Global Journal of Science Frontier Research
Volume 11 Issue 1 Version 1.0 February 2011
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc (USA)
ISSN: 0975-4350

Schur Convexity and Hadamard's Inequality

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SCHUR CONVEXITY AND HADAMARD S INEQUALITY

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Schur Convexity and Hadamard's Inequality

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I. Introduction

A function $f : I \subset \Re \to \Re$ is convex on I, *if*

$$f\left(\lambda x + (1-\lambda)y\right) \le \lambda f(x) + (1-\lambda) f(y), \quad \forall\, x, y \in I,\ \lambda \in [0,1]. \tag{1.1}$$

Suppose that I is an open interval with nonempty interior. A real valued function $F : I^n \to \Re$ is Schur convex on I^n if

$$F\left(x_1,\ldots x_n\right) \le F\left(y_1,\ldots, y_n\right) \tag{1.2}$$

for each $n-tuples$ $x = \left(x_1,\ldots x_n\right),\ y = \left(y_n,\ldots y_n\right) \in I^n$ with $x \prec y$, that is

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]},\ k = 1,2,\ldots,n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the i-th largest component in x.

Any function g is concave if $-g$ is convex. Also we set $G_x = \dfrac{\partial G}{\partial x}$.

Let $f : I \subset \Re \to \Re$ be a convex function. The well-known Hadamards inequality is the following

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \quad a, b \in I. \tag{1.3}$$

The following results are known :

Theorem 1.1[3]. *Suppose that $I \subset R$ is an interval and $\varphi : I^n \to R$ is a continuous symmetric function. If φ is differentiable on I^n, then φ is Schur convex (concave)on I^n if and only if*

$$\left(x_i - x_j\right)\left(\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j}\right) \ge (\le) 0, \tag{1.4}$$

for all $x_i, x_j \in I, \; i, j = 1,2,...,n$.

Theorem 1.2[2].. *Let* f *be a continuous function on an interval I, and*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)\,dt, & x, y \in I, \; x \neq y, \\ f(x), & x = y \in I. \end{cases}$$

Then $F(x,y)$ *is Schur convex on* I^2 *if and only if* f *convex on I.*

Theorem 1.3[1]. *Suppose that* I *is an open interval and* $f : I \to \Re$ *is a continuous function. If*

$$G(x,y) = \begin{cases} \frac{f(x)+f(y)}{2} - \frac{1}{y-x}\int_x^y f(t)\,dt, & x, y \in I, \; x \neq y, \\ 0, & x = y \in I. \end{cases} \qquad (1.5)$$

then $G(x,y)$ *is Schur convex (concave) on* I^2 *if and only if* f *is convex (concave) on I.*

Lemma 1.4[4]. *Let* $I \subset \Re$ *be an open interval and* $f : I \to \Re$ *be a continuous function, then* f *is convex (concave) on* I *if and only if*

$$\frac{1}{y-x}\int_x^y f(t)\,dt \le (\ge) \frac{f(x)+f(y)}{2}$$

or

$$\frac{1}{y-x}\int_x^y f(t)\,dt \ge (\le)\, f\left(\frac{x+y}{2}\right)$$

for all $x, y \in I$ *with* $x \neq y$.

We define the following functions

$$G(x,y) = \begin{cases} \frac{1}{2} f\left(\frac{x+y}{2}\right) + \frac{\left(f(x)+f(y)\right)}{4} - \frac{1}{y-x}\int_x^y f(t)\,dt, & x, y \in I, \; x \neq y, \\ 0, & x = y \in I. \end{cases} \qquad (1.6)$$

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f\left(tu + (1-t)\frac{x+y}{2}\right)du, & x, y \in I, \; x \neq y, \\ f(x), & x = y \in I. \end{cases} \qquad (1.7)$$

2. Lemmas

The following lemmas are needed for our aim

Lemma 2.1. *Let* $I \subset \Re$ *be an open interval and* $f : I \to \Re$ *be a continuous function , then* f *is convex (concave) on* I *if and only if*

$$\frac{1}{y-x}\int_x^y f(t)\,dt \le (\ge)\ \frac{1}{2} f\left(\frac{x+y}{2}\right)+\frac{f(x)+f(y)}{4} \tag{2.1}$$

for all $x, y \in I$ *with* $x \ne y$.

Proof. Suppose first f is convex, then by lemma 1.4,

$$\frac{1}{y-x}\int_x^y f(t)\,dt = \frac{1}{y-x}\left(\int_x^{\frac{x+y}{2}} f(t)\,dt + \int_{\frac{x+y}{2}}^{y} f(t)\,dt\right)$$

$$\le \frac{1}{2}\left(\frac{f\left(\frac{x+y}{2}\right)+f(x)}{2}+\frac{f(y)+f\left(\frac{x+y}{2}\right)}{2}\right)$$

$$= \frac{1}{2} f\left(\frac{x+y}{2}\right)+\frac{f(x)+f(y)}{4}.$$

Now suppose that (2.1) is satisfied, and f is not convex. Then by lemma 1.4, there exits $p, q \in I$ with $p < q$ such that the inequalities

$$\frac{2}{q-p}\int_p^{\frac{p+q}{2}} f(t)\,dt \le \frac{1}{2}\left(f\left(\frac{p+q}{2}\right)+f(p)\right), \quad \frac{2}{q-p}\int_{\frac{p+q}{2}}^{q} f(t)\,dt \le \frac{1}{2}\left(f\left(\frac{p+q}{2}\right)+f(q)\right)$$

are not valid. Therefore via adding the above inequalities, the inequality

$$\frac{1}{q-p}\int_p^q f(t)\,dt \le \frac{1}{2} f\left(\frac{p+q}{2}\right)+\frac{f(p)+f(q)}{4}$$

is not valid, which contradicts (2.1).

Lemma 2.2. *Let* $G(x,y)$ *be as defined in (1.6), If* f *has continuous third derivatives on* I, *then*

$$G_x(x,x) = 0,$$

for all $x \in I$.

Proof.

$$G_x(x,y) = \frac{1}{4(y-x)^2}\left((y-x)^2\left(f'(x)+f'\left(\frac{x+y}{2}\right)\right)+4(y-x)f(x)-4\int_x^y f(t)\,dt\right).$$

Making use of L'Hopitals rule, we have

$$G_x(x,x) = \lim_{x\to y}\frac{1}{8(y-x)}\times$$

$$(y-x)^2\left(f''(x)+\frac{1}{2}f''\left(\frac{x+y}{2}\right)\right)-2(y-x)\left(f'(x)+f'\left(\frac{x+y}{2}\right)\right)+4(y-x)f'(x)$$

$$=\lim_{x\to y}\frac{1}{-8}\left((y-x)^2\left(f'''(x)+\frac{1}{4}f'''\left(\frac{x+y}{2}\right)\right)-2(y-x)\left(f''(x)+\frac{1}{2}f''\left(\frac{x+y}{2}\right)\right)\right.$$

$$\left.+2\left(f'(x)+f'\left(\frac{x+y}{2}\right)\right)-4(y-x)f'''(x)-4f'(x)-2(y-x)\left(f''(x)+f''\left(\frac{x+y}{2}\right)\right)\right)=0.$$

Lemma 2.3. *Let* $G(x,y)$ *be as defined in (1.6), and let*

$$H(x,y)=G_y(x,y)-G_x(x,y),$$

If f has contiuous *second order derivatives on I, with* $x<y$, *then* $H(x,y)\ge 0$ *for all*
$x,y\in I$.

Proof. We have

$$G_x=\frac{1}{4}f'\left(\frac{x+y}{2}\right)+\frac{1}{4}f'(x)+\frac{f(x)}{y-x}-\frac{1}{(y-x)^2}\int_x^y f(t)\,dt,$$
$$G_y=\frac{1}{4}f'\left(\frac{x+y}{2}\right)+\frac{1}{4}f'(y)-\frac{f(x)}{y-x}+\frac{1}{(y-x)^2}\int_x^y f(t)\,dt.$$

Therefore,

$$H(x,y)=\frac{1}{4}\left(f'(y)-f'(x)\right)-\frac{f(x)+f(y)}{2}+\frac{2}{(y-x)^2}\int_x^y f(t)\,dt$$
$$=\frac{1}{(y-x)^2}\left(\frac{1}{4}(y-x)^2\left(f'(y)-f'(x)\right)-(y-x)\left(f(x)+f(y)+2\int_x^y f(t)\,dt\right)\right)$$
$$=\frac{1}{(y-x)^2}h(x,y).$$

Now,

$$h_y(x,y)=\frac{1}{4}(y-x)^2f''(y)+\frac{1}{2}(y-x)\left(f'(y)-f'(x)\right)-(y-x)f'(y)+f(y)-f(x)$$
$$\ge f(y)-f(x)-(y-x)f'(y)$$
$\ge f(y)-f(x)-(y-x)f'(s)$, for some $s\in(a,b)$, by the mean value theorem
≥ 0. Therefore $H(x,y)\ge 0$.

Lemma 2.4. *Let* $F(x,y)$ *be as defined in (1,7), if* f *has continuous second order derivatives on* I, *then*

$$K(x,y)=F_y-F_x\ge 0,$$

for all $x,y\in I$ with $x<y$.

Proof. We have

$$K(x,y)=\frac{1}{y-x}\left(f\left(ty+(1-t)\frac{x+y}{2}\right)+f\left(tx+(1-t)\frac{x+y}{2}\right)\right)-\frac{2}{(y-x)^2}\int_x^y f\left(tu+(1-t)\frac{x+y}{2}\right)du$$

$$=\frac{1}{(y-x)^2}\left((y-x)\left(\left(f\left(ty+(1-t)\frac{x+y}{2}\right)+f\left(tx+(1-t)\frac{x+y}{2}\right)\right)\right)-2\int_x^y f\left(tu+(1-t)\frac{x+y}{2}\right)du\right)$$

$$=\frac{k(x,y)}{(y-x)^2}.$$

Now,

$$k_x(x,y)=(y-x)\left(f'\left(ty+(1-t)\frac{x+y}{2}\right)\left(\frac{1-t}{2}\right)+f\left(tx+(1-t)\frac{x+y}{2}\right)\left(\frac{1+t}{2}\right)\right)$$
$$+f\left(tx+(1-t)\frac{x+y}{2}\right)-f\left(ty+(1-t)\frac{x+y}{2}\right)$$
$$=0,\ \text{ if }\ x=y.$$

$$k_{xx}(x,y)=(y-x)\left(f''\left(ty+(1-t)\frac{x+y}{2}\right)\left(\frac{1-t}{2}\right)^2+f\left(tx+(1-t)\frac{x+y}{2}\right)\left(\frac{1+t}{2}\right)^2\right)$$

$$+(t-1)f'\left(ty+(1-t)\frac{x+y}{2}\right).$$

$k(x,x)\ \big]_{x=y}=+(t-1)f'(x)\geq 0$. This shows that $k(x,y)$ attains its minimum at $x=y$ which is 0. Therefore $k(x,y)\geq 0$, which implies $K(x,y)\geq 0$.

3. Main Results

Theorem 3.1. *Let $f:I\to\Re$ be a continuous function and $G(x,y)$ be as defined in (1.6). Then $G(x,y)$ is Schur convex (concave) on I^2 if and only if f is convex (concave) on I.*

Necessity. If $G(x,y)$ is Schur convex on I^2, then from (1.6) and definition of Schur convex function together with the fact that $\left(\frac{x+y}{2},\frac{x+y}{2}\right)\prec(x,y)$ we obtain

$$\frac{1}{y-x}\int_x^y f(t)\,dt\leq\frac{1}{2}f\left(\frac{x+y}{2}\right)+\frac{f(x)+f(y)}{4},\tag{3.1}$$

for all $x,y\in I$ with $x\neq y$. The convexity of f on I follows from (3.1) and lemma (2.1).

Sufficiency. We have two cases to be considered :

Case 1. $x = y \in I$. Lemma (2.2) implies

$$(y-x)\left(G_y(x,y) - G_x(x,y)\right) = 0. \tag{3.2}$$

Case 2. $x \neq y,\ x, y \in I$. Lemma (2.3) leads to

$$(y-x)\left(G_y(x,y) - G_x(x,y)\right) \geq 0. \tag{3.3}$$

Therefore, $G(x,y)$ is Schur convex on I^2 follows from (3.2), (3.3) and theorem (1.1).

Theorem 3.2. *Let $f : I \to \Re$ be a continuous non-increasing function and $F(x,y)$ be as defined in (1.7). Then $F(x,y)$ is Schur convex (concave) on I^2 if and only if f is convex (concave) on I.*

Proof. Clearly $F(x,y)$ is symmetric. The result follows via lemmas (2.1), (2.4) together with theorem (1.1).

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