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By S. Yadav, D. L. Suthar

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A Quarter Symmetric Non-metric Connection in a Generalized Co-symplectic Manifolds

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I. INTRODUCTION


II. PRELIMINARIES

An $n$-dimensional differentiable manifold $M_n$ is an almost contact manifold if it admits a tensor field of type $F$, a vector field $U$ and 1-form $u$ satisfying for arbitrary vector field $X$.

\[(a) \quad \overline{X} + X = A(X) \quad (b) \quad U = 0 \quad (2.1)\]

where $\overline{X} \equiv FX$

Again (2.1) (a) and (2.1) (b), gives

\[(a) \quad u(\overline{X}) = 0 \quad (b) \quad u(U) = 1 \quad (2.2)\]

An almost contact manifold $M_n$ in which a Riemannian metric tensor $g$ of type $(0, 2)$ satisfies

\[(a) \quad g(\overline{X}, \overline{Y}) = g(X, Y) - u(X)u(Y) \quad (b) \quad g(X, U) = u(X) \quad (2.3)\]

for arbitrary vector field $X$ and $Y$, is called an almost contact metric manifold.

Let us put

\[\ ^{I}F(X, Y) = g(\overline{X}, Y) \]

Then, we obtain

\[(a) \quad \ ^{I}F(\overline{X}, \overline{Y}) = \ ^{I}F(X, Y) \quad (b) \quad \ ^{I}F(X, Y) = g(\overline{X}, Y) = -g(X, \overline{Y}) = -\ ^{I}F(Y, X) \quad (2.4)\]

An almost contact metric manifold satisfying

\[(D_{X}^{\ ^{I}F})(Y, Z) = u(Y)(D_{X}u)(Z) - u(Z)(D_{X}u)(Y) \quad (2.5)\]

\[(D_{X}^{\ ^{I}F})(Y, Z) + (D_{Y}^{\ ^{I}F})(Z, X) + (D_{Z}^{\ ^{I}F})(X, Y) + u(X)[(D_{Y}u)(Z) - (D_{Z}F)(\overline{Y})] \quad (2.6)\]

\[+ u(Y)[(D_{Z}u)(\overline{Z}) - (D_{Y}u)(\overline{Z})] + u(Z)[(D_{X}u)(\overline{Y}) - (D_{Y}u)(\overline{X})] = 0\]

for arbitrary vector field $X$, $Y$, $Z$ are respectively called generalized co-symplectic and generalized quasi-Sasakian manifolds[11].

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If on any manifold $U$, satisfies

(a) $\left(D_{\chi}u\right)(Y) = -\left(D_{\xi}u\right)(Y) = \left(D_{\eta}u\right)(X)$ \hspace{1cm} (2.7)

(b) $\left(D_{\xi}u\right)(Y) = -\left(D_{\tau}u\right)(Y) = \left(D_{\eta}u\right)(X)$ and

(c) $\left(D_{U_{1}}F\right) = 0$

then $U_{1}$ is said to be the first class.

If on an almost contact metric manifold $U$ satisfies

(a) $\left(D_{\chi}u\right)(Y) = \left(D_{\tau}u\right)(Y) = -\left(D_{\eta}u\right)(X)$ \hspace{1cm} (2.8)

(b) $\left(D_{\xi}u\right)(Y) = -\left(D_{\tau}u\right)(Y) = \left(D_{\eta}u\right)(X)$ and

(c) $\left(D_{U_{2}}F\right) = 0$

then $U_{2}$ is said to be the second class.

The Nijenhuis tensor in a generalized co-symplectic manifold is given by

(a) $N(X,Y) = \left(D_{\chi}F\right)Y - \left(D_{\tau}F\right)(X) - \left(D_{\eta}F\right)(Y) + \left(D_{\xi}F\right)(X)$ \hspace{1cm} (2.9)

(b) $\left(D_{\chi}F\right)(Y,Z) = \left(D_{\tau}F\right)(Y,Z) - \left(D_{\eta}F\right)(X,Z) + \left(D_{\xi}F\right)(X,Z) - \left(D_{\xi}F\right)(X,Z)$

**III. Quarter Symmetric Non-Metric Connection in a Generalized Co-Symplectic Manifold**

Let $(M^n, g)$ be a generalized co-symplectic manifold with Riemannian connection $D$. We define a linear connection $B$ on $(M^n, g)$ by

$$B_XY = D_X Y + u(Y)X + a(X)FY$$ \hspace{1cm} (3.1)

where $u$ and $a$ is 1-form associated with vector field $\xi$ and $A$ on $(M^n, g)$ that is

(a) $g(X, U) = u(X)$ and

(b) $g(X, A) = a(X)$ \hspace{1cm} (3.2)

for all vector field $X \in \mathcal{X}(M^n)$, where $\mathcal{X}(M^n)$ is the set of all differentiable vector fields on $(M^n, g)$.

Using (3.1) the torsion tensor $T$ of $(M^n, g)$ with respect to connection $B$ is given by

$$T(X,Y) = u(Y)X - u(X)FY + a(X)FY - a(Y)FX$$ \hspace{1cm} (3.3)

A linear connection satisfying (3.3) is called Quarter-symmetric connection and metric tensor $g$ satisfies [10].

$$\left(B_X g\right)(Y,Z) = -u(Y)g(FX,Z) - u(Z)g(FY,Y) - 2a(X)g(FY,Z) - 2\left(H(X,Y)\right)$$ \hspace{1cm} (3.4)

for arbitrary vector field $X, Y, Z$.

Then a linear connection $B$ defined by (3.1) satisfies (3.3) and (3.4) is called a quarter-symmetric non-metric connection.

If we put

$$B_XY = D_X Y + H(X,Y)$$

Where $H$ is a tensor of type $(0, 2)$, then we have

$$\begin{align*}
(i) & \quad H(X,Y) = u(Y)X + a(X)FY \\
(ii) & \quad H(X,Y,Z) = u(Y)g(X,Z) + a(X)g(FY,Z) \\
(iii) & \quad T(X,Y,Z) = u(Y)g(FX,Z) - u(X)g(FY,Y) + a(X)g(FY,Z) - a(Y)g(FX,Z) \\
(iv) & \quad \left(B_X u\right)(Y) = \left(D_X u\right)(Y) + g(X,Y) + H(X,Y)
\end{align*}$$ \hspace{1cm} (3.5)
where

\[ H(X,Y,Z) = \frac{1}{2} g(H(X,Y)Z) \]

\[ T(X,Y,Z) = \frac{1}{2} g(T(X,Y)Z) \]

we have

\[ X'(F,Y,Z) = (D_X F)(Y,Z) + (D_Y F)(Z,X) + F(Y,D_X Z) \]

\[ = (B_X F)(Y,Z) + (B_Y F)(Z,X) + F(Y,D_X Z) \]

Using (3.1) in the above equation, we get

\[ X'(F,Y,Z) = (B_X F)(Y,Z) + (B_Y F)(Z,X) + F(Y,D_X Z) + F(Y,F,Z) \]

(3.6)

The Nijenhuis tensor \( N \) in term of quarter symmetric non metric connection \( B \) is given by

\[
\begin{cases}
(i) & N(X,Y) = (B_X F)(Y) - (B_Y F)(X) + (B_X F)(Y) - (B_Y F)(X) \\
(ii) & N(X,Y,Z) = (B_X F)(Y,Z) - (B_Y F)(X,Z) - (B_X F)(Y,\bar{Z}) - (B_Y F)(X,\bar{Z})
\end{cases}
\]

(3.7)

**Theorem 3.1**: A generalized co-symplectic manifold admitting quarter symmetric non-metric connection such that \( B_X F = 0 \), then \( F \) is locally killing provided the vector fields \( X,Y,Z \) are orthogonal to \( U \).

**Proof**: From (3.6), we have

\[ (B_X F)(Y,Z) = (D_X F)(Y,Z) + u(Y) F(X,Z) + a(X) F(\bar{Y},Z) + u(Z) F(Y,X) + a(X) F(Y,\bar{Z}) \]

Since \( B_X F = 0 \), we get

\[ (D_X F)(Y,Z) = -u(Y) F(X,Z) - a(X) F(\bar{Y},Z) - u(Z) F(Y,Z) - a(X) F(Y,\bar{Z}) \]

(3.8)

Similarly

\[ (D_Y F)(X,Z) = -u(X) F(Y,Z) - a(Y) F(\bar{X},Z) - u(Z) F(X,Z) - a(Y) F(X,\bar{Z}) \]

(3.9)

By virtue of equation (3.8) and (3.9), we get

\[ (D_X F)(Y,Z) + (DF)(X,Z) = [u(Y) + u(Z)] F(Y,Z) - [u(Y) + u(Z)] F(Y,Z) \]

(3.10)

Taking the vector field \( X,Y,Z \) orthogonal to \( U \), we get

we get the required result.

**Theorem 3.2**: A generalized co-symplectic manifolds admitting quarter symmetric non-metric connection is locally closed with respect to this connection \( D \) if and only if it is locally closed with respect to Riemannian connection provided the vector fields \( X,Y,Z \) orthogonal to \( \xi \).

**Proof**: We have

\[ X'(F,Y,Z) = (D_X F)(Y,Z) + (D_Y F)(Z,X) + F(Y,D_X Z) \]

\[ = (B_X F)(Y,Z) + (B_Y F)(Z,X) + F(Y,D_X Z) \]

Using (3.1), we get

\[ (B_X F)(Y,Z) = (D_X F)(Y,Z) + u(Y) F(X,Z) + a(X) F(\bar{Y},Z) + u(Z) F(Y,X) + a(X) F(Y,\bar{Z}) \]

(3.11)

from (3.11), we obtained

\[ (B_X F)(Y,Z) + (B_Y F)(Z,X) + (B_Z F)(X,Y) = (D_X F)(Y,Z) + (D_Y F)(Z,X) + (D_Z F)(X,Y) \]

\[ + 2u(Y) F(X,Z) + a(X) F(\bar{Y},Z) + F(Y,\bar{Z}) \]

\[ + 2u(Z) F(Y,X) + a(Y) F(\bar{Z},X) + F(Z,\bar{X}) \]

\[ + 2u(X) F(Z,Y) + F(Z,Y) + a(Z) F(\bar{X},Y) + F(X,\bar{Y}) \]

Using (2.4)(b) in above, we have

\[ (B_X F)(Y,Z) + (B_Y F)(Z,X) + (B_Z F)(X,Y) = (D_X F)(Y,Z) + (D_Y F)(Z,X) + (D_Z F)(X,Y) \]

\[ + 2u(Y) F(X,Z) + u(Y) F(X,Z) + u(Z) F(Y,X) \]

\[ + u(X) F(Z,Y) + u(Y) F(X,Z) + u(Z) F(Y,X) \]
Taking the vector field $X, Y, Z$ orthogonal to $U$, we get
\[
(B_X F)(Y, Z) + (B_Y F)(Z, X) + (B_Z F)(X, Y)
= (D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) = 0
\] (3.12)

**Theorem 3.3**: A generalized co-symplectic manifolds admitting quarter symmetric non-metric connection satisfies the following relations

i. $(B_X F)(Y) = (D_X F)(Y)$
ii. $(B_X F)(\bar{Y}) = (D_X F)(\bar{Y})$
iii. $N(X, Y) = 0$ (Complete integrable) if $D_X F = 0$

**Proof**: From (3.1), we have
\[
B_X Y = D_X Y + H(X, Y)
\] (3.13)
where
\[
H(X, Y) = u(Y)X + a(X)FY
\]
for any vector field for $\bar{Y}$, equation (3.13) can be written as
\[
(B_X F)(Y) = (D_X F)(Y) - B_X \bar{Y} + (D_X \bar{Y} - a(X)Y + A(Y)a(X)T)
\] (3.14)
Operating both side equation (3.13) by
\[
\frac{B_X \bar{Y} - D_X \bar{Y}}{\bar{Y}} = u(Y)X - a(X)Y + a(X)A(Y)T
\] (3.15)
Using (3.15) in (3.14), we get
\[
(B_X F)(Y) = (D_X F)(Y) - u(Y)\bar{X}
\] (3.16)
Barring $X, Y$ and respectively in (3.16) and using (2.2) (a), we get the required result (i, ii).

Since $(D_X F)(Y) = 0$,
Then from (3.16), we get
\[
(B_X F)(Y) = -u(Y)\bar{X}
\] (3.17)
Barring $X$ and using (2.1)(a), we get
\[
(B_X F)(Y) = u(Y)X - u(Y)A(X)T
\] (3.18)
Interchanging $X$ and $Y$, we get
\[
(B_Y F)(X) = u(X)Y - u(X)A(Y)T
\] (3.19)
Operating both side equation (3.17) by $F$, we get
\[
B_X F)(Y) = u(Y)X - u(Y)A(X)T
\] (3.20)
Interchanging $X$ and $Y$ we get
\[
B_Y F)(\bar{X}) = u(X)Y - u(X)A(Y)T
\] (3.21)
Using (3.18),(3.19),(3.20)and(3.21)in (3.7)(i), we get the result (iii).

Finally barring $X, Y, Z$ in (3.5) (ii, iii), we get the result (iv).

**Theorem 3.4**: If $U$ is a killing on generalized co-symplectic manifolds with quarter symmetric non-metric connection then
\[
\{N(X, Y, Z) - (B_X F)(Y, Z) - (B_Y F)(X, Z) - (B_Z F)(X, Y) = n\}
(F(\bar{X}, Z)
\]

**Proof**: From (3.11) and (2.9) (b), we have
\[
\{N(X, Y, Z) - (B_X F)(Y, Z) - (B_Y F)(X, Z) - (B_Z F)(X, Y) = (D_X F)(Y, Z) - (D_Y F)(X, Z)
- (D_Z F)(X, Y) + u(Y)F(\bar{X}, Z) + u(Z)F(\bar{Y}, X) - u(X)F(X, \bar{Y}) - u(Z)F(\bar{Z}, X)
\]
Again using (2.5) in above equation, we get

\[-u(X)^i F(Y, Z)\]

\[
\begin{aligned}
&-N(X, Y, Z) = \langle B_x, F \rangle(Y, Z) - (B_y, F)(X, Z) - (B_z, F)(X, Y)
\quad = -u(X) \left[ (D_x u)(Z) + (D_y u)(Y) \right] \\
&+ u(Y) \left[ (D_x u)(Z) + (D_z u)(X) \right] + u(Z) \left[ (D_x u)(Z) - (D_y u)(X) \right] \\
&+ (u(Y) + u(Z)) F(\bar{X}, Z) 
\end{aligned}
\]

Since \( U \) is a killing then putting \( \langle D_x u \rangle(Y) + (D_y u)(x) = 0 \) in above equation, we obtained

\[
\begin{aligned}
(B_x, F)(Y, Z) + (B_y, F)(\bar{Z}, X) + (B_z, F)(X, Y)
&= -N(X, Y, Z) - u(Z)[(D_x u)(\bar{X}) - (D_y u)(\bar{Y})] \\
&- (u(Y) + u(Z)) F(\bar{X}, Z) 
\end{aligned}
\]

(3.22)

By virtue of equation (2.7) (b), equation (3.22) will be reduces

\[
\begin{aligned}
(B_x, F)(Y, Z) + (B_y, F)(\bar{Z}, X) + (B_z, F)(X, Y)
&= -N(X, Y, Z) - u(Z)^i F(\bar{X}, Z) 
\end{aligned}
\]

we get the required result.

**Theorem 3.5**: A generalized co-symplectic manifold of first class with quarter symmetric non-metric connection satisfy

\[
(B_x, F)(\bar{Y}, Z) + (B_y, F)(Z, \bar{X}) + (B_z, F)(\bar{X}, \bar{Y}) = 0
\]

**Proof**: From equation (2.5) and (3.11), we get

\[
\begin{aligned}
(B_x, F)(Y, Z) = u(Y) \left[ (D_x u)(Z) + (D_y u)(\overline{Z}) \right] \\
&- u(Z)(D_x u)(\overline{Y}) + a(X)^i F(\overline{Y}, Z)
\end{aligned}
\]

(3.23)

Taking cyclic sum of equation (3.23) in \( X, Y, \) and \( Z \) we get

\[
\begin{aligned}
(B_x, F)(Y, Z) + (B_y, F)(Z, X) + (B_z, F)(X, Y)
&= u(X) \left[ (D_x u)(\overline{Y}) - (D_y u)(\overline{Z}) \right] \\
&+ u(Y) \left[ (D_x u)(\overline{Z}) - (D_z u)(X) \right] \\
&+ u(Z) \left[ (D_x u)(\overline{X}) - (D_y u)(Y) \right] \\
&+ (u(Y) + u(Z)) F(\overline{X}, Z) + u(Z) F(\overline{Y}, X) \\
&+ (u(X)^i F(\overline{Z}, Y)
\end{aligned}
\]

(3.24)

Barring \( X, Y \) and \( Z \) in equation (3.24) and using (2.2)(a), we get

\[
\begin{aligned}
(B_x, F)(\overline{Y}, Z) + (B_y, F)(\overline{Z}, X) + (B_z, F)(\overline{X}, \overline{Y}) = 0
\end{aligned}
\]

we get the required result.

**Theorem 3.6**: An almost contact metric manifold admitting quarter symmetric non-metric connection \( B \) is a generalized co-symplectic if

\[
\begin{aligned}
(B_x, F)(Y, Z) = u(Y) \left[ (B_x u)(\overline{Z}) + (B_y u)(\overline{X}) - (B_z u)(\overline{Y}) \right] + u(Z) \left[ 2^i F(X, Y) - (B_x u)(\overline{Y}) + F(\overline{X}, \overline{Y}) \right]
\end{aligned}
\]

**Proof**: From equation (2.5) and (3.11), we have

\[
\begin{aligned}
(B_x, F)(Y, Z) = u(Y) \left[ (B_x u)(\overline{Z}) + (B_y u)(\overline{X}) - (B_z u)(\overline{Y}) \right] + u(Z) \left[ F(X, Y) - (B_x u)(\overline{Y}) \right]
\end{aligned}
\]

(3.25)

Using (2.4)(b), (3.5)(iv) in (3.25), we obtain

\[
\begin{aligned}
(B_x, F)(Y, Z) = u(Y) \left[ (B_x u)(\overline{Z}) + (B_y u)(\overline{X}) - (B_z u)(\overline{Y}) \right] + u(Z) \left[ 2^i F(X, Y) - (B_x u)(\overline{Y}) + F(\overline{X}, \overline{Y}) \right]
\end{aligned}
\]

(3.26)

we get the required result.

**Theorem 3.7**: A quasi-Sasakian manifold is normal if and only if

\[
\begin{aligned}
(B_x, F)(Y, Z) = u(Y) \left[ (B_x u)(\overline{Z}) + (B_y u)(\overline{X}) - (B_z u)(\overline{Y}) \right] + u(Z) \left[ -2^i F(X, Y) - (B_x u)(\overline{Y}) \right]
\end{aligned}
\]

**Proof**: The necessary and sufficient conditions for a quasi-Sasakian manifold to be normal [11] is

\[
\begin{aligned}
(D_x, F)(Y, Z) = u(Y) (D_z u)(\overline{X}) + u(Z) (D_x u)(Y)
\end{aligned}
\]

(3.27)

Using (3.11) in (3.27), we get

\[
\begin{aligned}
(B_x, F)(Y, Z) = u(Y) \left[ (D_z u)(\overline{X}) + (B_x u)(\overline{Z}) \right] + u(Z) \left[ (D_x u)(Y) + F(\overline{X}, \overline{Y}) \right]
\end{aligned}
\]

(3.28)
Theorem 3.8: A generalized co-symplectic manifold is quasi-Sasakian manifold if

\[ (B_\chi u)(\overline{Z}) = (B_\chi u)(\overline{X}) + 2'F(X,Y) \]

where \( B \) is the quarter symmetric non-metric connection.

Proof: From (3.11), we have

\[
- 2u(Y) F(X,Z) - a(X) \left[ F(\overline{Y},Z) + F(Y,\overline{Z}) \right]
- 2u(Z) F(Y,X) - a(Y) \left[ F(\overline{Z},X) + F(Z,\overline{X}) \right]
- 2u(X) \left[ F(Z,Y) + F(\overline{Z},\overline{Y}) \right]
\]  

By virtue of equation (3.26) and (3.29), we get

\[
(D_\chi F)(Y,Z) + (D_Y F)(Z,X) + (D_Z F)(X,Y) = u(Y) (B_\chi u)(\overline{Z}) - 2'F(X,Z) - (B_\chi u)(\overline{X})
+ u(Z) (B_\chi u)(\overline{Y}) - (B_Y u)(\overline{Y}) - 2'F(Y,X)
+ 2u(X) (B_\chi u)(\overline{Y}) - (B_\chi u)(\overline{Z}) - 2'F(Z,Y)
\]

Since the manifold is quasi-Sasakian, therefore

\[(D_\chi F)(Y,Z) + (D_Y F)(Z,X) + (D_Z F)(X,Y) = 0\]

From equation (3.30) and (3.31), we get

\[ (B_\chi u)(\overline{Z}) = (B_\chi u)(\overline{X}) + 2'F(X,Y) \]

we get the required result.

Theorem 3.9: If the generalized co-symplectic manifold is of first class with respect to Riemannian connection \( D \), then it is also first class with respect to the quarter symmetric non-metric connection \( B \) and satisfies

\[ (B_\chi F)(Y,Z) = u(Y) (B_\chi u)(\overline{Z}) + 2'F(X,Z) - (B_\chi u)(\overline{X}) \]

Proof: Barring \( X \) and \( Y \) in (3.5) (iv) respectively and using (2.3) (a), we have

\[ (D_\chi u)(Y) = (B_\chi u)(Y) - g(\overline{X},Y) - F(\overline{X},Y) \]  

and

\[ (D_\chi u)(\overline{Y}) = (B_\chi u)(\overline{Y}) - g(X,Y) - F(X,Y) \]

Adding (3.32), (3.33) and using (2.4) (b), we get

\[ (D_\chi u)(Y) + (D_\chi u)(\overline{Y}) = (B_\chi u)(Y) + (B_\chi u)(\overline{Y}) \]

By virtue of (2.7) (a) and (3.34), we have

\[ (B_\chi u)(Y) = -(B_\chi u)(\overline{Y}) \]  

Again in similar way, we have

\[ (B_\chi u)(\overline{Y}) = (B_\chi u)(\overline{X}) \]

From (3.35) and (3.36), we get

\[ (B_\chi u)(\overline{Y}) = (B_\chi u)(\overline{X}) \]

Taking covariant derivative of \( FY = \overline{Y} \) with respect to \( B \) and using (2.1) (a), (2.2) (a) and (3.11), we get

\[ (B_\chi F)(Y) = (D_\chi F)(Y) - u(Y) \overline{X} \]

Replacing \( X \) by \( U \) in (3.38) and using (2.1) (a), (2.8) (c), we obtain

\[ (B_\chi u)(Y) = 0 \]

By virtue of equation (2.5), (3.5) (iv) and (3.11), we get
We get the required result.

REFERENCES  RÉFÉRENCES REFERENCIAS
