right)_n}{z^{2n}}, \tag{1.2}$$

$$\psi_0(t, z, \alpha; q) = \frac{1}{(t)_{\infty}} \sum_{n=1}^{\infty} \frac{(t)_n q^{\frac{n(n+1)}{2}+\alpha} \left(-\frac{q^2}{z}; q\right)_{n-1}}{z^n}, \tag{1.3}$$

$$F_0(t, z, \alpha; q) = \frac{1}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_n q^{2n^2-5n+\alpha} z^{4n}}{\left(\frac{z^2}{q}; q^2\right)_n}, \tag{1.4}$$

$$f_1(t, z, \alpha; q) = \frac{1}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-2n+\alpha} z^{2n}}{(-z; q)_n}, \tag{1.5}$$

$$\phi_1(t, z, \alpha; q) = \frac{q^5}{z^4} \frac{1}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_n q^{n^2+3n+\alpha} \left(-\frac{q^3}{z^2}; q^2\right)_n}{z^{2n}}, \tag{1.6}$$

$$\psi_1(t, z, \alpha; q) = \frac{q}{z} \frac{1}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_n q^{\frac{n(n+1)}{2}+\alpha} \left(-\frac{q^2}{z}; q\right)_n}{z^n}, \tag{1.7}$$

$$F_1(t, z, \alpha; q) = \frac{1}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_n q^{2n^2-3n+\alpha} z^{4n}}{\left(\frac{z^2}{q}; q^2\right)_{n+1}}, \tag{1.8}$$

$$\chi_0(t, z, \alpha; q) = \frac{1}{(t)_{\infty}} \sum_{n=0}^{\infty} \frac{(t)_n q^{n\alpha} (z; q)_n}{(z^2 q^{-1}; q)_{2n}}, \tag{1.9}$$

$$\chi_1(t, z, \alpha; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n q^{n\alpha} (z; q)_n}{(z^2 q^{-1}; q)_{2n+1}}, \tag{1.10}$$

and the generalized seventh order mock theta functions are:

$$F_0(t, z, \alpha; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-3n+\alpha} (z; q)_n z^{2n}}{\left(\frac{z^2}{q}; q\right)_{2n}}, \tag{1.11}$$

$$F_1(t, z, \alpha; q) = 1 + \frac{1}{(t)_\infty} \sum_{n=1}^{\infty} \frac{(t)_n q^{n^2-3n+\alpha} (z; q)_{n-1} z^{2n}}{\left(\frac{z^2}{q}; q\right)_{2n-1}}, \tag{1.12}$$

$$F_2(t, z, \alpha; q) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n q^{n^2-2n+\alpha} (z; q)_n z^{2n}}{\left(\frac{z^2}{q}; q\right)_{2n+1}}. \tag{1.13}$$

For $t = 0, \alpha = 1$ and $z = q$ the above generalized functions reduce to Ramanujan's mock theta functions.

Now we define partial mock theta functions. By taking the partial sums of the series defined in (1.1)-(1.13) from 0 to N , we have the partial generalized mock theta functions of order five and seven as

$$f_{0,N}(t, z, \alpha; q) = \frac{1}{(t)_\infty} \sum_{n=0}^N \frac{(t)_n q^{n^2-3n+\alpha} z^{2n}}{(-z; q)_n}$$

Similarly for the other functions.

II. NOTATION

We shall use the following usual basic hypergeometric notations: For $|q^k| < 1$,

$$(a; q^k)_n = (1 - a)(1 - aq^k) \cdots (1 - aq^{k(n-1)}), \quad n \geq 1$$

$$(a; q^k)_0 = 1,$$

$$(a; q^k)_\infty = \prod_{j=0}^{\infty} (1 - aq^{kj}),$$

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right]_n = \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n$$

With $\binom{n}{2} = n(n-1)/2$, where $q \neq 0$ when $r > s + 1$.

III. A SUMMATION FORMULA AND A GENERAL THEOREM

We shall be using the following identity in developing the expansions. The identity is obtained by simple rearrangement of the series.

$$\sum_{r=0}^p \alpha_r \beta_r = \beta_{p+1} \sum_{r=0}^p \alpha_r + \sum_{m=0}^p (\beta_m - \beta_{m+1}) \sum_{r=0}^m \alpha_r. \tag{3.1}$$

Using the summation formula

$${}_2\phi_1 \left[\begin{matrix} a, b, q \\ e, f \end{matrix}; q \right]_n = \frac{(q-e)(e-abq)}{(aq-e)(e-bq)} \left[1 - \frac{(a, b)_{n+1}}{\left(\frac{e}{q}, \frac{abq}{e}\right)_{n+1}} \right] \tag{3.2}$$

and taking in (3.1),

$$\beta_m = \frac{(aq-e)(e-bq)}{(q-e)(e-abq)} \sum_{n=0}^{m-1} \frac{(a, b)_n q^n}{\left(\frac{e}{q}, \frac{abq^2}{e}\right)_n} = 1 - \frac{(a, b)_m}{\left(\frac{e}{q}, \frac{abq}{e}\right)_m} \tag{3.3}$$

we finally have

$$\begin{aligned} \sum_{r=0}^p \alpha_r \frac{(a, b)_r}{\left(\frac{e}{q}, \frac{abq}{e}\right)_r} &= \frac{(a, b)_{p+1}}{\left(\frac{e}{q}, \frac{abq}{e}\right)_{p+1}} \sum_{r=0}^p \alpha_r \\ &\quad - \frac{(aq-e)(e-bq)}{(q-e)(e-abq)} \sum_{m=0}^p \frac{(a, b)_m q^m}{\left(\frac{e}{q}, \frac{abq^2}{e}\right)_m} \sum_{r=0}^m \alpha_r. \end{aligned} \tag{3.4}$$

IV. CERTAIN RELATIONS BETWEEN MOCK THETA FUNCTIONS OF ORDER FIVE

In (3.1) putting $\beta_m = \sum_{r=0}^{m-1} q^{\lambda r} = \frac{q^{\lambda m} - 1}{q^{\lambda} - 1}$, where λ is a positive integer, we get

$$\begin{aligned} \frac{1}{q^{\lambda} - 1} \sum_{r=0}^p \alpha_r (q^{\lambda r} - 1) &= \frac{q^{\lambda(p+1)} - 1}{q^{\lambda} - 1} \sum_{r=0}^p \alpha_r - \sum_{m=0}^p q^{\lambda m} \sum_{r=0}^m \alpha_r. \end{aligned} \tag{4.1}$$

For various values of α_r and taking $\lambda=1$ or 2, (4.1) gives several interesting cases.

i. Taking $\alpha_r = \frac{(t)_r q^{r^2-3r+\alpha} z^{2r}}{(-z; q)_r}$ and $\lambda = 1$ in (4.1), we get

$$\begin{aligned} \sum_{r=0}^p \frac{(t)_r q^{r^2-3r+\alpha} z^{2r}}{(-z; q)_r} (1 - q^r) &= (1 - q^{p+1}) \sum_{r=0}^p \frac{(t)_r q^{r^2-3r+\alpha} z^{2r}}{(-z; q)_r} - (1 - q) \sum_{m=0}^p q^m \sum_{r=0}^m \frac{(t)_r q^{r^2-3r+\alpha} z^{2r}}{(-z; q)_r} \end{aligned}$$

or

$$\begin{aligned} \sum_{r=0}^p \frac{(t)_r q^{r^2-2r+\alpha} z^{2r}}{(-z; q)_r} &= q^{p+1} \sum_{r=0}^p \frac{(t)_r q^{r^2-3r+\alpha} z^{2r}}{(-z; q)_r} + (1 - q) \sum_{m=0}^p q^m \sum_{r=0}^m \frac{(t)_r q^{r^2-3r+\alpha} z^{2r}}{(-z; q)_r} \end{aligned}$$

which can be written as

$$(1 - q) \sum_{m=0}^p q^m f_{0,m}(t, z, \alpha; q) = -q^{p+1} f_{0,p}(t, z, \alpha; q) + f_{1,p}(t, z, \alpha; q). \tag{4.2}$$

ii. Taking $\alpha_r = \frac{{}_t r q^{r^2+r+r\alpha} \left(\frac{-q^3}{z^2}; q^2\right)_r}{z^{2r}}$ and $\lambda = 2$ in (4.1), we get

$$\begin{aligned} \sum_{m=0}^p q^{2m} \phi_{0,m}(t, z, \alpha; q) \\ = \frac{-q^{2p+2}}{1 - q^2} \phi_{0,p}(t, z, \alpha; q) + \frac{z^4}{q^5 (1 - q^2)} \phi_{1,p}(t, z, \alpha; q). \end{aligned} \tag{4.3}$$

iii. Taking $\alpha_r = \frac{{}_t r q^{\frac{r^2+r}{2}+r\alpha} \left(\frac{-q^2}{z}; q\right)_r}{z^r}$ and in $\lambda = 1$ (4.1), we get

$$\begin{aligned} \sum_{m=0}^p q^m \psi_{1,m}(t, z, \alpha; q) \\ = \frac{-q^{p+1}}{1 - q} \psi_{1,p}(t, z, \alpha; q) + \frac{1}{q(1 - q)} \psi_{0,p}(t, z, \alpha; q). \end{aligned} \tag{4.4}$$

iv. Taking $\alpha_r = \frac{{}_t r q^{r\alpha} (z; q)}{(z^2 q^{-1}; q)_{2r+1}}$ and $\beta_r = (1 - z^2 q^{2r-1})$ in (3.1), we get

$$\begin{aligned} \sum_{m=0}^p q^{2m-1} \chi_{1,m}(t, z, \alpha; q) \\ = \frac{(1 - z^2 q^{2p+1})}{z^2 (1 - q^2)} \chi_{1,p}(t, z, \alpha; q) - \frac{1}{z^2 (1 - q^2)} \chi_{0,p}(t, z, \alpha; q). \end{aligned} \tag{4.5}$$

v. Taking $\alpha_r = \frac{q^{2r^2+2r}}{(-q^2; q^2)_r}$, making $q \rightarrow q^2$ and setting $a = 0, b = -q^2, e = q^5$ in (3.4), we get

$$\begin{aligned} \sum_{r=0}^p \frac{q^{2r^2+2r}}{(-q^2; q^2)_r} \frac{(-q^2; q^2)_r}{(q^3; q^2)_r} &= \frac{(-q^2; q^2)_{p+1}}{(q^3; q^2)_{p+1}} \sum_{r=0}^p \frac{q^{2r^2+2r}}{(-q^2; q^2)_r} \\ &+ \frac{q^2(1 + q)}{(1 - q^3)} \sum_{m=0}^p \frac{(-q^2; q^2)_m q^{2m}}{(q^5; q^2)_m} \sum_{r=0}^m \frac{q^{2r^2+2r}}{(-q^2; q^2)_r} \end{aligned}$$

or

$$\begin{aligned} \sum_{r=0}^p \frac{q^{2r^2+2r}}{(q^3; q^2)_r} &= \frac{(-q^2; q^2)_{p+1}}{(q^3; q^2)_{p+1}} \sum_{r=0}^p \frac{q^{2r^2+2r}}{(-q^2; q^2)_r} \\ &+ \frac{q^2(1 + q)}{(1 - q^3)} \sum_{m=0}^p \frac{(-q^2; q^2)_m q^{2m}}{(q^5; q^2)_m} \sum_{r=0}^m \frac{q^{2r^2+2r}}{(-q^2; q^2)_r} \end{aligned}$$

and we have,

$$\begin{aligned}
 (1 - q)F_{1,p}(q) &= \frac{(-q^2; q^2)_{p+1}}{(q^3; q^2)_{p+1}} f_{1,p}(q^2) \\
 &+ \frac{q^2(1 + q)}{(1 - q^3)} \sum_{m=0}^p \frac{(-q^2; q^2)_m q^{2m}}{(q^5; q^2)_m} f_{1,m}(q^2).
 \end{aligned} \tag{4.6}$$

vi. Taking $\alpha_r = \frac{q^{2r^2+2r}}{(q^3; q^2)_r}$, $q \rightarrow q^2$, $a = 0$, $b = q^3$, $e = -q^4$ in (3.4), we get

$$\begin{aligned}
 f_{1,p}(q^2) &= \frac{(q^3; q^2)_{p+1}}{(-q^2; q^2)_{p+1}} (1 - q)F_{1,p}(q) \\
 &- \frac{q^2(1 - q^2)}{(1 + q^2)} \sum_{m=0}^p \frac{(q^3; q^2)_m q^{2m}}{(-q^4; q^2)_m} F_{1,m}(q).
 \end{aligned} \tag{4.7}$$

vii. Taking $\alpha_r = \frac{q^{2r^2}}{(-q^2; q^2)_r}$, $q \rightarrow q^2$, $a = 0$, $b = -q^2$, $e = q^3$ and in (3.4), we get

$$\begin{aligned}
 F_{0,p}(q) &= \frac{(-q^2; q^2)_{p+1}}{(q; q^2)_{p+1}} f_{0,p}(q^2) \\
 &+ \frac{q(1 + q)}{(1 - q)} \sum_{m=0}^p \frac{(-q^2; q^2)_m q^{2m}}{(q^3; q^2)_m} f_{0,m}(q^2).
 \end{aligned} \tag{4.8}$$

viii. Taking $\alpha_r = \frac{q^{2r^2}}{(q; q^2)_r}$, $q \rightarrow q^2$, $a = 0$, $b = q$, $e = -q^4$ and in (3.4), we get

$$f_{0,p}(q^2) = \frac{(q; q^2)_{p+1}}{(-q^2; q^2)_{p+1}} F_{0,p}(q) - \frac{q(1 + q)}{(1 + q^2)} \sum_{m=0}^p \frac{(q; q^2)_m q^{2m}}{(-q^4; q^2)_m} F_{0,m}(q). \tag{4.9}$$

V. EXPANSIONS FOR SEVENTH ORDER MOCK THETA FUNCTIONS

i. Taking $\alpha_r = \frac{(t)_r q^{r^2-3r+r\alpha} z^{2r}(z; q)_r}{\left(\frac{z^2}{q}; q\right)_{2r+1}}$ and $\lambda = 1$ in (4.1), we get

$$\begin{aligned}
 F_{2,p}(t, z, \alpha; q) &= q^{p+1} [F_{0,p}(t, z, \alpha; q) + F_{1,p+1}(t, z, \alpha; q)] \\
 &+ (1 - q) \sum_{m=0}^p q^m [F_{0,m}(t, z, \alpha; q) \\
 &+ F_{1,m+1}(t, z, \alpha; q)].
 \end{aligned} \tag{5.1}$$

ii. Taking $\alpha_r = \frac{(t)_r q^{r^2-2r+r\alpha} z^{2r}(z; q)_r}{\left(\frac{z^2}{q}; q\right)_{2r+1}}$ and $\lambda = 1$ in (4.1), we get

$$\begin{aligned} \sum_{m=0}^p q^m F_{2,m}(t, z, \alpha; q) &= \frac{-q^{p+1}}{(1-q)} F_{2,p}(t, z, \alpha; q) \\ &\quad - \frac{q}{(1-q)} [1 - F_{1,p+1}(t, z, \alpha; q)]. \end{aligned} \tag{5.2}$$

iii. Taking $\alpha_r = \frac{(t)_r q^{r^2-2r+r\alpha} z^{2r}(z;q)_r}{\left(\frac{z^2}{q};q\right)_{2r+1}}$ and $\beta_r = q^{-r}(1 - z^2 q^{2r-1})$ in (3.1), we get

$$\begin{aligned} F_{0,p}(t, z, \alpha; q) &= q^{-(p+1)}(1 - z^2 q^{2p+1}) F_{2,p}(t, z, \alpha; q) \\ &\quad - \frac{(1-q)}{q} \sum_{m=0}^p \frac{(1 + z^2 q^{2m})}{q^m} F_{2,m}(t, z, \alpha; q). \end{aligned} \tag{5.3}$$

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